Equitable coloring of Cartesian product of some graphs

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Abstract

A graph G is said to be equitably k-colorable if the vertex set of G can be divided into k independent sets for which any two sets differ in size at most one. The equitable chromatic number of G, $\chi_=(G)$, is the minimum k for which G is equitably k-colorable. The equitable chromatic threshold of G, $\chi_=^*(G)$, is the minimum k for which G is equitably k'-colorable for all $k' \geq k$. In this paper, the exact values of $\chi_=^*(P_{n',2}\square K_{m,n})$ and $\chi_=(P_{n',2}\square K_{m,n})$ are obtained except that $3 \leq \chi_=^*(P_{5,2}\square K_{m,n}) = \chi_=(P_{5,2}\square K_{m,n}) \leq 4$ when $m+n > 3min\{m,n\}+2$ or $m+n < 3min\{m,n\}-2$.

Keywords: Equitable coloring; equitable chromatic number; equitable chromatic threshold; Cartesian product

1 Introduction

All graphs considered in this paper are finite, undirected, loopless and without multiple edges. For a positive integer k and a real number x, let $[k] = \{1, 2, \cdots, k\}$, and $\lceil x \rceil$ and $\lfloor x \rfloor$ denote the smallest integer no less than x and the largest integer no more than x, respectively. We refer to [3] for other terminologies in graph theory, such as a Peterson graph, a bipartite graph, and so on.

A k-coloring of a graph G is called proper if every subset of vertices with the same color is an independent set, and graph G is called to be k-colorable if it has a proper k-coloring. A graph G is said to be equitably k-colorable if G has a proper k-coloring such that the sizes of any two color classes differ by at most one. The equitable chromatic number of G, $\chi_=(G)$, is the minimum k such that G is equitably k-colorable. The equitable chromatic threshold of G, $\chi_=^*(G)$, is the minimum k such that G is equitably k'-colorable for all integers $k' \geq k$.

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The concept of equitable colorability was first given by Meyer [22]. His motivation came from the problem of assigning one of the six days of the work week to each garbage collection route, where the vertices represent garbage collection routes and two such vertices are joined by an edge when the corresponding routes should not be run on the same day. So the problem of assigning one of the six days of the work week to each route actually transforms into the problem of 6-coloring of G. On practical reasons it might also be advisable to have an approximately equal number of routes run on each of the six days, so we have to color the graph in the equitable way [12].

Equitable coloring can be applied in many fields, such as in the scheduling, timetabling and transportation. For more applications of equitable coloring, please see [1, 4, 10, 12, 14, 15, 16].

The Equitable Coloring Conjecture [22] $\chi_=(G) \le \Delta(G)$ if $G \ne K_m$ and $G \ne C_{2m+1}$ was due to Meyer who verified the ECC successful only for graphs with six or fewer vertices. The Equitable Δ-Coloring Conjecture [8] $\chi^*_=(G) \le \Delta(G)$ was proved by Chen, Lih and Wu and it is true except for $G \in \{K_m, C_{2m+1}, K_{2m+1,2m+1}\}$. Lih and Wu have shown that the EΔCC is true for any connected bipartite graph [18]. The exact values of equitable chromatic numbers and equitable chromatic thresholds of trees [5, 6], complete multipartite graphs [2, 17], the Cartesian products of cycles and paths with bipartite graphs [21], the Cartesian products of square of cycles and paths with complete bipartite graphs [20], the Kronecker products of complete multipartite graphs and complete graphs [24], are determined. Recently, Chen and Lih [7] testified that two conjectures on equitable coloring are equivalent, and Chen and Yen [9] provided the necessary conditions for a graph G with $\Delta(G) \ge \chi(G)$ to be equitably $\Delta(G)$-colorable. One can refer to the survey by Lih [19] and Furmańczyk [12] for the progresses on the equitable coloring of graphs. The general problem of deciding if $\chi_=(G) \le 3$ is NP-complete [13].

The generalized Peterson graph $P_{n',k}$ is a graph whose vertex set and edge set are
$\{u_1, u_2, \cdots, u_{n'}, v_1, v_2, \cdots, v_{n'}\}$ and $\{u_iu_{i+1}, u_iv_i, v_iv_{i+k} | i \in [1, n']\}$, respectively, where the indices are all taken modular n'. We just consider $k = 2$ in this paper.

The Cartesian product of graphs G and H, denoted by $G\Box H$, is a graph whose vertex set and edge set are $\{(x, y) : x \in V(G), y \in V(H)\}$ and

$$\{(x,y)(x',y') : x = x' \text{ with } yy' \in E(H) \text{ or } y = y' \text{ with } xx' \in E(G)\},$$

respectively.

The equitable colorability of Cartesian products of graphs was studied by Chen, Lin and Yan [9] and Furmańczyk [11] at first. In [9], the authors testified the following result.

Theorem 1.1. *If G and H are equitably k-colorable, then so is $G\Box H$.*

In [23], the author proved the following results.

Theorem 1.2. *If G and H are graphs , then* $\chi(G\Box H)=max\{\chi(G),\chi(H)\}$.

It is obvious that $\chi(G) \leq \chi_=(G) \leq \chi_=^*(G)$ for any graph G from the definition. Lin and Chang deemed that it was possible to bound $\chi_=(G\Box H)$ by usual colorability of its factors and proposed the following conjecture at the end of [21].

Conjecture 1.3. $\chi_=(G\Box H) \leq \chi(G)\chi(H)$ *for any connected graphs* G *and* H.

2 Main results

Since $P_{1,2}$ and $P_{2,2}$ are isomorphic to P_2 and P_4, respectively, and the Cartesian product of a path with a complete bipartite graph has been discussed in [21], we just consider the equitable coloring of $P_{n',2}\Box K_{m,n}$ for $n' \geq 3$ in this paper.

Theorem 2.1. *Let* $n' \geq 3$, m *and* n *be positive integers. Then*

$$\chi_=^*(P_{n',2}\Box K_{m,n}) = \chi_=(P_{n',2}\Box K_{m,n}) = 3,$$

except that

$$3 \leq \chi_=^*(P_{5,2}\Box K_{m,n}) = \chi_=(P_{5,2}\Box K_{m,n}) \leq 4,$$

when $m+n > 3min\{m,n\}+2$ *or* $m+n < 3min\{m,n\}-2$.

We will show that $P_{n',2}\Box K_{m,n}$ is equitably k-colorable for $k \geq 4$ according to n' about congruence of modules 4 and whether it is equitably 3-colorable according to n' about congruence of modules 3. It is not difficult to verify that the following lemma holds.

Lemma 2.2. *Let* n', m *and* n *be positive integers with* $n' \geq 3$. *Then*

$$\chi_=^*(P_{n',2}\Box K_{m,n}) \geq \chi_=(P_{n',2}\Box K_{m,n}) \geq \chi(P_{n',2}\Box K_{m,n}) \geq \chi(P_{n',2}) = 3.$$

In the following, we always assume that $K_{m,n}$ has bipartition $X = \{x_1, x_2, \cdots, x_m\}$ and $Y = \{y_1, y_2, \cdots, y_n\}$.

Lemma 2.3. *Let* l, m, n, *and* k *be positive integers. Then* $P_{4l,2}\Box K_{m,n}$ *is equitably* k*-colorable for* $k \geq 4$.

Proof. Suppose that $V(P_{4l,2}) = \{u_1, u_2, \cdots, u_{4l}, v_1, v_2, \cdots, v_{4l}\}$.
Arrange the vertices of the product graph $P_{4l,2}\Box K_{m,n}$ as follows.

$$\begin{array}{llll}(u_1,x_1), & (u_1,x_2), & \cdots, & (u_1,x_m),\\ (u_2,y_1), & (u_2,y_2), & \cdots, & (u_2,y_n),\end{array}$$

$$
\begin{array}{cccc}
\vdots & \vdots & & \vdots \\
(u_{2t+1}, x_1), & (u_{2t+1}, x_2), & \cdots, & (u_{2t+1}, x_m), \\
(u_{2t+2}, y_1), & (u_{2t+2}, y_2), & \cdots, & (u_{2t+2}, y_n), \\
\vdots & \vdots & & \vdots \\
(u_{4l-1}, x_1), & (u_{4l-1}, x_2), & \cdots, & (u_{4l-1}, x_m), \\
(u_{4l}, y_1), & (u_{4l}, y_2), & \cdots, & (u_{4l}, y_n), \\
(v_1, x_1), & (v_1, x_2), & \cdots, & (v_1, x_m), \\
(v_2, y_1), & (v_2, y_2), & \cdots, & (v_2, y_n), \\
(v_3, y_1), & (v_3, y_2), & \cdots, & (v_3, y_n), \\
(v_4, x_1), & (v_4, x_2), & \cdots, & (v_4, x_m), \\
\vdots & \vdots & & \vdots \\
(v_{4s+1}, x_1), & (v_{4s+1}, x_2), & \cdots, & (v_{4s+1}, x_m), \\
(v_{4s+2}, y_1), & (v_{4s+2}, y_2), & \cdots, & (v_{4s+2}, y_n), \\
(v_{4s+3}, y_1), & (v_{4s+3}, y_2), & \cdots, & (v_{4s+3}, y_n), \\
(v_{4s+4}, x_1), & (v_{4s+4}, x_2), & \cdots, & (v_{4s+4}, x_m), \\
\vdots & \vdots & & \vdots \\
(v_{4l-3}, x_1), & (v_{4l-3}, x_2), & \cdots, & (v_{4l-3}, x_m), \\
(v_{4l-2}, y_1), & (v_{4l-2}, y_2), & \cdots, & (v_{4l-2}, y_n), \\
(v_{4l-1}, y_1), & (v_{4l-1}, y_2), & \cdots, & (v_{4l-1}, y_n), \\
(v_{4l}, x_1), & (v_{4l}, x_2), & \cdots, & (v_{4l}, x_m), \\
(u_1, y_1), & (u_1, y_2), & \cdots, & (u_1, y_n), \\
(u_2, x_1), & (u_2, x_2), & \cdots, & (u_2, x_m), \\
\vdots & \vdots & & \vdots \\
(u_{2t+1}, y_1), & (u_{2t+1}, y_2), & \cdots, & (u_{2t+1}, y_n), \\
(u_{2t+2}, x_1), & (u_{2t+2}, x_2), & \cdots, & (u_{2t+2}, x_m), \\
\vdots & \vdots & & \vdots \\
(u_{4l-1}, y_1), & (u_{4l-1}, y_2), & \cdots, & (u_{4l-1}, y_n), \\
(u_{4l}, x_1), & (u_{4l}, x_2), & \cdots, & (u_{4l}, x_m), \\
(v_1, y_1), & (v_1, y_2), & \cdots, & (v_1, y_n), \\
(v_2, x_1), & (v_2, x_2), & \cdots, & (v_2, x_m), \\
(v_3, x_1), & (v_3, x_2), & \cdots, & (v_3, x_m), \\
(v_4, y_1), & (v_4, y_2), & \cdots, & (v_4, y_n), \\
\vdots & \vdots & & \vdots \\
(v_{4s+1}, y_1), & (v_{4s+1}, y_2), & \cdots, & (v_{4s+1}, y_n), \\
(v_{4s+2}, x_1), & (v_{4s+2}, x_2), & \cdots, & (v_{4s+2}, x_m), \\
(v_{4s+3}, x_1), & (v_{4s+3}, x_2), & \cdots, & (v_{4s+3}, x_m), \\
(v_{4s+4}, y_1), & (v_{4s+4}, y_2), & \cdots, & (v_{4s+4}, y_n), \\
\vdots & \vdots & & \vdots
\end{array}
$$

$$\begin{array}{cccc}
(v_{4l-3},y_1), & (v_{4l-3},y_2), & \cdots, & (v_{4l-3},y_n),\\
(v_{4l-2},x_1), & (v_{4l-2},x_2), & \cdots, & (v_{4l-2},x_m),\\
(v_{4l-1},x_1), & (v_{4l-1},x_2), & \cdots, & (v_{4l-1},x_m),\\
(v_{4l},y_1), & (v_{4l},y_2), & \cdots, & (v_{4l},y_n),
\end{array}$$

where s and t are positive integers. Note that any set consisting of consecutive vertices in the ordering of size no more than $2l(m+n)$ is an independent set.

For $k \geq 4$, let

$$\sigma_t = \lfloor \frac{8l(m+n)+t-1}{k} \rfloor$$

for $t \in [k]$. Since

$$\sigma_k = \lfloor \frac{8l(m+n)+k-1}{k} \rfloor \leq 2l(m+n),$$

we can partition the vertex set of the product graph into k independent sets of sizes $\sigma_1, \sigma_2, \cdots, \sigma_k$ consecutively in the ordering. Hence the product graph is equitably k-colorable.

Lemma 2.4. *Let l, m, n, and k be positive integers. Then $P_{4l+1,2}\Box K_{m,n}$ is equitably k-colorable for $k \geq 4$.*

Proof. Suppose that $V(P_{4l+1,2}) = \{u_1, u_2, \cdots, u_{4l+1}, v_1, v_2, \cdots, v_{4l+1}\}$.

For $k \geq 5$, we arrange the vertices of the product graph $P_{4l+1,2}\Box K_{m,n}$ as follows.

$$\begin{array}{cccc}
(u_1,x_1), & (u_1,x_2), & \cdots, & (u_1,x_m),\\
(u_2,y_1), & (u_2,y_2), & \cdots, & (u_2,y_n),\\
\vdots & \vdots & & \vdots\\
(u_{2t+1},x_1), & (u_{2t+1},x_2), & \cdots, & (u_{2t+1},x_m),\\
(u_{2t+2},y_1), & (u_{2t+2},y_2), & \cdots, & (u_{2t+2},y_n),\\
\vdots & \vdots & & \vdots\\
(u_{4l-1},x_1), & (u_{4l-1},x_2), & \cdots, & (u_{4l-1},x_m),\\
(u_{4l},y_1), & (u_{4l},y_2), & \cdots, & (u_{4l},y_n),\\
(v_{4l+1},x_1), & (v_{4l+1},x_2), & \cdots, & (v_{4l+1},x_m),\\
(v_1,x_1), & (v_1,x_2), & \cdots, & (v_1,x_m),\\
(v_2,y_1), & (v_2,y_2), & \cdots, & (v_2,y_n),\\
(v_3,y_1), & (v_3,y_2), & \cdots, & (v_3,y_n),\\
(v_4,x_1), & (v_4,x_2), & \cdots, & (v_4,x_m),\\
\vdots & \vdots & & \vdots\\
(v_{4s+1},x_1), & (v_{4s+1},x_2), & \cdots, & (v_{4s+1},x_m),\\
(v_{4s+2},y_1), & (v_{4s+2},y_2), & \cdots, & (v_{4s+2},y_n),\\
(v_{4s+3},y_1), & (v_{4s+3},y_2), & \cdots, & (v_{4s+3},y_n),
\end{array}$$

$$\begin{array}{cccc}
(v_{4s+4}, x_1), & (v_{4s+4}, x_2), & \cdots, & (v_{4s+4}, x_m), \\
\vdots & \vdots & & \vdots \\
(v_{4l-3}, x_1), & (v_{4l-3}, x_2), & \cdots, & (v_{4l-3}, x_m), \\
(v_{4l-2}, y_1), & (v_{4l-2}, y_2), & \cdots, & (v_{4l-2}, y_n), \\
(v_{4l-1}, y_1), & (v_{4l-1}, y_2), & \cdots, & (v_{4l-1}, y_n), \\
(u_{4l}, x_1), & (u_{4l}, x_2), & \cdots, & (u_{4l}, x_m), \\
(u_1, y_1), & (u_1, y_2), & \cdots, & (u_1, y_n), \\
(u_2, x_1), & (u_2, x_2), & \cdots, & (u_2, x_m), \\
\vdots & \vdots & & \vdots \\
(u_{2t+1}, y_1), & (u_{2t+1}, y_2), & \cdots, & (u_{2t+1}, y_n), \\
(u_{2t+2}, x_1), & (u_{2t+2}, x_2), & \cdots, & (u_{2t+2}, x_m), \\
\vdots & \vdots & & \vdots \\
(u_{4l-1}, y_1), & (u_{4l-1}, y_2), & \cdots, & (u_{4l-1}, y_n), \\
(v_{4l}, x_1), & (v_{4l}, x_2), & \cdots, & (v_{4l}, x_m), \\
(u_{4l+1}, y_1), & (u_{4l+1}, y_2), & \cdots, & (u_{4l+1}, y_n), \\
(v_1, y_1), & (v_1, y_2), & \cdots, & (v_1, y_n), \\
(v_2, x_1), & (v_2, x_2), & \cdots, & (v_2, x_m), \\
(v_3, x_1), & (v_3, x_2), & \cdots, & (v_3, x_m), \\
(v_4, y_1), & (v_4, y_2), & \cdots, & (v_4, y_n), \\
\vdots & \vdots & & \vdots \\
(v_{4s+1}, y_1), & (v_{4s+1}, y_2), & \cdots, & (v_{4s+1}, y_n), \\
(v_{4s+2}, x_1), & (v_{4s+2}, x_2), & \cdots, & (v_{4s+2}, x_m), \\
(v_{4s+3}, x_1), & (v_{4s+3}, x_2), & \cdots, & (v_{4s+3}, x_m), \\
(v_{4s+4}, y_1), & (v_{4s+4}, y_2), & \cdots, & (v_{4s+4}, y_n), \\
\vdots & \vdots & & \vdots \\
(v_{4l-3}, y_1), & (v_{4l-3}, y_2), & \cdots, & (v_{4l-3}, y_n), \\
(v_{4l-1}, x_1), & (v_{4l-1}, x_2), & \cdots, & (v_{4l-1}, x_m), \\
(v_{4l-2}, x_1), & (v_{4l-2}, x_2), & \cdots, & (v_{4l-2}, x_m), \\
(v_{4l+1}, y_1), & (v_{4l+1}, y_2), & \cdots, & (v_{4l+1}, y_n), \\
(u_{4l+1}, x_1), & (u_{4l+1}, x_2), & \cdots, & (u_{4l+1}, x_m),
\end{array}$$

$$(v_{4l}, y_1), \quad (v_{4l}, y_2), \quad \cdots, \quad (v_{4l}, y_n),$$

where s and t are positive integers. Note that any set consisting of consecutive vertices in the ordering of size no more than $2l(m+n)$ is an independent set.

Let

$$\sigma_t = \lfloor \frac{(8l+2)(m+n)+t-1}{k} \rfloor$$

for $t \in [k]$. Since

$$\sigma_k = \lfloor \frac{(8l+2)(m+n)+k-1}{k} \rfloor \leq 2l(m+n),$$

we can partition the vertex set of the product graph into k independent subsets of sizes $\sigma_1, \sigma_2, \cdots, \sigma_k$ consecutively in the ordering. Hence the product graph is equitably k-colorable.

Table 1. The equitable 4-coloring of $P_{4l+1,2} \Box K_{m,n}$

$\Box$		$x_1 \cdots x_a$	$x_{a_1} \cdots x_m$	$y_1 \cdots y_b$	$y_{b_1} \cdots y_n$
	v_1	$2 \cdots 2$	$2 \cdots 2$	$4 \cdots 4$	$4 \cdots 4$
u_1		$1 \cdots 1$	$1 \cdots 1$	$3 \cdots 3$	$3 \cdots 3$
	v_2	$4 \cdots 4$	$1 \cdots 1$	$2 \cdots 2$	$3 \cdots 3$
u_2		$2 \cdots 2$	$3 \cdots 3$	$4 \cdots 4$	$1 \cdots 1$
	v_3	$4 \cdots 4$	$4 \cdots 4$	$2 \cdots 2$	$2 \cdots 2$
u_3		$1 \cdots 1$	$1 \cdots 1$	$3 \cdots 3$	$3 \cdots 3$
	v_4	$3 \cdots 3$	$3 \cdots 3$	$1 \cdots 1$	$1 \cdots 1$
u_4		$2 \cdots 2$	$2 \cdots 2$	$4 \cdots 4$	$4 \cdots 4$
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
	v_{4s+1}	$2 \cdots 2$	$2 \cdots 2$	$4 \cdots 4$	$4 \cdots 4$
u_{4s+1}		$1 \cdots 1$	$1 \cdots 1$	$3 \cdots 3$	$3 \cdots 3$
	v_{4s+2}	$4 \cdots 4$	$4 \cdots 4$	$2 \cdots 2$	$2 \cdots 2$
u_{4s+2}		$3 \cdots 3$	$3 \cdots 3$	$1 \cdots 1$	$1 \cdots 1$
	v_{4s+3}	$4 \cdots 4$	$4 \cdots 4$	$2 \cdots 2$	$2 \cdots 2$
u_{4s+3}		$1 \cdots 1$	$1 \cdots 1$	$3 \cdots 3$	$3 \cdots 3$
	v_{4s+4}	$2 \cdots 2$	$2 \cdots 2$	$4 \cdots 4$	$4 \cdots 4$
u_{4s+4}		$3 \cdots 3$	$3 \cdots 3$	$1 \cdots 1$	$1 \cdots 1$
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
	v_{4l-3}	$2 \cdots 2$	$2 \cdots 2$	$4 \cdots 4$	$4 \cdots 4$
u_{4l-3}		$1 \cdots 1$	$1 \cdots 1$	$3 \cdots 3$	$3 \cdots 3$
	v_{4l-2}	$4 \cdots 4$	$4 \cdots 4$	$2 \cdots 2$	$2 \cdots 2$
u_{4l-2}		$3 \cdots 3$	$3 \cdots 3$	$1 \cdots 1$	$1 \cdots 1$
	v_{4l-1}	$4 \cdots 4$	$4 \cdots 4$	$2 \cdots 2$	$2 \cdots 2$
u_{4l-1}		$1 \cdots 1$	$1 \cdots 1$	$3 \cdots 3$	$3 \cdots 3$
	v_{4l}	$3 \cdots 3$	$3 \cdots 3$	$1 \cdots 1$	$1 \cdots 1$
u_{4l}		$2 \cdots 2$	$2 \cdots 2$	$4 \cdots 4$	$4 \cdots 4$
	v_{4l+1}	$3 \cdots 3$	$3 \cdots 3$	$1 \cdots 1$	$1 \cdots 1$
u_{4l+1}		$4 \cdots 4$	$4 \cdots 4$	$2 \cdots 2$	$2 \cdots 2$

For $k = 4$, we show that $P_{4l+1,2} \Box K_{m,n}$ is equitably 4-colorable by giving it an equitable 4-coloring with $a = a_1 - 1 = \lfloor \frac{m}{2} \rfloor$ and $b = b_1 - 1 = \lceil \frac{n}{2} \rceil$

as in Table 1, where the part of the cycle is the coloring of corresponding eight lines for $v_{4s+1}, u_{4s+1}, \cdots, v_{4s+4}, u_{4s+4}$.

Obviously, the sizes of vertex sets in color 1, 2, 3, and 4 are $2l(m+n)+\lceil\frac{m}{2}\rceil+\lfloor\frac{n}{2}\rfloor$, $2l(m+n)+\lfloor\frac{m}{2}\rfloor+\lceil\frac{n}{2}\rceil$, $2l(m+n)+\lceil\frac{m}{2}\rceil+\lfloor\frac{n}{2}\rfloor$, and $2l(m+n)+\lfloor\frac{m}{2}\rfloor+\lceil\frac{n}{2}\rceil$, respectively. Hence $P_{4l+1,2}\square K_{m,n}$ is equitably 4-colorable. $\square$

Similarly, we can obtain the following Lemmas 2.5 and 2.6.

Lemma 2.5. *Let l, m, n, and k be positive integers. Then $P_{4l+2,2}\square K_{m,n}$ is equitably k-colorable for $k \geq 4$.*

Lemma 2.6. *Let l, m, n, and k be positive integers. Then $P_{4l+3,2}\square K_{m,n}$ is equitably k-colorable for $k \geq 4$.*

Lemma 2.7. *Let l, m, and n be positive integers. Then $P_{3l,2}\square K_{m,n}$ is equitably 3-colorable.*

Proof. Suppose that $V(P_{3l,2}) = \{u_1, u_2, \cdots, u_{3l}, v_1, v_2, \cdots, v_{3l}\}$. It is clear that $P_{3l,2}\square K_{m,n}$ consists of three independent sets

$$V_1 = \{(v_{3t-2}, x_i), (v_{3t-1}, y_j), (u_{3t-1}, x_i), (u_{3t}, y_j) : t \in [l], i \in [m], j \in [n]\},$$

$$V_2 = \{(v_{3t-2}, y_j), (u_{3t-2}, x_i), (u_{3t-1}, y_j), (v_{3t}, x_i) : t \in [l], i \in [m], j \in [n]\},$$

and

$$V_3 = \{(u_{3t-2}, y_j), (v_{3t-1}, x_i), (v_{3t}, y_j), (u_{3t}, x_i) : t \in [l], i \in [m], j \in [n]\}$$

with same size $2l(m+n)$. Hence $P_{3l,2}\square K_{m,n}$ is equitably 3-colorable.

Lemma 2.8. *Let l, m, and n be positive integers. Then $P_{3l+1,2}\square K_{m,n}$ is equitably 3-colorable.*

Proof. We deal with the problem into three cases.

Case 1. $l \geq 3$.

We show that $P_{3l+1,2}\square K_{m,n}$ is equitably 3-colorable by giving it an equitable 3-coloring with $a+1=a_1, b+1=b_1, a'+1=a'_1$ and $b'+1=b'_1$ as in Table 2,where the part of the cycle is the coloring of corresponding six lines for $u_{3s+1}, v_{3s+2}, \cdots, u_{3s+3}, v_{3s+4}$.

Subcase 1.1. $m \equiv 0(mod\ 3)$ and $n \equiv 0, 1(mod\ 3)$.

Let $a = b-a = \lfloor\frac{m}{3}\rfloor$ and $a' = b'-a' = \lfloor\frac{n}{3}\rfloor$.

Subcase 1.2. $m \equiv 0(mod\ 3)$ and $n \equiv 2(mod\ 3)$.

Let $a = b-a = \lfloor\frac{m}{3}\rfloor, b'-a' = \lfloor\frac{n}{3}\rfloor$ and $a' = \lfloor\frac{n}{3}\rfloor+1$.

Subcase 1.3. $m \equiv 1(mod\ 3)$ and $n \equiv 1(mod\ 3)$.

Let $a = b-a = \lfloor\frac{m}{3}\rfloor, a' = \lfloor\frac{n}{3}\rfloor$ and $b'-a' = \lfloor\frac{n}{3}\rfloor+1$.

Subcase 1.4. $m \equiv 1(mod\ 3)$ and $n \equiv 2(mod\ 3)$.

Let $a = b-a = \lfloor\frac{m}{3}\rfloor$ and $a' = b'-a' = \lfloor\frac{n}{3}\rfloor+1$.

Subcase 1.5. $m \equiv 2(mod\ 3)$ and $n \equiv 2(mod\ 3)$.

Let $a = \lfloor\frac{m}{3}\rfloor, b-a = \lfloor\frac{m}{3}\rfloor+1$ and $a' = b'-a' = \lfloor\frac{n}{3}\rfloor+1$.

Table 2. The equitable 3-coloring of $P_{3l+1,2}\square K_{m,n}$ for $l \geq 3$

$\square$		$x_1 \cdots x_a$	$x_{a_1} \cdots x_b$	$x_{b_1} \cdots x_m$	$y_1 \cdots y_{a'}$	$y_{a'_1} \cdots y_{b'}$	$y_{b'_1} \cdots y_n$
	v_1	$1 \cdots 1$	$1 \cdots 1$	$1 \cdots 1$	$2 \cdots 2$	$2 \cdots 2$	$2 \cdots 2$
u_1		$2 \cdots 2$	$2 \cdots 2$	$2 \cdots 2$	$3 \cdots 3$	$1 \cdots 1$	$1 \cdots 1$
	v_2	$3 \cdots 3$	$3 \cdots 3$	$3 \cdots 3$	$1 \cdots 1$	$1 \cdots 1$	$1 \cdots 1$
u_2		$1 \cdots 1$	$1 \cdots 1$	$1 \cdots 1$	$2 \cdots 2$	$2 \cdots 2$	$3 \cdots 3$
	v_3	$2 \cdots 2$	$2 \cdots 2$	$2 \cdots 2$	$3 \cdots 3$	$3 \cdots 3$	$3 \cdots 3$
u_3		$3 \cdots 3$	$3 \cdots 3$	$3 \cdots 3$	$1 \cdots 1$	$1 \cdots 1$	$1 \cdots 1$
	v_4	$1 \cdots 1$	$1 \cdots 1$	$1 \cdots 1$	$2 \cdots 2$	$2 \cdots 2$	$2 \cdots 2$
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
u_{3s+1}		$2 \cdots 2$	$2 \cdots 2$	$2 \cdots 2$	$3 \cdots 3$	$3 \cdots 3$	$3 \cdots 3$
	v_{3s+2}	$1 \cdots 1$	$1 \cdots 1$	$1 \cdots 1$	$2 \cdots 2$	$2 \cdots 2$	$2 \cdots 2$
u_{3s+2}		$3 \cdots 3$	$3 \cdots 3$	$3 \cdots 3$	$1 \cdots 1$	$1 \cdots 1$	$1 \cdots 1$
	v_{3s+3}	$2 \cdots 2$	$2 \cdots 2$	$2 \cdots 2$	$3 \cdots 3$	$3 \cdots 3$	$3 \cdots 3$
u_{3s+3}		$1 \cdots 1$	$1 \cdots 1$	$1 \cdots 1$	$2 \cdots 2$	$2 \cdots 2$	$2 \cdots 2$
	v_{3s+4}	$3 \cdots 3$	$3 \cdots 3$	$3 \cdots 3$	$1 \cdots 1$	$1 \cdots 1$	$1 \cdots 1$
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
u_{3l-5}		$2 \cdots 2$	$2 \cdots 2$	$2 \cdots 2$	$3 \cdots 3$	$3 \cdots 3$	$3 \cdots 3$
	v_{3l-4}	$1 \cdots 1$	$1 \cdots 1$	$1 \cdots 1$	$2 \cdots 2$	$2 \cdots 2$	$2 \cdots 2$
u_{3l-4}		$3 \cdots 3$	$3 \cdots 3$	$3 \cdots 3$	$1 \cdots 1$	$1 \cdots 1$	$1 \cdots 1$
	v_{3l-3}	$2 \cdots 2$	$2 \cdots 2$	$2 \cdots 2$	$3 \cdots 3$	$3 \cdots 3$	$3 \cdots 3$
u_{3l-3}		$1 \cdots 1$	$1 \cdots 1$	$1 \cdots 1$	$2 \cdots 2$	$2 \cdots 2$	$2 \cdots 2$
	v_{3l-2}	$2 \cdots 2$	$2 \cdots 2$	$2 \cdots 2$	$3 \cdots 3$	$3 \cdots 3$	$3 \cdots 3$
u_{3l-2}		$3 \cdots 3$	$3 \cdots 3$	$3 \cdots 3$	$1 \cdots 1$	$1 \cdots 1$	$1 \cdots 1$
	v_{3l-1}	$3 \cdots 3$	$3 \cdots 3$	$3 \cdots 3$	$1 \cdots 1$	$1 \cdots 1$	$1 \cdots 1$
u_{3l-1}		$1 \cdots 1$	$1 \cdots 1$	$1 \cdots 1$	$2 \cdots 2$	$2 \cdots 2$	$2 \cdots 2$
	v_{3l}	$3 \cdots 3$	$3 \cdots 3$	$3 \cdots 3$	$1 \cdots 1$	$1 \cdots 1$	$1 \cdots 1$
u_{3l}		$2 \cdots 2$	$2 \cdots 2$	$2 \cdots 2$	$3 \cdots 3$	$3 \cdots 3$	$3 \cdots 3$
	v_{3l+1}	$2 \cdots 2$	$2 \cdots 2$	$1 \cdots 1$	$3 \cdots 3$	$3 \cdots 3$	$3 \cdots 3$
u_{3l+1}		$1 \cdots 1$	$3 \cdots 3$	$3 \cdots 3$	$2 \cdots 2$	$2 \cdots 2$	$2 \cdots 2$

Now we take Subcase 1.3 for an example to verify that it is the equitable

3-coloring of $P_{3l+1,2}\Box K_{m,n}$. In Subcase 1.3, the sizes of vertex sets in color 1, 2, and 3 are $2l(m+n)+2\lfloor\frac{m}{3}\rfloor+2\lfloor\frac{n}{3}\rfloor+2$, $2l(m+n)+2\lfloor\frac{m}{3}\rfloor+2\lfloor\frac{n}{3}\rfloor+1$, $2l(m+n)+2\lfloor\frac{m}{3}\rfloor+2\lfloor\frac{n}{3}\rfloor+1$, respectively. Similarly, we can show that the coloring is an equitable 3-coloring of $P_{3l+1,2}\Box K_{m,n}$ in every other subcase. Hence, $P_{3l+1,2}\Box K_{m,n}$ is equitably 3-colorable when $l \geq 3$.

Table 3. The equitable 3-coloring of $P_{7,2}\Box K_{m,n}$

$\Box$	$x_1\cdots x_a$	$x_{a_1}\cdots x_b$	$x_{b_1}\cdots x_m$	$y_1\cdots y_{a'}$	$y_{a'_1}\cdots y_{b'}$	$y_{b'_1}\cdots y_n$
v_1	$1\cdots 1$	$1\cdots 1$	$1\cdots 1$	$2\cdots 2$	$2\cdots 2$	$2\cdots 2$
u_1	$2\cdots 2$	$2\cdots 2$	$2\cdots 2$	$3\cdots 3$	$1\cdots 1$	$1\cdots 1$
v_2	$3\cdots 3$	$3\cdots 3$	$3\cdots 3$	$1\cdots 1$	$1\cdots 1$	$1\cdots 1$
u_2	$1\cdots 1$	$1\cdots 1$	$1\cdots 1$	$2\cdots 2$	$2\cdots 2$	$3\cdots 3$
v_3	$2\cdots 2$	$2\cdots 2$	$2\cdots 2$	$3\cdots 3$	$3\cdots 3$	$3\cdots 3$
u_3	$3\cdots 3$	$3\cdots 3$	$3\cdots 3$	$1\cdots 1$	$1\cdots 1$	$1\cdots 1$
v_4	$1\cdots 1$	$1\cdots 1$	$1\cdots 1$	$2\cdots 2$	$2\cdots 2$	$2\cdots 2$
u_4	$2\cdots 2$	$2\cdots 2$	$2\cdots 2$	$3\cdots 3$	$3\cdots 3$	$3\cdots 3$
v_5	$1\cdots 1$	$1\cdots 1$	$1\cdots 1$	$2\cdots 2$	$2\cdots 2$	$2\cdots 2$
u_5	$3\cdots 3$	$3\cdots 3$	$3\cdots 3$	$1\cdots 1$	$1\cdots 1$	$1\cdots 1$
v_6	$3\cdots 3$	$3\cdots 3$	$3\cdots 3$	$1\cdots 1$	$1\cdots 1$	$1\cdots 1$
u_6	$2\cdots 2$	$2\cdots 2$	$1\cdots 1$	$3\cdots 3$	$3\cdots 3$	$3\cdots 3$
v_7	$2\cdots 2$	$2\cdots 2$	$2\cdots 2$	$3\cdots 3$	$3\cdots 3$	$3\cdots 3$
u_7	$1\cdots 1$	$3\cdots 3$	$3\cdots 3$	$2\cdots 2$	$2\cdots 2$	$2\cdots 2$

Case 2. $l = 2$.

We show that $P_{7,2}\Box K_{m,n}$ is equitably 3-colorable by giving it an equitable 3-coloring with $a+1=a_1, b+1=b_1, a'+1=a'_1$ and $b'+1=b'_1$ as in Table 3 and their values are given in every subcase below.

Subcase 2.1. $m \equiv 0(mod\ 3)$ and $n \equiv 0, 1(mod\ 3)$.

Let $a = b-a = \lfloor\frac{m}{3}\rfloor$ and $a' = b'-a' = \lfloor\frac{n}{3}\rfloor$.

Subcase 2.2. $m \equiv 0(mod\ 3)$ and $n \equiv 2(mod\ 3)$.

Let $a = b-a = \lfloor\frac{m}{3}\rfloor, b'-a' = \lfloor\frac{n}{3}\rfloor$ and $a' = \lfloor\frac{n}{3}\rfloor+1$.

Subcase 2.3. $m \equiv 1(mod\ 3)$ and $n \equiv 1(mod\ 3)$.

Let $a = b-a = \lfloor\frac{m}{3}\rfloor, a' = \lfloor\frac{n}{3}\rfloor$ and $b'-a' = \lfloor\frac{n}{3}\rfloor+1$.

Subcase 2.4. $m \equiv 1(mod\ 3)$ and $n \equiv 2(mod\ 3)$.

Let $a = b-a = \lfloor\frac{m}{3}\rfloor$ and $a' = b'-a' = \lfloor\frac{n}{3}\rfloor+1$.

Subcase 2.5. $m \equiv 2(mod\ 3)$ and $n \equiv 2(mod\ 3)$.

Let $a = \lfloor \frac{m}{3} \rfloor, b - a = \lfloor \frac{m}{3} \rfloor + 1$ and $a' = b' - a' = \lfloor \frac{n}{3} \rfloor + 1$.

Now we take Subcase 2.2 for an example to testify that it is an equitable 3-coloring of $P_{7,2} \square K_{m,n}$. In Subcase 2.2, the sizes of vertex sets in color 1, 2, and 3 are $4(m+n) + 2\lfloor \frac{m}{3} \rfloor + 2\lfloor \frac{n}{3} \rfloor + 1$, $4(m+n) + 2\lfloor \frac{m}{3} \rfloor + 2\lfloor \frac{n}{3} \rfloor + 1$, $4(m+n) + 2\lfloor \frac{m}{3} \rfloor + 2\lfloor \frac{n}{3} \rfloor + 2$. Similarly, in every other subcase, we can show that the coloring is an equitable 3-coloring of $P_{7,2} \square K_{m,n}$. Hence, $P_{7,2} \square K_{m,n}$ is equitably 3-colorable.

Case 3. $l = 1$.

We show that $P_{4,2} \square K_{m,n}$ is equitably 3-colorable by giving it an equitable 3-coloring with $a + 1 = a_1, b + 1 = b_1, a' + 1 = a'_1$ and $b' + 1 = b'_1$ as in Fig. 4 and their values are given in every subcase below.

Subcase 3.1. $m \equiv 0(mod\ 3)$ and $n \equiv 0, 1(mod\ 3)$.

Let $a = b - a = \lfloor \frac{m}{3} \rfloor$ and $a' = b' - a' = \lfloor \frac{n}{3} \rfloor$.

Subcase 3.2. $m \equiv 0(mod\ 3)$ and $n \equiv 2(mod\ 3)$.

Let $a = b - a = \lfloor \frac{m}{3} \rfloor, b' - a' = \lfloor \frac{n}{3} \rfloor$ and $a' = \lfloor \frac{n}{3} \rfloor + 1$.

Subcase 3.3. $m \equiv 1(mod\ 3)$ and $n \equiv 1(mod\ 3)$.

Let $a = b - a = \lfloor \frac{m}{3} \rfloor, a' = \lfloor \frac{n}{3} \rfloor$ and $b' - a' = \lfloor \frac{n}{3} \rfloor + 1$.

Subcase 3.4. $m \equiv 1(mod\ 3)$ and $n \equiv 2(mod\ 3)$.

Let $a = b - a = \lfloor \frac{m}{3} \rfloor$ and $a' = b' - a' = \lfloor \frac{n}{3} \rfloor + 1$.

Subcase 3.5. $m \equiv 2(mod\ 3)$ and $n \equiv 2(mod\ 3)$.

Let $a = \lfloor \frac{m}{3} \rfloor, b - a = \lfloor \frac{m}{3} \rfloor + 1$ and $a' = b' - a' = \lfloor \frac{n}{3} \rfloor + 1$.

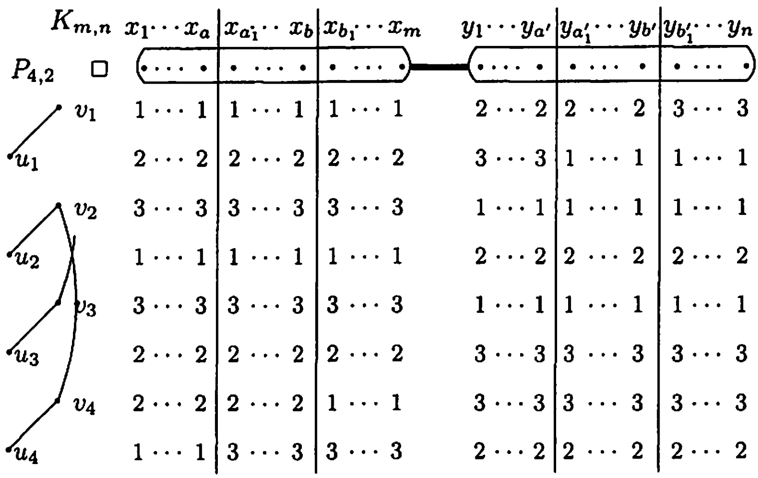

Figure 4. The equitable 3-coloring of $P_{4,2} \square K_{m,n}$

Now we take Subcase 3.2 for an example to testify that it is an equitable 3-coloring of $P_{4,2} \square K_{m,n}$. In Subcase 3.2, the sizes of vertex sets in color

1, 2, and 3 are $2(m+n)+2\lfloor\frac{m}{3}\rfloor+2\lfloor\frac{n}{3}\rfloor+1$, $2(m+n)+2\lfloor\frac{m}{3}\rfloor+2\lfloor\frac{n}{3}\rfloor+1$, $2(m+n)+2\lfloor\frac{m}{3}\rfloor+2\lfloor\frac{n}{3}\rfloor+2$. In every other subcase, we can show that the coloring is an equitable 3-coloring of $P_{4,2}\square K_{m,n}$ similarly. Hence, $P_{4,2}\square K_{m,n}$ is equitably 3-colorable. □

Similarly, we obtain the following Lemma 2.9.

Lemma 2.9. *Let l, m, and n be positive integers. Then $P_{3l+2,2}\square K_{m,n}$ is equitably 3-colorable for $l > 1$.*

Lemma 2.10. *Let m and n be positive integers. Then $P_{5,2}\square K_{m,n}$ is equitably 3-colorable when*

$$3min\{m,n\} - 2 \leq m + n \leq 3min\{m,n\} + 2. \tag{1}$$

Proof. Without loss of generality, assume that $n \leq m$. Then (1) is equivalent to

$$2n - 2 \leq m \leq 2n + 2.$$

We can divide the vertices of the graph $P_{5,2}$ into three independent sets $V_1 = \{u_2, u_4, v_1, v_5\}$, $V_2 = \{u_1, v_3, v_4\}$ and $V_3 = \{u_3, u_5, v_2\}$, and divide the vertices of the graph $K_{m,n}$ into three independent sets $V_1' = \{x_1, x_2, \cdots, x_{\lfloor\frac{m}{2}\rfloor}\}$, $V_2' = \{x_{\lfloor\frac{m}{2}\rfloor+1}, x_{\lfloor\frac{m}{2}\rfloor+2}, \cdots, x_m\}$ and $V_3' = \{y_1, y_2, \cdots, y_n\}$. Then $P_{5,2}$ and $K_{m,n}$ are all equitably 3-colorable.

According to Theorem 1.1, the lemma holds. □

Applying Lemmas 2.2-2.10, we prove Theorem 2.1 immediately.

By the results above, we obtain that Conjecture 1.3 holds for Cartesian product of the generalized Peterson graph $P_{n',2}$ with complete bipartite graphs.

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