Recursive Formulae for the Chromatic Polynomials of Complete r-uniform Mixed Interval Hypergraphs

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Abstract

In response to a problem of Voloshin, we find recursive formulae for the chromatic polynomials of complete r-uniform interval hypergraphs and cohypergraphs. We also give recursive formulae for the chromatic polynomials of complete 3-uniform and 4-uniform interval bihypergraphs and comment on the challenges for general r. Our method is to exploit the uniform and complete structure of these hypergraphs to extend the standard splitting-contraction algorithm to all partitions of a set of order $r-1$. We then turn the computation of the chromatic polynomials into an exercise in matrix multiplication and comment that this method can be extended to compute the chromatic polynomial of any complete uniform pattern interval hypergraph.

Keywords: chromatic polynomials, chromatic spectrum, hypergraph coloring, interval hypergraph, partitions, pattern hypergraph.

1 Introduction and Notation

A *hypergraph* $H = (X, E)$ is a collection of vertices $X = \{x_i \mid i \in I\}$ and a collection of hyperedges $E = \{e_j \subseteq X \mid j \in J\}$. A *coloring*, or a λ-coloring, of H is a mapping $f : X \to \{1, ..., \lambda\}$. The inverse image $f^{-1}(k)$ is a color set defined by the coloring f and the collection of the nonempty color sets defined by f gives a partition of X. For any subset $U \subset X$, the inverse image $f_U^{-1}(k)$ of the restricted mapping $f_U : U \to \{1, ..., \lambda\}$ is a color set of the induced subhypergraph $H[U]$. The nonempty color sets defined by f_U form a partition of U which we refer to as a *restricted partition.* Two distinct colorings f and g of H which define different partitions of X may define equivalent restricted partitions for a proper subset $U \subset X$. An important case of restricted partitions occurs when the subset U is the vertex set of some hyperedge e. We may omit the term restricted when the context is clear. Our results come from studying partitions which have *restricted equivalence* relative to a certain specified subset of vertices.

In traditional hypergraph coloring, which extends classical graph coloring, a proper coloring of a hypergraph requires that no hyperedge is colored monochromatically. Voloshin first considered the inverse setting of *cohypergraphs*, where no hyperedge is colored polychromatically (in a rainbow) [9]. He also considered *bihypergraphs*, where no hyperedge is colored monochromatically or polychromatically, and even mixed hypergraphs in which hyperedges of different edge-types may appear. More recently, Dvořák et al [4] have extended the subject to pattern hypergraphs, wherein each hyperedge e_j is assigned an edge-type π_j. An *edge-type* is a collection of partitions of a set of order $|e_j|$ which constitute the restricted partitions to which the color sets must adhere for the coloring to be *proper*. We write $\pi_j = D$ for the edge-type of hyperedge e_j which is the collection of all partitions of a set of order $|e_j|$ with at least two nonempty sets. We write $\pi_j = C$ for the edge-type which is the collection of all partitions of a set of order $|e_j|$ with fewer than $|e_j|$ nonempty sets. We write $\pi_j = B$ for the edge-type which is the collection of all partitions of a set of order $|e_j|$ with between two and $|e_j|-1$ nonempty sets. Our main results are for hypergraphs, cohypergraphs, and bihypergraphs, which have all hyperedges with the same edge-type D, C, or B respectively. However, our computational method will use a decomposition that incorporates superhypergraphs which are more general pattern hypergraphs.

It is well-known, see [8, 9], that the chromatic polynomial $P = P(H) = P(H, \lambda)$, which is the function that counts the number of proper λ-colorings of H, can be expressed in the form

$$P = \sum_{i=1}^{n} R_i \lambda^{\underline{i}} \tag{1}$$

where $|X| = n$ and $\lambda^{\underline{i}} = \lambda(\lambda-1)\cdots(\lambda-i+1)$ is the falling factorial. Note that the falling factorial counts the number of colorings of the complete graph on i vertices. Throughout this note λ is reserved for the variable of the chromatic polynomials, representing the maximum number of possible colors used in a coloring. As such, we choose to suppress the λ symbol from the name of the polynomials to simplify expressions.

The coefficients R_i in (1) count the number of possible, or feasible, partitions of X using i nonempty subsets. The subject of coloring problems places conditions on what partitions are allowed, as in the above, giving meaning to the hyperedges. Without coloring conditions, R_i is the Stirling number of the second kind $S(n,i)$ and their sum is the n^{th} Bell number B_n counting the number of partitions of a set of order n [5]. With coloring conditions, we have the trivial bound $R_i \leq S(n,i)$. The collection of these coefficients $\{R_1, ..., R_n\}$ is called the chromatic spectrum of H. We state our main results as recursive formulae for chromatic polynomials of a certain family of hypergraphs, but these formulae obviously imply relationships between the spectral values. In that light, we state all chromatic polynomials with respect to the falling factorial basis. The recursive formulae on the chromatic polynomials can naturally be translated to recursive relationships for spectral values. We state some of these relationships for the spectral values in this note and leave others to the reader. In order to state those expressions more concisely, we expand (1) to the expression

$$P = \sum_{i=1}^{\infty} R_i^n \lambda^{\underline{i}} \tag{2}$$

where $R_i^n = 0$ for $i > n$. Additionally, putting $R_i^n = 0$ for $i \leq 0$ allows for general expressions without worrying about zero or negative indexes.

Mixed hypergraph coloring has many diverse applications. Voloshin's monograph [8] gives an overview of many of these applications, such as problems in resource allocation, data base management, and molecular biology, where an interval mixed hypergraph (defined below) may be used to model a DNA molecule. There are also applications of mixed hypergraph coloring to issues regarding cybersecurity. One such reference is a recent Ph.d thesis [7] giving algorithms for scalable fault tolerance.

Many authors have studied the (lower) chromatic number of H (the smallest index of the nonzero spectral values), the upper chromatic number (the largest index of the nonzero spectral values), and whether there are gaps in the spectrum (zero spectral values with indexes between the lower and upper chromatic numbers). For a thorough survey of work, we recommend Volsohin's monograph [8] and his webpage [11]. It is generally a hard problem to compute the spectral values, and equivalently, to obtain explicit expressions for the chromatic polynomials of hypergraphs. Voloshin gives a splitting-contraction algorithm [8, 10] that finds the chromatic polynomial of any mixed hypergraph, but it has a high level of computational complexity that makes it impractical to use in large hypergraphs. In this note we study a class of hypergraphs with a specific structure that allows for an extension of the splitting-contraction algorithm, and this method helps us find recursive formulae for the chromatic polynomials for (D)-hypergraphs and cohypergraphs with this structure. The splitting-contraction algorithm computes chromatic polynomials by repeated case work corresponding to coloring pairs of vertices with either the same or different colors. This case work also corresponds to considering feasible partitions which have restricted partitions, when restricted to a pair of vertices, which are either a single two element set (same color) or two singleton sets (different colors). Our extension to the splitting-contration algorithm looks at all possible restricted partitions of a larger set of vertices. Labeling these restricted partitions requires a larger collection of terms which essentially replace the coarse values in the chromatic spectrum with a finer collection of values that distinguishes partitions on a more detailed level. For some classes of hypergraphs there is enough structure present so that these more detailed values reveal relationships, not only between themselves but also between the coarser spectral values. This note investigates complete interval hypergraphs which have enough structure to illustrate this theory.

An *interval hypergraph* on n vertices has an ordering on its vertex set $X = \{x_1, ..., x_n\}$ so that any hyperedge is a set of vertices indexed by consecutive integers. In a r-uniform hypergraph all hyperedges have order r. For the remainder of this note T^n will denote a complete r-uniform interval hypergraph, for which every set $e_j = \{x_j, ..., x_{j+r-1}\}$ for $j = 1, ..., n-r+1$ appears in the hyperedge set E. If all hyperedges have the same edge-type, we specify the edge-type by writing T^n_D, T^n_C, or T^n_B for a complete r-uniform interval (D)-hypergraph, cohypergraph, or bihypergraph respectively. When the order r of the hyperedges needs to be specified, we write $T^n_*(r)$. See Figure 1. In this note we only give one example of a complete uniform mixed interval hypergraph, however a convenient extension to the notation is $T^n(r, s)$, where s is a sequence whose j^{th} element gives the edge-type π_j. In a private communication, Voloshin asked if there were relationships between the chromatic polynomials of these hypergraphs, cohypergraphs, or bihypergraphs, knowing some relationships for small r. (Note that some things are already known about these mixed hypergraphs, such as that there are no gaps in their chromatic spectra [6].)

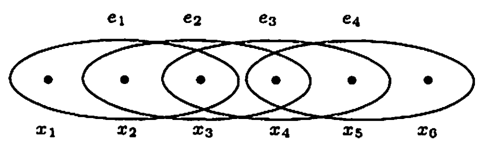

Figure 1: The complete 3-uniform interval hypergraph $T^6(3)$

By decomposing the chromatic polynomials of complete r-uniform interval mixed hypergraphs over all partitions of sets of order $r-1$, we develop an algebraic method for computing their chromatic polynomials. The equations that result from this method helped us find recursive formulae for all r for hypergraphs and cohypergraphs. The formula for hypergraphs has an easy combinatorial proof. However, the combinatorial proof for cohypergraphs is hidden within algebraic manipulations that result in the recursive formula. To navigate the details of the computational method, we need to set some notation to work with and compare partitions. The proofs of the recursive formulae in the main results in sections 3 and 4 can be read without the details of the computational method, as only polynomials that correspond to monochromatic and polychromatic colorings of a hyperedge of a certain superhypergraph are needed for these proofs.

Choose an index set $J = \{1, ..., B_{r-1}\}$ and an ordering of the partitions Y_j of $Y = \{y_1, ..., y_{r-1}\}$ following the finer-than partial ordering on the set of these partitions, so that $Y_j \geq Y_k$ when $j < k$ and Y_j and Y_k are comparable. Hence, Y_1 is the trivial partition comprised of one subset $\{\{y_1, ..., y_{r-1}\}\}$, $Y_{B_{r-1}}$ is comprised of all singletons $\{\{y_1\}, \{y_2\}, ...\{y_{r-1}\}\}$, and the partitions of the same size $|Y_j|$ (the number of subsets forming the partition) are grouped together in a specified order. See Table 1 for an example of this ordering and for the restrictions discussed below.

The chromatic polynomial $P^n = P(T^n)$ trivially decomposes as the sum $\sum_{j=1}^{B_{r-1}} P_j^n$ where P_j^n counts the colorings of T^n so that the restriction of the nonempty color sets to the last $r-1$ vertices $\{x_{n-r+2}, ..., x_n\}$ of T^n agrees with the partition Y_j applied to $\{x_{n-r+2}, ..., x_n\}$ under the bijection $y_i = x_{n-r+1+i}$.

In preparation for comparisons of partitions, as we move from T^{n-1} to T^n, we use X_j^n for the partition Y_j applied to $\{x_{n-r+2}, ..., x_n\}$, X_j^{n-1} for the partition Y_j applied to $\{x_{n-r+1}, ..., x_{n-1}\}$, and so on. Note P_j^n is the chromatic polynomial of the pattern interval hypergraph T_j^n which agrees with T^n with the given hyperedges and edge-types, but with the additional hyperedge $\{x_{n-r+2}, ..., x_n\}$ of (pattern)-edge-type given by X_j^n . It is clear that a partition Y_j on Y restricts to a partition $Y_j|_Z$ on any nonempty subset $Z \subset Y$. If Y_j is a partition on Y, Y'_k is a partition on Y', and $Z = Y \cap Y'$, then we say $Y_j = Y'_k$ if $Y_j|_Z = Y'_k|_Z$. We write $X_j^{n-1} = X_k^n$ for the understood restricted equality $X_j^{n-1}|_Z = X_k^n|_Z$ for $Z = \{x_{n-r+2}, ..., x_{n-1}\}$, or $X_j^* = X_k^*$ when the equality is on the intersection of the sets of which they partition.

This decomposition allows us to write P_k^n as a linear combination of P_j^{n-1} with polynomial coefficients corresponding to the edge-type π_n. The computation of the chromatic polynomial P^n then becomes the computation of a bilinear form obtained by multiplying $n-r+1$ matrices, the coefficient matrices determined by each hyperedge e_j, in the appropriate order and evaluating the bilinear form at a specific ordered pair of B_{r-1}-vectors. We make a series of observations, based on standard counting arguments, in the next section to determine the form of these linear combinations, and hence the form of these coefficient matrices.

2 Method of Computation of the Chromatic Polynomial of a Complete Uniform Mixed Interval Hypergraph

In this section we create an algebraic method for computing the chromatic polynomial of a complete r-uniform mixed interval hypergraph where edge-types of any hyperedge are either D, C, or B.

Observation 1: Consider the cases $X_j^{n-1} = X_k^n$ with any index j not equal to either 1 or B_{r-1} and any index k. A coloring counted in P_j^{n-1} extends to colorings counted in P_k^n, since $\{x_{n-r+1}, ..., x_{n-1}\}$ would be colored neither monochromatically nor polychromatically and trivially satisfies the coloring condition π_n for any of our three edge-types. If $\{x_n\}$ is a singleton in the partition X_k^n, then a coloring counted by P_j^{n-1} extends to $(\lambda - |Y_k| + 1)$ colorings counted by P_k^n, since x_n can be colored with any color not coloring any of the vertices $x_{n-r+2}, ..., x_{n-1}$ and $|Y_k| - 1$ colors must be used to color these vertices to satisfy X_k^n. If $\{x_n\}$ is not a singleton in the partition X_k^n , then a coloring counted by P_j^{n-1} extends to one coloring counted by P_k^n, since x_n must be colored with a color already specified for some vertex in $\{x_{n-r+2}, ..., x_{n-1}\}$.

Observation 2: The case $X_1^{n-1} = X_k^n$ happens only twice, when $k = 1$ and when $X_k^n = \{\{x_{n-r+2}, ..., x_{n-1}\}, \{x_n\}\}$. In the latter case, the coloring condition π_n is met for any of our three edge-types and a coloring in P_1^{n-1} extends to $(\lambda - 1)$ colorings counted by P_k^n. The edge-type π_n must be considered when $k = 1$. Colorings in P_1^{n-1} would only extend to a coloring in P_1^n when $\pi_n = C$, since the requirements of X_1^{n-1} and X_1^n force e_n to be monochromatic.

Observation 3: Consider cases when $X_{B_{r-1}}^{n-1} = X_k^n$. For $k < B_{r-1}$ the coloring condition π_n is met for any of our three edge-types and a coloring in $P_{B_{r-1}}^{n-1}$ extends to one coloring counted by P_k^n. The edge-type π_n must be considered when $k = B_{r-1}$. The vertices $\{x_{n-r+1}, ..., x_{n-1}\}$ would be colored polychromatically and colorings in $P_{B_{r-1}}^{n-1}$ would extend to a single coloring in $P^n_{B_{r-1}}$ when $\pi_n = C$ or B, since the requirements of $X^{n-1}_{B_{r-1}}$ and $X^n_{B_{r-1}}$ force x_n and x_{n-r+1} to share a color. However, when $\pi_n = D$ the coloring condition on e_n is already met by the requirements of $X^{n-1}_{B_{r-1}}$, and the additional requirement of $X^n_{B_{r-1}}$ forces x_n to be colored using one of the $(\lambda - r + 2)$ colors not already used to color $\{x_{n-r+2}, ..., x_{n-1}\}$.

Remark 2.1: The observations above completely determine the (k, j)-entries of three $B_{r-1} \times B_{r-1}$ coefficient matrices A_D, A_C, and A_B with polynomial entries from the set $\{0, 1, (\lambda - 1), ..., (\lambda - r + 2)\}$ indexed by the indexing set J chosen above. A different choice of J will permute the rows of each of the three matrices. The three matrices only differ in the $(1,1)$ and the (B_{r-1}, B_{r-1}) entries. The zero entries, except for the $(1,1)$ entry in A_D and A_B, correspond to cases when $X^{n-1}_j \neq X^n_k$.

Theorem 2.1. *The chromatic polynomial of the complete r-uniform mixed interval hypergraph T^n, for $n \geq r$, is given by*

$$P^n = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \cdot Q \cdot v \tag{3}$$

where $Q = A_{\pi_{n-r+1}} \cdot \ldots \cdot A_{\pi_1}$ and $v = \begin{bmatrix} \lambda & \cdots & \lambda^{\underline{|Y_j|}} & \cdots & \lambda^{\underline{r-1}} \end{bmatrix}^T$ is the B_{r-1}-vector whose j^{th} entry counts the colorings of the complete graph on $|Y_j|$ vertices.

Proof. We can write the decomposition

$$P^n = \sum_{j=1}^{B_{r-1}} P^n_j \text{ as } P^n = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \cdot \begin{bmatrix} P^n_1 \\ \vdots \\ P^n_{B_{r-1}} \end{bmatrix}$$

The vector v gives the decompostion of P^{r-1}, where T^{r-1} is the empty graph on the first $r - 1$ vertices. Taking the edge-type π_{n-r+1} into consideration and using observations 1-3 to create the appropriate coefficient matrix $A_{\pi_{n-r+1}}$, we express P^n in terms of $P^{n-1}_1, ..., P^{n-1}_{B_{r-1}}$

as $\begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \cdot A_{\pi_{n-r+1}} \cdot \begin{bmatrix} P^{n-1}_1 \\ \vdots \\ P^{n-1}_{B_{r-1}} \end{bmatrix}$. The matrix Q is formed inductively and the result follows.

□

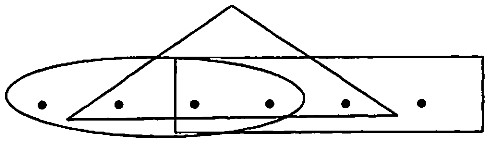

Figure 2: The complete 4-uniform mixed interval hypergraph $T^6(4, DBC)$ of Example 2.1.

Example 2.1: Let $T^6(4, DBC)$ be a complete 4-uniform mixed interval hypergraph with edge-types in the order D, B, and C. See figure 2. To compute the chromatic polynomial of $T^6(4, DBC)$, we could compute $Q = A_C \cdot A_B \cdot A_D$ and then evaluate the bilinear form as in (3). Instead, we compute the individual vectors to illustrate more of the computational technique.

Partitions of $\{y_1, y_2, y_3\}$ Y_j	applied to $\{x_2, x_3, x_4\}$ restricted to $\{x_2, x_3\}$ $X_k^4\|_{\{x_2,x_3\}}$	applied to $\{x_1, x_2, x_3\}$ restricted to $\{x_2, x_3\}$ $X_j^3\|_{\{x_2,x_3\}}$	Restricted equality of row k with column $j = (\ldots)$
$\{\{y_1, y_2, y_3\}\}$	$\{\{x_2, x_3\}\}$	$\{\{x_2, x_3\}\}$	$(1, 2)$
$\{\{y_1\}, \{y_2, y_3\}\}$	$\{\{x_2\}, \{x_3\}\}$	$\{\{x_2, x_3\}\}$	$(3, 4, 5)$
$\{\{y_2\}, \{y_1, y_3\}\}$	$\{\{x_2\}, \{x_3\}\}$	$\{\{x_2\}, \{x_3\}\}$	$(3, 4, 5)$
$\{\{y_3\}, \{y_1, y_2\}\}$	$\{\{x_2, x_3\}\}$	$\{\{x_2\}, \{x_3\}\}$	$(1, 2)(\lambda - 1)$
$\{\{y_1\}, \{y_2\}, \{y_3\}\}$	$\{\{x_2\}, \{x_3\}\}$	$\{\{x_2\}, \{x_3\}\}$	$(3, 4, 5)(\lambda - 2)$

Table 1: The restricted equivalence of the ordered partitions for complete 4-uniform mixed interval hypergraphs to find the coefficient matrices that will compute $P(T^n)$ from $P(T^{n-1})$.

We order the partitions as shown in Table 1. The fourth column indicates the columns corresponding to the restricted equivalence of X_j^{n-1} with the partition X_k^n (with $n = 4$ for convenience) in the given row. These entries are understood to be one, except for the $(1, 1)$ entry that depends on edge-type, or $(\lambda - m)$, except for the $(5, 5)$ entry that depends on edge-type, as indicated.

Other orderings that fit the finer-than partial ordering would permute the middle three partitions of the first column of Table 1. With the given ordering, the coefficient matrices are of the following form, only differing in the (1, 1) and (5, 5) entries:

$$A(\alpha,\beta) = \begin{bmatrix} \alpha & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ \lambda-1 & \lambda-1 & 0 & 0 & 0 \\ 0 & 0 & \lambda-2 & \lambda-2 & \beta \end{bmatrix} \text{ with } A_D = A(0,\lambda-2),$$

$A_B = A(0,1)$, and $A_C = A(1,1)$.

The computation begins with the initial vector $v = \begin{bmatrix} \lambda & \lambda^{\underline{2}} & \lambda^{\underline{2}} & \lambda^{\underline{2}} & \lambda^{\underline{3}} \end{bmatrix}^T$ and the product

$$A_D \cdot v = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ \lambda-1 & \lambda-1 & 0 & 0 & 0 \\ 0 & 0 & \lambda-2 & \lambda-2 & \lambda-2 \end{bmatrix} \cdot v = \begin{bmatrix} \lambda^{\underline{2}} \\ 2\lambda^{\underline{2}} + \lambda^{\underline{3}} \\ 2\lambda^{\underline{2}} + \lambda^{\underline{3}} \\ 2\lambda^{\underline{2}} + \lambda^{\underline{3}} \\ 3\lambda^{\underline{3}} + \lambda^{\underline{4}} \end{bmatrix}$$

Notice the sum of the components gives the chromatic polynomial $7\lambda^{\underline{2}} + 6\lambda^{\underline{3}} + \lambda^{\underline{4}}$ of a 4-hyperedge, as expected.

The next product is $A_B \cdot A_D \cdot v =$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ \lambda-1 & \lambda-1 & 0 & 0 & 0 \\ 0 & 0 & \lambda-2 & \lambda-2 & 1 \end{bmatrix} \cdot A_D \cdot v = \begin{bmatrix} 2\lambda^{\underline{2}} + \lambda^{\underline{3}} \\ 4\lambda^{\underline{2}} + 5\lambda^{\underline{3}} + \lambda^{\underline{4}} \\ 4\lambda^{\underline{2}} + 5\lambda^{\underline{3}} + \lambda^{\underline{4}} \\ 3\lambda^{\underline{2}} + 5\lambda^{\underline{3}} + \lambda^{\underline{4}} \\ 9\lambda^{\underline{3}} + 3\lambda^{\underline{4}} \end{bmatrix}$$

The sum of the preceding components gives the chromatic polynomial of the 4-uniform mixed interval hypergraph with edge-types in the sequence D then B. The interested reader can easily check that reversing the order of the computations produces the same sum, but different components. We expect the sum to be the same, since reversing the sequence of edge-types produces an isomorphic hypergraph. The components of the decompositions of the isomorphic hypergraphs are different due to the specific partitions corresponding to the terms of the decomposition and the coloring requirements of the different edge-types.

Lastly, $A_C \cdot A_B \cdot A_D \cdot v =$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ \lambda-1 & \lambda-1 & 0 & 0 & 0 \\ 0 & 0 & \lambda-2 & \lambda-2 & 1 \end{bmatrix} \cdot A_B \cdot A_D \cdot v = \begin{bmatrix} 6\lambda^{\underline{2}} + 6\lambda^{\underline{3}} + \lambda^{\underline{4}} \\ 7\lambda^{\underline{2}} + 19\lambda^{\underline{3}} + 5\lambda^{\underline{4}} \\ 7\lambda^{\underline{2}} + 19\lambda^{\underline{3}} + 5\lambda^{\underline{4}} \\ 6\lambda^{\underline{2}} + 18\lambda^{\underline{3}} + 9\lambda^{\underline{4}} + \lambda^{\underline{5}} \\ 26\lambda^{\underline{3}} + 17\lambda^{\underline{4}} + 2\lambda^{\underline{5}} \end{bmatrix}$$

which produces the chromatic polynomial $P(T^6(4, DBC)) = 26\lambda^{\underline{2}} + 88\lambda^{\underline{3}} + 37\lambda^{\underline{4}} + 3\lambda^{\underline{5}}$

We can compute the matrix Q first and then evaluate the bilinear form (3) determined by Q to go immediately to the chromatic polynomial of T^n. If we wish to keep the expressions in terms of the falling factorial basis, to compute the chromatic spectrum, we use the trivial relationship $(\lambda - j)\lambda^{\underline{i}} = (i - j)\lambda^{\underline{i}} + \lambda^{\underline{i+1}}$ as we have above.

As an extension to example 2.1, with $Q = A_C \cdot A_B \cdot A_D =$

$$\begin{bmatrix} \lambda-1 & \lambda-1 & \lambda & \lambda & \lambda \\ (\lambda-1)^2 & \lambda(\lambda-1) & 4(\lambda-2)+2 & 4(\lambda-2)+2 & 4(\lambda-2)+2 \\ (\lambda-1)^2 & \lambda(\lambda-1) & 4(\lambda-2)+2 & 4(\lambda-2)+2 & 4(\lambda-2)+2 \\ (\lambda-1)^2 & (\lambda-1)^2 & \lambda(\lambda-1) & \lambda(\lambda-1) & \lambda(\lambda-1) \\ 2(\lambda-1)(\lambda-2) & 3(\lambda-1)(\lambda-2) & 2\lambda(\lambda-2) & 2\lambda(\lambda-2) & 2\lambda(\lambda-2) \end{bmatrix}$$

we can then find the chromatic polynomial of $T^{3(k+1)}(4, DBCDBC\ldots)$ via the decomposition $Q^k \cdot v$.

One interested in any complete r-uniform pattern interval hypergraph, can add to observations 1-3 in consideration of other edge-types to broaden the scope of the construction of the factors of the operator Q. The zero entries of A_C will continue to be zero for any pattern-edge-type of the hyperedge, but any of the nonzero values of A_C can change depending on the edge-type. Then the proof of Theorem 2.1 extends without change to give a method for computing the chromatic polynomials of such pattern interval hypergraphs.

3 Recursive formulae for the Chromatic Polynomials of Complete r-uniform Interval (D)-Hypergraphs

We make some remarks on the elements of the coefficient matrices for complete uniform interval mixed hypergraphs to first obtain a well-understood combinatorial relationship, verifying the details of their construction.

Remark 3.1: If there are m subsets in the partition $X_j^{n-1}|_{\{x_{n-r+2},\ldots,x_{n-1}\}}$, then the j^{th} column of the coefficient matrix will have $m+1$ nonzero entries, corresponding to the $m+1$ possible places x_n could take in X_k^n, except for $j=1$ and the $(1,1)$ entry is zero for the D and B coefficient matrices. If $X_i^{n-1}|_{\{x_{n-r+2},\ldots,x_{n-1}\}} = X_j^{n-1}|_{\{x_{n-r+2},\ldots,x_{n-1}\}}$, the i^{th} and j^{th} columns of the coefficient matrix will be the same vector, except for possibly when either is the first or the last column. There will be $m+1$ such equivalent columns corresponding to the $m+1$ possible places for x_{n-r+1}. The sum of the entries of each column (except perhaps the first and last) of the coefficient matrix for any of the three edge-types is λ, since an entry of the form $(\lambda - m)$ in a column guarantees there must be exactly m other nonzero entries equal to one in the same column.

Remark 3.2: If there are m subsets in the partition $X_k^n|_{\{x_{n-r+2},\ldots,x_{n-1}\}}$, then the k^{th} row of the coefficient matrix will have $m+1$ nonzero entries that are either all 1 or all $(\lambda - m)$, with the usual possible exception for $k = B_{r-1}$, depending on edge-type.

It now follows, with the appropriate $(1,1)$ and (B_{r-1}, B_{r-1}) entries in place for each of the three edge-types,

Theorem 3.1. *The chromatic polynomials for the complete r-uniform interval hypergraphs in each of the three edge-types satisfy the relationships*

$$
\begin{aligned}
&for\ \ T_D^n: && P^n = \lambda P^{n-1} - P_1^{n-1} && (4)\\
&for\ \ T_C^n: && P^n = \lambda P^{n-1} - (\lambda - r + 1)P_{B_{r-1}}^{n-1} && (5)\\
&for\ \ T_B^n: && P^n = \lambda P^{n-1} - P_1^{n-1} - (\lambda - r + 1)P_{B_{r-1}}^{n-1} && (6)
\end{aligned}
$$

□

Equations (4), (5), and (6) are trivial counting relationships which amount to multiplying the colorings of the first $n-1$ vertices by the λ possible colors of the n^{th} vertex, and then subtracting either the number of colorings that color e_n monochromatically (given by P_1^{n-1}), the number of colorings that color e_n polychromatically (given by $(\lambda - r + 1)P_{B_{r-1}}^{n-1}$), or both. As such, our larger decomposition gives a complicated confirmation of these basic counting relationships. However, having P_j^n for $j = 2, ..., B_{r-1} - 1$ provides a more robust computation technique, Theorem 2.1. In section 5 we convert (6) to recursive formulae for complete 3-uniform and 4-uniform interval bihypergraphs using algebraic manipulations that require the full decomposition.

Lemma 3.1. *The chromatic polynomials for the family of hypergraphs $T_D^n(r)$ satisfy, for $r \geq 3$ and $n \geq 2r - 2$,*

$$P_1^n = (\lambda - 1)P^{n-r+1}$$

Proof. The colorings of T_D^n counted by P_1^n color the vertices $x_{n-r+2}, ..., x_n$ with the same color. To satisfy the edge-type D, that color must be different from the color assigned to vertex x_{n-r+1}. Taking any coloring of the first $n - r + 1$ vertices counted by P^{n-r+1} and assigning one of the remaining $\lambda - 1$ colors not assigned to x_{n-r+1} to $x_{n-r+2}, ..., x_n$ produces any possible coloring counted by P_1^n. □

Combining Lemma 3.1 with (4) proves

Theorem 3.2. *The chromatic polynomials for the family of hypergraphs $T_D^n(r)$ satisfy, for $r \geq 3$ and $n \geq 2r - 1$,*

$$P^n = \lambda P^{n-1} - (\lambda - 1)P^{n-r}$$

□

Using the notation established in (2) and the subsequent comment, we can state this recursion for the individual spectral values as

Corollary 3.2.1. *The coefficients R_i^n of the chromatic polynomial of the family $T_D^n(r)$, for $n \geq 2r - 1$, satisfy*

$$R_i^n = iR_i^{n-1} - (i-1)R_i^{n-r} + R_{i-1}^{n-1} - R_{i-1}^{n-r}$$

□

4 Recursive formulae for the Chromatic Polynomials of Complete r-uniform Interval Cohypergraphs

We first define a series of families of partitions and a nested family of indexing sets based on restricted equivalence.

Choose index sets J_k for $k = 1, ..., r-1$ with indexes $j_{ki} \in J_k$ and a corresponding family of partitions $Y_{j_{ki}}$ so that

a) $J_1 = \{1\}$ with $Y_{j_{11}} = Y_1 = \{\{y_1, ..., y_{r-1}\}\}$
b) $J_{k-1} \subset J_k$ for $k = 2, ..., r-1$ with $Y_{j_{k1}} = Y_1$
c) $|J_k| = B_k$ with $Y_{j_{kB_k}} = \{\{y_1\}, ...\{y_{k-1}\}, \{y_k, ..., y_{r-1}\}\}$
d) the family of partitions $Y_{j_{ki}}$ is formed from the family $Y_{j_{(k-1)i}}$ by increasing each vertex index by 1, removing y_r, and appending a new y_1 to the partition by either adding it to an existing subset or appending the singleton $\{y_1\}$. This process ensures, for each $i = 1, ..., B_k$, the restricted equivalence of the partitions $X^{n-1}_{j_{ki}} = X^n_{j_{(k-1)l}}$ for exactly one $l = 1, ..., B_{k-1}$.

Remark 4.1: To illustrate the construction of the above family of partitions, note that $P^{n-k+2}_{j_{(k-1)l}} = \sum_{j_{ki}} P^{n-k+1}_{j_{ki}}$ where $X^{n-k+1}_{j_{ki}} = X^{n-k+2}_{j_{(k-1)l}}$ for each of the $|X^{n-k+2}_{j_{(k-1)l}}| + 1$ terms in the sum.

In comparison with Lemma 3.1, we have the following lemma.

Lemma 4.1. *For T^n_C, the nested family of partitions and indexing sets above, and $k = 1, ..., r-1$*

$$P^n_1 = \sum_{i=1}^{B_k} P^{n-k+1}_{j_{ki}} = P^{n-r+2}$$

Proof. The result follows from Remark 4.1 and induction on $k = 1, ..., r-1$. Note that the equality $P^n_1 = P^{n-r+2}$ is trivial by a counting argument like the one used in Lemma 3.1. □

Remark 4.1 and Lemma 4.1 show more of the counting relationships that correspond to natural combinatorial case work which is implicit in the relationships between the different partitions, but only part of Lemma 4.1 is needed in the proof of the recursive formula via the key step given by the next lemma. The combinatorial proof of the recursive formula only requires defining the partitions given in part (c) of the construction, which correspond to coloring polychromatically a hyperedge with the vertices with the $r-k$ highest indexes contracted.

Lemma 4.2. *For T^n_C, the nested family of partitions and indexing sets above, and $k = 1, ..., r-2$*

$$P^{n-k} - P^{n-1}_{j_{(r-k)B_{(r-k)}}} = \tag{7}$$
$$(r-k-1)P^{n-k-1} + (\lambda - r + k + 1)[P^{n-k-1} - P^{n-1}_{j_{(r-k-1)B_{(r-k-1)}}}]$$

Proof. We prove the lemma by counting the colorings of a new hypergraph H in two ways. Form the hypergraph H by adding the cohyperedge $\{x_{n-r+1}, \ldots, x_{n-k}\}$ to the set of hyperedges E of the complete r-uniform interval cohypergraph T^{n-k}. See figure 3, which depicts the case when $k = 2$ and $r = 5$

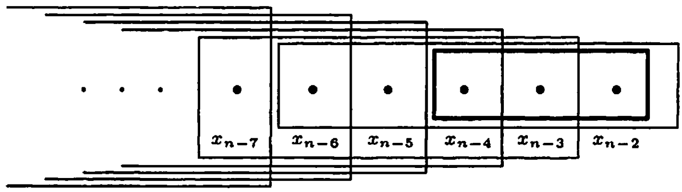

Figure 3: T_C^{n-k} and the cohypergraph H, with the additional bold cohyperedge, used in the proof of Lemma 4.2.

The polynomial P^{n-k} counts the colorings of the original interval cohypergraph T^{n-k}, which includes some colorings that color the vertices $x_{n-r+1}, \ldots, x_{n-k}$ polychromatically. By the above construction of the family of partitions, $P^{n-1}_{j_{(r-k)}B_{(r-k)}}$ counts those colorings where $x_{n-r+1}, \ldots, x_{n-k}$ are colored polychromatically. Hence the left hand side of (7) counts the colorings of H.

Alternatively, λP^{n-k-1} counts the colorings of another new hypergraph which is the union of T^{n-k-1} and an isolated vertex x_{n-k}. Again by our construction above, the polynomial $P^{n-1}_{j_{(r-k-1)}B_{(r-k-1)}}$ counts the colorings of T^{n-k-1} (vertices x_{n-k-1} through x_{n-1} have been contracted) which color the $r-k-1$ vertices $x_{n-r+1}, \ldots, x_{n-k-1}$ polychromatically. Hence, $(\lambda - r + k + 1)P^{n-1}_{j_{(r-k-1)}B_{(r-k-1)}}$ counts the colorings of T^{n-k} in which the vertices $x_{n-r+1}, \ldots, x_{n-k}$ are polychrome, and $\lambda P^{n-k-1} - (\lambda - r + k + 1)P^{n-1}_{j_{(r-k-1)}B_{(r-k-1)}}$ counts the colorings of H. The result (7) follows after regrouping the terms. □

Theorem 4.1. *The chromatic polynomials for the family of cohypergraphs $T_C^n(r)$ satisfy, for $r \geq 3$ and $n \geq 2r-2$,*

$$P^n = (r-1)P^{n-1} + (r-2)(\lambda-r+1)P^{n-2} + \ldots + (\lambda-r+1)\cdots(\lambda-2)P^{n-r+1} \tag{8}$$

$$= \sum_{k=1}^{r-1}(r-k)(\lambda-r+k-1)^{\underline{k-1}}P^{n-k}$$

Proof. Regroup terms in equation (5) to obtain

$$P^n = \lambda P^{n-1} - (\lambda - r + 1)P^{n-1}_{B_{r-1}} = (r-1)P^{n-1} + (\lambda - r + 1)(P^{n-1} - P^{n-1}_{B_{r-1}})$$

We now expand $P^{n-1} - P^{n-1}_{B_{r-1}}$ using (7) repeatedly to obtain

$$P^{n-1} - P^{n-1}_{B_{r-1}} = P^{n-1} - P^{n-1}_{j_{(r-1)B_{r-1}}}$$

$$= (r-2)P^{n-2} + (\lambda - r + 2)(P^{n-2} - P^{n-1}_{j_{(r-2)B_{r-2}}})$$

$$\vdots$$

$$= (r-2)P^{n-2} + (\lambda - r + 2)[(r-3)P^{n-3} + (\lambda - r + 3)[(r-4)P^{n-4} + \ldots$$

where the innermost term will be

$$(\lambda - 2)[P^{n-r+1} + (\lambda - 1)(P^{n-r+1} - P^{n-1}_{j_1 B_1})]\ldots]$$

By Lemma 4.1, $P^{n-1}_{j_1 B_1} = P^{n-r+1}$ and the expansion concludes in equation (8).

□

Since there are multiple shifts of the coefficients and $r-1$ polynomials involved in the recursion, we state an example of the resulting recursion for the spectral values for the case $r = 4$.

Corollary 4.1.1. *The coefficients R^n_i of the chromatic polynomial of the family $T^n_C(4)$, for $n \geq 6$, satisfy*

$$R^n_i = 3R^{n-1}_i + 2(i-3)R^{n-2}_i + 2R^{n-2}_{i-1} + (i-3)(i-2)R^{n-3}_i + (2i-5)R^{n-3}_{i-1} + R^{n-3}_{i-2}$$

□

5 Complete Uniform Interval Bihypergraphs

The algebraic treatments for complete r-uniform interval hypergraphs and cohypergraphs, left as exercises, are relatively simple with the case $r = 3$ being trivial. With bihypergraphs we immediately have more complex manipulations to arrive at recursive formulae. Below we show the manipulations that produce recursive formulae for complete 3-uniform and 4-uniform interval bihypergraphs, and we comment on the difficulty in generalizing these formulae to $r > 4$.

The coefficient matrix A_B, or the observations that produce it, gives the relationships

$$\begin{aligned} P_1^n &= P_2^{n-1} && (9) \\ P_2^n &= (\lambda-1)P_1^{n-1} + P_2^{n-1} = (\lambda-2)P_1^{n-1} + P^{n-1} && (10) \end{aligned}$$

for the chromatic polynomial decompositions of the complete 3-uniform interval byhypergraphs T_B^n.

Adding equations (9) and (10) and iterating produces

$$\begin{aligned} P^n &= (\lambda-1)P_1^{n-1} + 2P_2^{n-1} \\ &= 2P^{n-1} + (\lambda-3)P_1^{n-1} && (11) \\ &= 2P^{n-1} + (\lambda-3)P^{n-3} + (\lambda-3)(\lambda-2)P_1^{n-3} && (12) \end{aligned}$$

Using equation (11) for $n-2$ and multiplying by $(\lambda-2)$ gives

$$(\lambda-2)P^{n-2} = 2(\lambda-2)P^{n-3} + (\lambda-3)(\lambda-2)P_1^{n-3} \qquad (13)$$

Subtracting (13) from (12) proves

Theorem 5.1. *The chromatic polynomials for the family of bihypergraphs $T_B^n(3)$ satisfy, for $n \geq 5$,*

$$P^n = 2P^{n-1} + (\lambda-2)P^{n-2} + (\lambda-1)P^{n-3}$$

□

Proceeding to the case $r = 4$, using the order of partitions given in Table 1, we find

$$\begin{aligned} P_1^n &= P_2^{n-1} \\ P_2^n &= P_3^{n-1} + P_4^{n-1} + P_5^{n-1} \\ P_3^n &= P_3^{n-1} + P_4^{n-1} + P_5^{n-1} && (14) \\ P_4^n &= (\lambda-1)(P_1^{n-1} + P_2^{n-1}) \\ P_5^n &= (\lambda-2)(P_3^{n-1} + P_4^{n-1}) + P_5^{n-1} \end{aligned}$$

from which we find the following special case of the general relationship (6)

$$P^n = \lambda P^{n-1} - P_1^{n-1} - (\lambda-3)P_5^{n-1} \qquad (15)$$

Note the above relationships in (14) immediately imply the relationships

$$\begin{aligned} P_1^n + P_2^n &= P^{n-1} - P_1^{n-1} \\ P_3^n + P_4^n &= (\lambda-2)(P_1^{n-1}+P_2^{n-1}) + P^{n-1} \\ P_3^n + P_4^n + P_5^n &= (\lambda-1)P^{n-1} + (\lambda-3)P_5^{n-1} \end{aligned} \tag{16}$$

which will expedite the proof of

Theorem 5.2. *The chromatic polynomials for the family of bihypergraphs $T_B^n(4)$ satisfy, for $n \geq 8$*

$$P^n = \lambda P^{n-1} - (\lambda-3)(\lambda-2)P^{n-3} - ((\lambda-3)(\lambda-2)(\lambda-1) + (\lambda-1))P^{n-4} - (\lambda-3)(\lambda-2)(\lambda-1)P^{n-5} \tag{17}$$

Proof. To eliminate the P_j^* terms from (15), we manipulate it as follows:

$$P^n = \lambda P^{n-1} - P_2^{n-2} - (\lambda-3)(\lambda-2)(P_3^{n-1}+P_4^{n-1}) - (\lambda-3)P_5^{n-2}$$

$$= \lambda P^{n-1} - (P_3^{n-3}+P_4^{n-3}+P_5^{n-3}) - (\lambda-3)(\lambda-2)^2(P_1^{n-3}+P_2^{n-3}) - (\lambda-3)(\lambda-2)P^{n-3} - (\lambda-3)(\lambda-2)(P_3^{n-3}+P_4^{n-3}) - (\lambda-3)P_5^{n-3}$$

$$= \lambda P^{n-1} - (\lambda-3)(\lambda-2)P^{n-3} - (\lambda-1)P^{n-4} + (\lambda-3)P_5^{n-4} - (\lambda-3)(\lambda-2)^2(P_1^{n-3}+P_2^{n-3}) - (\lambda-3)(\lambda-2)(P_3^{n-3}+P_4^{n-3}) - (\lambda-3)(\lambda-2)(P_3^{n-4}+P_4^{n-4}) - (\lambda-3)P_5^{n-4}$$

$$= \lambda P^{n-1} - (\lambda-3)(\lambda-2)P^{n-3} - (\lambda-1)P^{n-4} - (\lambda-3)(\lambda-2)^2(P_1^{n-3}+P_2^{n-3}) - (\lambda-3)(\lambda-2)^2(P_1^{n-4}+P_2^{n-4}) - (\lambda-3)(\lambda-2)P^{n-4} - (\lambda-3)(\lambda-2)^2(P_1^{n-5}+P_2^{n-5}) - (\lambda-3)(\lambda-2)P^{n-5}$$

$$= \lambda P^{n-1} - (\lambda-3)(\lambda-2)P^{n-3} - ((\lambda-3)(\lambda-2) + (\lambda-1) + (\lambda-3)(\lambda-2)^2)P^{n-4} - (\lambda-3)(\lambda-2)P^{n-5} - (\lambda-3)(\lambda-2)^2P_2^{n-4} - (\lambda-3)(\lambda-2)^2(P_1^{n-5}+P_2^{n-5})$$

To obtain the last equation above, we use the first equation of (16) on $P_1^{n-3} + P_2^{n-3}$, producing a P_1^{n-4} term that cancels with the one already present. For the remaining P_2^{n-4} term, we use the second equation of (14) and the fact that $P_1^{n-5}+P_2^{n-5}$ is $P^{n-5}-(P_3^{n-5}+P_4^{n-5}+P_5^{n-5})$ to complete the process of eliminating the P_j^* terms and arrive at the result (17).

□

Neither technique used for $T_B^n(3)$ or $T_B^n(4)$ seems to extend to the case when $r = 5$, much less the case for general r. It is desirable to find a simpler process for the proof of (17) that can more easily be extended to higher r. It would be more desirable to find a simpler recursion than that given by (17). In the cases of $T_D^n(r)$ and $T_C^n(r)$, we were able to replace most of the algebraic manipulations used in the discovery of the recursive formulae, similar to those seen above, with counting arguments. It would be most desirable to find a consistent pattern of recursive formulae for $T_B^n(r)$ that would permit a counting argument for the formula itself, as in the hypergraph case, or for a key algebraic step, as in the cohypergraph case. As a first indication of the increased difficulty of the bihypergraph case, we have not found an analog to Lemmas 3.1 and 4.1, expressing P_1^n only in terms of previous chromatic polyomials, without requiring terms of our extended decomposition.

6 Spectra for Complete Uniform Interval Hypergraphs of Constant Edge-type for Small r up to $n = 10$

In each of the cases given in this paper, if we want to use the recursive formulae to compute chromatic polynomials, we must first use the method of Theorem 2.1 to compute the first few polynomials. To illustrate these computations, we include the decompositions of the first eight polynomials for the family $T_B^n(4)$.

We generate the terms of the decomposition of the chromatic polynomials for $n = 4, ..., 8$, given in Table 2, duplicating the work of Example 2.1 but with each coefficient matrix a copy of A_B.

Adding the columns in Table 2, we find the first five chromatic polynomials of the complete 4-uniform interval byhypergraphs are

	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
P_1^n	0, 1	0, 2, 1	0, 4, 5	0, 7, 19, 3	0, 13, 63, 23, 1
P_2^n	0, 2, 1	0, 4, 5	0, 7, 19, 3	0, 13, 63, 23, 1	0, 24, 198, 125, 14
P_3^n	0, 2, 1	0, 4, 5	0, 7, 19, 3	0, 13, 63, 23, 1	0, 24, 198, 125, 14
P_4^n	0, 2, 1	0, 3, 5, 1	0, 6, 18, 6	0, 11, 59, 33, 3	0, 20, 184, 160, 30, 1
P_5^n	0, 0, 3	0, 0, 9, 2	0, 0, 26, 14, 1	0, 0, 76, 69, 10	0, 0, 222, 330, 78, 4

Table 2: The coefficients of the terms of the chromatic polynomials of complete 4-uniform interval bihypergraphs of order n. The entry 0, 0, 222, 330, 78, 4 indicates the polynomial $222\lambda^{\underline{3}} + 330\lambda^{\underline{4}} + 78\lambda^{\underline{5}} + 4\lambda^{\underline{6}}$.

$$
\begin{aligned}
P^4 &= 7\lambda^{\underline{2}} + 6\lambda^{\underline{3}} \\
P^5 &= 13\lambda^{\underline{2}} + 25\lambda^{\underline{3}} + 3\lambda^{\underline{4}} \\
P^6 &= 24\lambda^{\underline{2}} + 87\lambda^{\underline{3}} + 26\lambda^{\underline{4}} + \lambda^{\underline{5}} \\
P^7 &= 44\lambda^{\underline{2}} + 280\lambda^{\underline{3}} + 151\lambda^{\underline{4}} + 15\lambda^{\underline{5}} \\
P^8 &= 81\lambda^{\underline{2}} + 865\lambda^{\underline{3}} + 736\lambda^{\underline{4}} + 137\lambda^{\underline{5}} + 5\lambda^{\underline{6}}
\end{aligned}
$$

Using $P^3 = \lambda + 3\lambda^{\underline{2}} + \lambda^{\underline{3}}$, one can confirm that P^8 is obtained from the recursion (17) and the polynomials P^3 through P^7.

Using a computer algebra system such as Maple 16, as we have to obtain some of the results below, for small r it is a simple process to use the computational method of Theorem 2.1 to obtain the chromatic polynomials and the corresponding spectral values. For $r = 3$ there is no significant computational effort saved by using the recursive relationships over computing the full decomposition. For $r = 4$ we found it convenient to use the recursive relationships to compute by hand and check our computations using Maple to compute the full decomposition. For $r = 5$ and $r = 6$ we only used Maple, since the recursive formulae are not applicable until $n = 9$ or $n = 10$. Were one interested in carrying the computations forward beyond $n = 10$, the recursive formula would save some computational effort. Of course, we have yet to find a recursive formula for the bihypergraph case, so the full decompostion must be used in that case for all n. As r grows and the corresponding Bell numbers grow, the benefit of the recursive formulae grows. We record in Tables 3, 4, ,5, and 6 the chromatic spectra for complete uniform interval hypergraphs in each of our three edge-types for $r = 3, 4, 5$ and 6 for orders up to $n = 10$ for reference, and to illustrate some final remarks about some of the recurring spectral values.

Remark 6.1: When fewer than r colors are used, each hyperedge of $T^n(r)$ must have two or more vertices which share a color. Hence, the cohypergraph coloring condition is trivially met and the first $r-1$ coefficients of $P(T^n_C(r))$ are the Stirling numbers of the second kind $S(n,1),...,S(n,r-1)$. Hence, $P(T^n_C(r-1))$ and $P(T^n_C(r))$ have the same first $r-2$ coefficients.

Remark 6.2: When more than $n-r+1$ colors are used, each hyperedge of $T^n(r)$ must have two or more vertices colored differently. Hence, the (D)-hypergraph coloring condition is trivially met and the last $r-1$ coefficients of $P(T^n_D(r))$ are the Stirling numbers of the second kind $S(n,n-r+2), S(n,n-r+3),...,S(n,n)$. Hence, $P(T^n_D(r-1))$ and $P(T^n_D(r))$ have the same last $r-2$ coefficients.

Remark 6.3: It immediately follows from the previous two remarks that the first $r-1$ coefficients of $P(T^n_B(r))$ are the same as the first $r-1$ coefficients of $P(T^n_D(r))$, since the cohypergraph condition is trivially met. Likewise the last $r-1$ coefficients of $P(T^n_B(r))$ are the same as the last $r-1$ coefficients of $P(T^n_C(r))$. Furthermore, when $n \leq 2r-3$ all three polynomials $P(T^n_D(r)), P(T^n_C(r))$, and $P(T^n_B(r))$ will share the coefficients $S(n,n-r+2), S(n,n-r+3),...,S(n,r-1)$. For example, in Table 6 row $n=9$, 6951 is the fifth coefficient for all three polynomials.

n	Hypergraphs	Cohypergraphs	Bihypergraphs
3	$0,3,1$	$1,3,0$	$0,3,0$
4	$0,5,6,1$	$1,7,1,0$	$0,5,1,0$
5	$0,8,22,10,1$	$1,15,5,0,0$	$0,8,4,0,0$
6	$0,13,69,61,15,1$	$1,31,18,1,0,0$	$0,13,11,1,0,0$
7	$0,21,203,304,135,21,1$	$1,63,56,7,0,0,0$	$0,21,27,5,0,0,0$
8	$0,34,578,1367,965,260,$ $28,1$	$1,127,161,34,1,0,$ $0,0$	$0,34,62,19,1,0,$ $0,0$
9	$0,55,1617,5794,6071,2505,$ $455,36,1$	$1,255,441,138,9,0,$ $0,0,0$	$0,65,137,61,6,0,$ $0,0,0$
10	$0,89,4479,23678,35305,$ $20861,5663,742,45,1$	$1,511,1170,505,55,$ $1,0,0,0,0$	$0,109,295,180,30,$ $1,0,0,0,0$

Table 3: The chromatic spectra for the family $T^n(3)$ up to $n=10$ for each of our three edge-types.

n	Hypergraphs	Cohypergraphs	Bihypergraphs
4	$0,7,6,1$	$1,7,6,0$	$0,7,6,0$
5	$0,13,25,10,1$	$1,15,25,3,0$	$0,13,25,3,0$
6	$0,24,87,65,15,1$	$1,31,90,26,1,0$	$0,24,87,26,1,0$
7	$0,44,280,346,140,$ $21,1$	$1,63,301,153,$ $15,0,0$	$0,44,280,151,$ $15,0,0$
8	$0,81,865,1655,1045,$ $266,28,1$	$1,127,966,762,$ $138,5,0,0$	$0,81,865,736,$ $137,5,0,0$
9	$0,149,2613,7430,6866,$ $2640,462,36,1$	$1,255,3025,3457,$ $980,77,1,0,0$	$0,149,2613,3261,$ $961,77,1,0,0$
10	$0,274,779,32051,41635,$ $22686,5873,750,45,1$	$1,511,9330,14798,$ $6019,780,28,0,0,0$	$0,274,7790,13629,$ $5807,772,28,0,0,0$

Table 4: The chromatic spectra for the family $T^n(4)$ up to $n = 10$ for each of our three edge-types.

n	Hypergraphs	Cohypergraphs	Bihypergraphs
5	$0,15,25,10,1$	$1,15,25,10,0$	$0,15,25,10,0$
6	$0,29,90,65,15,1$	$1,31,90,65,6,0$	$0,29,90,65,6,0$
7	$0,56,298,350,$ $140,21,1$	$1,63,301,350,$ $76,3,0$	$0,56,298,350,$ $76,3,0$
8	$0,108,945,1697,$ $1050,266,28,1$	$1,127,966,1701,$ $638,62,1,0$	$0,108,945,1697,$ $638,62,1,0$
9	$0,208,2924,7724,$ $6946,2646,462,36,1$	$1,255,3025,7770,$ $4444,757,35,0,0$	$0,208,2924,7724,$ $4441,757,35,0,0$
10	$0,401,8915,33765,$ $42440,22821,5880,$ $750,45,1$	$1,511,9330,34105,$ $27826,7246,642,$ $15,0,0$	$0,401,8915,33765,$ $27767,7244,642,$ $15,0,0$

Table 5: The chromatic spectra for the family $T^n(5)$ up to $n = 10$ for each of our three edge-types.

n	Hypergraphs	Cohypergraphs	Bihypergraphs
6	$0,31,90,65,15,1$	$1,31,90,65,15,0$	$0,31,90,65,15,0$
7	$0,61,301,350,$ $140,21,1$	$1,63,301,350,$ $1140,10,0$	$0,61,301,350,$ $140,10,0$
8	$0,120,963,1701,$ $1050,266,28,1$	$1,127,966,1701,$ $1050,171,6,0$	$0,120,963,1701,$ $1050,171,6,0$
9	$0,236,3004,7766,$ $6951,2646,462,36,1$	$1,255,3025,7770,$ $6951,1905,161,3,0$	$0,236,3004,7766,$ $6951,1905,161,3,0$
10	$0,464,9229,34059,$ $42520,22827,5880,$ $750,45,1$	$1,511,9330,34105,$ $42525,17368,2528,$ $121,1,0$	$0,464,9229,34059,$ $42520,17368,2528,$ $121,1,0$

Table 6: The chromatic spectra for the family $T^n(6)$ up to $n = 10$ for each of our three edge-types.

Acknowledgements

The author thanks Voloshin for posing the question that lead to this work and for his collaboration and encouragment. The author also thanks his colleague Allagan for many helpful discussions and for his enthusiastic inspiration and support.

The author thanks the anonymous referee for suggestions which helped improve the exposition of this paper.

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