Bounds of the Sum-connectivity Energy of Graphs *

Fuqin Zhan[a,b] Youfu Qiao[a,b,†] Junliang Cai[c]

[a] *School of Mathematics and Statistics, Zhaoqing University, Zhaoqing 526061, P.R.China*

[b] *Department of Mathematics, Hechi University, Yizhou 546300, P.R.China*

[c] *College of mathematics, Beijing Normal University, Beijing 100875, P.R.China*

(zhanfq@mail.bnu.edu.cn; qiaoyf78@163.com; caijunliang@bnu.edu.cn)

Abstract

The sum-connectivity energy of a graph is defined as the sum of the absolute value of all the eigenvalues of its sum-connectivity matrix. In this paper, we give further lower and upper bounds for the sum-connectivity energy in terms of number of vertices, number of edges, the harmonic index, and determinant of sum-connectivity matrix. We also show that among connected graphs with n vertices, the star graph $K_{1,n-1}$ has the minimum sum-connectivity energy.

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1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G)$, $|E(G)| = m$. In what follows we say that G is an (n, m)-graph. Let d_i be the degree of the vertex v_i for $i = 1, 2, \dots, n$. If the vertices v_i and v_j are adjacent, we denote this by $v_i \sim v_j$, i.e., $v_iv_j \in E(G)$. The eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of the adjacency matrix $\mathbf{A}$, assumed in non-increasing order, are the eigenvalues of the graph G [1]. The energy of a

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† Corresponding author.

graph G is defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

This concept was introduce by Gutman [2], in connection to the so-called total π-electron energy. For more details on the theory of graph energy see the monograph [3]. The sum-connectivity matrix of G (denoted by $\mathbf{S}$) defined by Bo Zhou *et al.* [4]

$$s_{ij} = \begin{cases} \frac{1}{\sqrt{d_i+d_j}} & \text{if } v_i \text{ and } v_j \text{ are adjacent in } G, \\ 0 & \text{otherwise.} \end{cases}$$

Denote the eigenvalues of the sum-connectivity matrix $\mathbf{S} = \mathbf{S}(G)$ by $\mu_1, \mu_2, \ldots, \mu_n$ and arranged in a non-increasing order. The multiset $Sp_S = Sp_S(G) = \{\mu_1, \mu_2, \ldots, \mu_n\}$ will be referred to as the S-spectrum of G.

Recently, the sum-connectivity energy of the graph G is defined as [4]

$$SE(G) = \sum_{i=1}^{n} |\mu_i|.$$

For further study on this graph invariant the reader is referred to [4]. In this paper, we obtain some new upper and lower bounds on the sum-connectivity energy $SE(G)$ of the graph G. Moreover, we characterize the graph with minimal sum-connectivity energy in the connected graphs with n vertices.

2. Bounding the sum-connectivity energy

In this section, we obtain some lower and upper bounds on $SE(G)$ of graph G. First we need a chemical index, namely, harmonic index of a graph. The harmonic index $H(G)$ is defined as [5]

$$H = H(G) = \sum_{uv \in E(G)} \frac{2}{d(u) + d(v)}.$$

The harmonic index has recently attracted much attention [6–8].

Lemma 2.1(see [8]) Let G be a connected graph with n vertices. Then

$$H(G) \geq \frac{2(n-1)}{n}$$

with equality if and only if G is a star graph $K_{1,n-1}$.

Now we recall some known bounds. In 2010, Zhou [4] discovered the first upper and lower bound for $SE(G)$ as follows:

$$SE(G) \leq \sqrt{2n \sum_{i \sim j} \frac{1}{d_i + d_j}} \tag{1}$$

with equality if and only if G is an empty graph or a regular graph of degree one. Further,

$$SE(G) \geq 2\sqrt{\sum_{i\sim j}\frac{1}{d_i+d_j}} \tag{2}$$

with equality if and only if G is an empty graph or a complete bipartite graph with possibly isolated vertices.

We now give upper and lower bound on $SE(G)$ in terms of $|E(G)|$ and $|V(G)|$.

Theorem 2.2 Let G be an (n,m)-graph with no isolated vertex. Then

$$2\sqrt{\frac{m}{n}} \leq SE(G) \leq \sqrt{2}m. \tag{3}$$

Equality holds in the right side if and only if G is regular of degree 1, and equality holds in the left side if and only if G is complete bipartite graph.

Proof. Upper bound: We start with relation (1)

$$SE(G) \leq \sqrt{nH}.$$

For a graph with m edges and no isolated vertex, we have $n \leq 2m$. Therefore

$$SE(G) \leq \sqrt{2mH}.$$

We know that $d_i + d_j \geq 2$ for edge $ij \in E(G)$, then $H \leq m$. So

$$SE(G) \leq \sqrt{2}m.$$

All inequalities occuring in the above reasoning become equalities in the case of regular graphs of degree 1. Consequently, $SE(G) = \sqrt{2}m$ holds for regular graphs of degree 1. For all other graphs $H = m$ and $n = 2m$ cannot hold at the same time. Therefore $SE(G) = \sqrt{2}m$ holds if and only if G is regular of degree 1.

Lower bound: By inequality (2), we have

$$SE(G) \geq \sqrt{2H}.$$

For a graph with n vertices, we know $d_i + d_j \leq n$ for all edges $ij \in E(G)$. Then

$$H = H(G) \geq \frac{2m}{n}.$$

The equality holds if and only if $d_i + d_j = n$ for all edges $ij \in E(G)$. Therefore, the results follows. □

Theorem 2.3 Among all connected graphs of order n, the star graph $K_{1,n-1}$ attains the minimum sum-connectivity energy.

Proof. By inequality (2), we have $SE(G) \geq \sqrt{2H}$. The result follows from Lemma 2.1. □

Lemma 2.4(see [9]) Let a_i and b_i, $1 \le i \le n$, be real numbers for which there exist real constants a, b, A and B, so that for each i, $i = 1, 2, \dots, n$, $a \le a_i \le A$ and $b \le b_i \le B$. Then

$$\left| n\sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \right| \le s(n)(A-a)(B-b), \tag{4}$$

where $s(n) = n\left[\frac{n}{2}\right]\left(1 - \frac{1}{n}\left[\frac{n}{2}\right]\right)$. Equality in (4) holds if and only if $a_1 = \cdots = a_n$ and $b_1 = \cdots = b_n$.

Theorem 2.5 Let G be a graph with n vertices and m edges. Then

$$SE(G) \ge \sqrt{2m - s(n)(x_n - x_1)^2}, \tag{5}$$

where $s(n) = n[\frac{n}{2}](1 - \frac{1}{n}[\frac{n}{2}])$, x_1 and x_n are minimum and maximum values of the set $\{|\mu_i| | 1 \le i \le n\}$. Equality holds in (5) if and only if G is an empty graph or a regular graph of degree one.

Proof. Setting $a_i = |\mu_i|$, $b_i = |\mu_i|$, $a = b = x_1 = \min_{1\le i\le n} |\mu_i|$ and $A = B = x_n = \max_{1\le i\le n} |\mu_i|$ in Lemma 2.4, we obtain

$$\left| n\sum_{i=1}^{n} |\mu_i|^2 - \sum_{i=1}^{n} |\mu_i| \sum_{i=1}^{n} |\mu_i| \right| \le s(n)(x_n - x_1)^2.$$

Note that $\sum_{i=1}^{n} \mu_i^2 = H$, we have

$$\left| nH - SE^2(G) \right| \le s(n)(x_n - x_1)^2.$$

By the relations $SE(G) \le \sqrt{nH}$ and $H \ge \frac{2m}{n}$, then

$$SE(G) \ge \sqrt{2m - s(n)(x_n - x_1)^2}.$$

Now we suppose that equality holds in (5). Then all inequalities in the above argument must be equalities. From equality in (4), we get $|\mu_1| = \cdots = |\mu_n|$. Hence, we are reduced to two possibilities: either G has only one eigenvalue, in which case G must be an empty graph, or G has two eigenvalues with equal absolute values, implying that for any component H of G. $\mathbf{S}(H)$ has exactly two distinct eigenvalues μ_1 and $-\mu_1$. By the Perron-Frobenius theorem, the multiplicity of μ_1 as an eigenvalue of $\mathbf{S}(H)$ is one. Then we have $|V(G)| = 2$ and thus G is a regular graph of degree one. Conversely, one can see easily that the equality holds in (5) for empty graph and regular graph of degree one.

This completes the proof. □

Lemma 2.6(see [10]) Let a_i and b_i, $1 \leq i \leq n$, be real numbers for which there exist real constants r and s so that for each i, $i = 1, 2, \ldots, n$ holds $ra_i \leq b_i \leq sa_i$. Then

$$\sum_{i=1}^{n} b_i^2 + rs \sum_{i=1}^{n} a_i^2 \leq (r+s) \sum_{i=1}^{n} a_i b_i.$$

Equality holding if and only if for at least one i, $1 \leq i \leq n$, holds $ra_i = b_i = sa_i$.

Theorem 2.7 Let G be an (n,m)-graph. Then

$$SE(G) \geq \frac{2m + n^2 x_1 x_n}{n(x_1 + x_n)}, \tag{6}$$

where x_1 and x_n are minimum and maximum values of the set $\{|\mu_i| | 1 \leq i \leq n\}$. Moreover, the equality in (6) holds if and only if G is an empty graph.

Proof. Setting $b_i = |\mu_i|$, $a_i = 1$, $r = x_1$ and $s = x_n$, $1 \leq i \leq n$, in Lemma 2.6, we have

$$\sum_{i=1}^{n} |\mu_i|^2 + n x_1 x_n \leq (x_1 + x_n) \sum_{i=1}^{n} |\mu_i|.$$

Note that $\sum\limits_{i=1}^{n} |\mu_i|^2 = H$. Because $H \geq \frac{2m}{n}$, then

$$SE(G) \geq \frac{\frac{2m}{n} + n x_1 x_n}{x_1 + x_n}.$$

Now suppose that equality holds in (6). Then for some i holds $|\mu_i| = x_1 = x_n$, which implies $|\mu_i| \leq |\mu_j| \leq |\mu_i|$ for each j, $j \neq i$. Therefore, equality holds if and only if $|\mu_1| = \cdots = |\mu_n|$. □

Lemma 2.8(see [11]) Let $a_1, a_2, \ldots, a_n$ be non-negative numbers. Then

$$n\Big[\frac{1}{n}\sum_{i=1}^{n} a_i - \Big(\prod_{i=1}^{n} a_i\Big)^{\frac{1}{n}}\Big] \leq n\sum_{i=1}^{n} a_i - \Big(\sum_{i=1}^{n}\sqrt{a_i}\Big)^2 \leq n(n-1)\Big[\frac{1}{n}\sum_{i=1}^{n} a_i - \Big(\prod_{i=1}^{n} a_i\Big)^{\frac{1}{n}}\Big].$$

Theorem 2.9 Let G be a graph with $n \geq 2$ vertices. Then

$$\sqrt{H + n(n-1)\Delta^{\frac{2}{n}}} \leq SE(G) \leq \sqrt{(n-1)H + n\Delta^{\frac{2}{n}}}, \tag{7}$$

where Δ is the determinant of sum-connectivity matrix **S**.

Proof. By setting $a_i = \mu_i^2$, $1 \leq i \leq n$, in Lemma 2.8, we have

$$B \leq n\sum_{i=1}^{n} \mu_i^2 - \Big(\sum_{i=1}^{n} |\mu_i|\Big)^2 \leq (n-1)B,$$

where $B = n\Big[\frac{1}{n}\sum_{i=1}^{n}\mu_i^2 - \big(\prod_{i=1}^{n}\mu_i^2\big)^{\frac{1}{n}}\Big] = \sum_{i=1}^{n}\mu_i^2 - n\big(\prod_{i=1}^{n}\mu_i^2\big)^{\frac{1}{n}}$.

Note that $\sum_{i=1}^{n}\mu_i^2 = H$ and $\prod_{i=1}^{n}\mu_i = det(\mathbf{S}) = \Delta$. Then

$$H + n(n-1)\Delta^{\frac{2}{n}} \leq nH - SE^2(G) \leq (n-1)H + n\Delta^{\frac{2}{n}}.$$

Hence the result. □

Remark 2.10 The upper bound (7) is always better than the upper bound in (1). This is because by using arithmetic-geometric mean inequality, we have

$$2\sum_{i\sim j}\frac{1}{d_i + d_j} \geq n\Delta^{\frac{2}{n}}$$

and bearing in mind the upper bound in (7), we arrive at

$$SE(G) \leq \sqrt{nH}$$

which is the upper bound in (2).

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