Dynamical 2-domination in Graphs

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Abstract

In this paper a domination-type parameter, called dynamical 2-domination number, will be introduced. Let $G = (V(G), E(G))$ be a graph. A subset $D \subseteq V(G)$ is called a 2-dominating set in G if every vertex in $V(G) \setminus D$ is adjacent to at least two vertices in D, and in this paper D is called a *dynamical 2-dominating set* if there exists a sequence of sets $D = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_k = V(G)$ such that, for each i, V_{i-1} is a 2-dominating set in $< V_i >$, the induced subgraph generated by V_i. Also for a given graph G, the size of its dynamical 2-dominating sets of minimum cardinality will be called *dynamical 2-domination number* of G and will be denoted by $\tilde{\gamma}_2(G)$. We study some basic properties of dynamical 2-dominating sets and compute $\tilde{\gamma}_2(G)$ for some graph classes. Also some results about $\tilde{\gamma}_2$ of a number of binary operations on graphs are proved. A characterization of graphs with extreme values of $\tilde{\gamma}_2$ is presented. Finally, we study this concept for trees and give an upper bound and a lower bound for dynamical 2-domination number of trees.

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The research of Ch. Eslahchi was supported by a grant from Shahid Beheshti Univ.
The research of H. R. Maimani was supported by a grant from IPM (No. 91050214).
The research of R. Torabi was supported by a grant from IPM (No. 89050044).
The research of R.Tusserkani was in part supported by a grant from IPM.
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2000 *Mathematics subject classification.* 05C35.
Key words and phrases. Nearly perfect sets, Domination, 2-domination number.

1 Introduction

Domination and its variations provide an extremely promising area of study in modern graph theory [11]. In this work a domination-type parameter is studied which is a natural generalization of 2-domination number of graphs.

Consider a network in which a piece of information is to be sent to all nodes. Also suppose that each node is able to obtain the information, if one of its neighbors has got it. Obviously, we do not have to put the information on all nodes as it is sufficient to find a subset of nodes such that it dominates all vertices. This application leads us to the concept of domination in graphs: in a graph, a vertex is said to *dominate* itself and all of its neighbors. A *dominating set* of $G = (V(G), E(G))$ is a subset $D \subseteq V(G)$ in which every vertex in $V(G)$ is dominated by at least one vertex in D.

Now suppose that the communication is not totally safe and some errors are likely to occur. Therefore there is a need for a procedure to decrease the errors during the propagation. There are different methods for solving this problem such as sending parity checks and/or sending the information twice. Clearly, if the source of information is damaged, retransmission of information cannot help. The alternative method could be sending the information from two different sources adjacent to destination nodes. In this case a subset of nodes has to be found so that it dominates all other nodes of the network at least twice. This application leads us to the concept of 2-domination in a graph [10]: a subset $D \subseteq V(G)$ is called a *2-dominating set* of $G = (V(G), E(G))$ if every vertex in $V(G) \setminus D$ is dominated by at least two vertices in D.

Once a node has obtained some information, it can be used as a new source. This leads to dynamical 2-dominating sets which is a natural generalization of 2-dominating sets and the main interest of the authors in this paper.

If $D = V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_k = V(G)$ is a sequence of subsets of $V(G)$, and for each i, V_{i-1} is a 2-dominating set in $< V_i >$, the induced subgraph generated by V_i, then D is called a *dynamical 2-dominating set* of G. In other words, we start with a set D and expand it by adding vertices of $V(G)$ that have at least two neighbors in D. Then we continue this procedure for the resulting set until it does not change anymore. The obtained set is the *closure* of D and this notion was introduced by Eslahchi et al. [9]. A set will be a dynamical 2-dominating set of G, when its closure is equal to the vertex set of G. In order to compute the *power domination number* of $P_m \Box P_n$ Dorbec et al. [8] introduced the concept of a *life winning set* of grids. In fact in these graphs the life winning sets are the same as the dynamical 2-dominating sets.

The definition of a dynamical 2-dominating set immediately results in two different interesting parameters of graphs. The first one, is the minimum cardinality of dynamical 2-dominating sets of a graph $G = (V(G), E(G))$. This number will be called *dynamical 2-domination number*, denoted by $\tilde{\gamma}_2(G)$. The second one, called *pace*, refers to the number of steps needed to dominate the vertex set by a minimum dynamical 2-dominating set. Such parameters are interesting when the cost of re-sending needs to be minimized. However, pace is not studied in this paper as the main focus is on $\tilde{\gamma}_2$.

It should be noted that a simple and totally natural generalization of dynamical 2-dominating sets are dynamical k-dominating sets, for $k \geq 2$. Studying these parameters is potentially an attractive subject for anyone interested in the k-domination number and related topics.

In this paper, we study some basic properties of dynamical 2-dominating sets and the dynamical 2-domination number. Furthermore some bounds for $\tilde{\gamma}_2$ of a graph and binary operations on graphs are presented.

2 Definitions and Preliminaries

Throughout this paper it is assumed that all graphs are finite, simple, and undirected.

Let $G = (V(G), E(G))$ be a graph with $\mid V \mid= n$. We use $\Delta(G) = \Delta$ to denote the maximum degree of vertices of G. For a non-empty subset $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $< S >$. For any vertex $v \in V(G)$, the open neighborhood of v is the set $N(v) = \{u : uv \in E\}$, while the closed neighborhood of v is the set $N[v] = N(v) \cup \{v\}$. For a subset $S \subseteq V(G)$ the open and closed neighborhoods of S are defined by $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = \bigcup_{v \in S} N[v]$, respectively.

Let G_1 and G_2 be two vertex-disjoint graphs. The *direct product* of G_1 and G_2, which is denoted by $G_1 \times G_2$, is a graph whose vertex set is $V(G_1) \times V(G_2)$ and two distinct vertices (x, y) and (x', y') are adjacent if and only if x is adjacent to x' in G_1 and y is adjacent to y' in G_2. The *cartesian product* of G_1 and G_2, which is denoted by $G_1 \square G_2$, is a graph whose vertex set is $V_1 \times V_2$ and (x, y) is adjacent to (x', y') whenever $x = x'$ and y is adjacent to y' or x is adjacent to x' and $y = y'$. The *join* of G_1 and G_2, which is denoted by $G_1 \vee G_2$, is a graph whose vertex set is $V(G_1) \bigcup V(G_2)$ and edge set is $E(G_1) \bigcup E(G_2) \bigcup \{xy : x \in V(G_1), y \in V(G_2)\}$. The *union* of G_1 and G_2 which is denoted by $G_1 \cup G_2$ is a graph whose vertex set is $V(G_1) \bigcup V(G_2)$ and edge set is $E(G_1) \bigcup E(G_2)$. If $G = G_1 \vee G_2 \vee \cdots \vee G_m$ and $G_i = H$ for any $1 \leq i \leq m$, we denote G by

$G = mH$.

Definitions 2.1. Let D be a subset of $V(G)$. We define a function c on the power set of $V(G)$ by $c(D) = \{x : x \in D$ or x has at least two neighbors in $D\}$. The set D is called *closed* if $c(D) = D$. Also we define the *closure* of D as the minimal closed subset containing D. If we define $c^{m+1} := c \circ c^m$ then $cl(D) = c^n(D)$ where $c^{n+1}(D) = c^n(D)$. A subset $D \subseteq V(G)$ is called a 2-*dominating set* of G if $c(D) = V$.

Now we introduce a domination-type parameter in graphs.

Definitions 2.2. A subset D of $V(G)$ is called a *dynamical* 2-*dominating set* of G, if $cl(D) = V$. The dynamical 2-domination number of G, $\tilde{\gamma}_2(G)$, is the minimum cardinality of all dynamical 2-dominating sets of G. Every dynamical 2-dominating set of G of size $\tilde{\gamma}_2(G)$ is called a $\tilde{\gamma}_2$-set.

Obviously the vertex set of a graph is a dynamical 2-dominating set, so $\tilde{\gamma}_2(G)$ is well defined. In the trivial graph K_1 there is nothing to investigate and $\tilde{\gamma}_2(K_1) = 1$ so throughout this paper it is assumed that $\mid V \mid \geq 2$. It is easy to see that if $\mid V \mid \geq 2$, then $\tilde{\gamma}_2(G)$ is at least 2.

The following proposition is an immediate consequence of the definition of a dynamical 2-dominating set.

Proposition 2.3. *Let G be a graph of order n. Then the following holds,*
a) For every minimum dynamical 2-dominating set D in G, $< D >$ is a forest whose components are K_1 or K_2.
b) If $V_1 := \{x : deg_G(x) \leq 1\}$ and D is an arbitrary dynamical 2-dominating set, then $V_1 \subseteq D$.

In the following example the values of $\tilde{\gamma}_2$ of several classes of graphs are given.

Example 2.4. We have
a) $\tilde{\gamma}_2(K_n) = 2$; for $n \geq 2$.
b) $\tilde{\gamma}_2(K_{1,n}) = n$; for $n \geq 2$.
c) $\tilde{\gamma}_2(K_{m,n}) = 2$; for $m, n \geq 2$.
d) $\tilde{\gamma}_2(C_n) = \lceil \frac{n}{2} \rceil$, for any cycle C_n.
e) $\tilde{\gamma}_2(P_n) = \lfloor \frac{n}{2} \rfloor + 1$, for any path on n vertices P_n.
f) $\tilde{\gamma}_2(W_n) = 2$; for any wheel W_n.

The next theorem shows that $\tilde{\gamma}_2$ has a close relation with the concept of nearly perfect sets and PN_p number of graphs, which have been studied in [4], [7] and [9]. First we need some definitions.

A set $S \subseteq V(G)$ is called *nearly perfect* (or *np*-set) if for every vertex $v \in V(G) \backslash S$, $|N(v) \cap S| \leq 1$. Moreover, if $S \neq V(G)$ then S is called a *proper nearly perfect set* (*pnp-set*, for short) in G. For any graph G, the maximum cardinality of *pnp*-sets in G is denoted by $PN_p(G)$; such a set is called a $PN_p - set$.

Theorem 2.5. *Let G be a graph with $\Delta(G) \geq 2$ and $PN_p(G) \geq 2$. Then $\tilde{\gamma}_2(G) \leq PN_p(G)$.*

Proof. Lęt $PN_p(G) = k$. Any subset of vertices of G of size $k+1$ is a $\tilde{\gamma}_2$-set. Let x be a vertex of degree $\Delta(G)$ and S be a k-subset of $V(G) \setminus \{x\}$ such that $|S \cap N(x)| \geq 2$. Therefore $x \in cl(S)$ and so $cl(S \cup \{x\}) \subseteq cl(S)$. Since $S \cup \{x\}$ has $k+1$ elements, then $V(G) = cl(S \cup \{x\}) = cl(S)$. It means that S is a $\tilde{\gamma}_2$-set in G of size k. Therefore $\tilde{\gamma}_2(G) \leq k$. □

Remark 2.6. The excluded cases of the theorem above are $\Delta(G) \leq 1$ and/or $PN_p(G) \leq 1$. If $\Delta(G) \leq 1$, obviously $PN_p(G) = n - 1$ and $\tilde{\gamma}_2(G) = n$. If $PN_p(G) = 1$ then every two vertices are forming a 2-dominating set and so $\tilde{\gamma}_2(G) = 2$. Therefore, in these cases $\tilde{\gamma}_2(G) = PN_p(G) + 1$.

In the following theorem we focus on binary operations on graphs and calculate $\tilde{\gamma}_2$ of the disjoint union and the (complete) join of two graphs. Also we present some upper bounds for the direct product and the cartesian product of two graphs.

Theorem 2.7. *Let G_1 and G_2 be two vertex-disjoint graphs. Then*
(a) $\tilde{\gamma}_2(G_1 \cup G_2) = \tilde{\gamma}_2(G_1) + \tilde{\gamma}_2(G_2)$.
(b) $\tilde{\gamma}_2(G_1 \vee G_2) = 2$, where $V(G_i) \geq 2$ for $i = 1, 2$.
(c) If G_1 and G_2 are connected, then $\tilde{\gamma}_2(G_1 \Box G_2) \leq min\{\tilde{\gamma}_2(G_1) \times |V(G_2)|, \tilde{\gamma}_2(G_2) \times |V(G_1)|\}$.
(d) If G_1 and G_2 are connected, then $\tilde{\gamma}_2(G_1 \times G_2) \leq \tilde{\gamma}_2(G_1) + \tilde{\gamma}_2(G_2) - 1$.

Proof. (a) and (b) are obvious.
(c) Let S be a $\tilde{\gamma}_2$-set of G_1. The proof becomes obvious when we consider one copy of S in each copy of G_1 in $G_1 \Box G_2$.
(d) Let S_1 and S_2 be two $\tilde{\gamma}_2$-sets of G_1 and G_2, respectively. Suppose $v_1 \in S_1$ and $v_2 \in S_2$ and set $S = \{(v_1, x) | x \in S_2\} \cup \{(x, v_2) | x \in S_1\}$. Since G_1 and G_2 are connected it is easy to check that S is a $\tilde{\gamma}_2$-set of $G_1 \times G_2$. □

Remark 2.8. (a) By Theorem 2.7(a), in calculating $\tilde{\gamma}_2$, it is enough to focus only on connected graphs and we will do so in the rest of the paper.
(b) In Theorem 2.7(b) the only remaining case is when at least one of the graphs is trivial. Suppose that $V(G_1) = \{x\}$ and $G_2^1, G_2^2, \ldots, G_2^k$ are connected components of G_2. Set $x_i \in G_2^i$. If $k = 1$, then $\{x, x_1\}$ is a $\tilde{\gamma}_2$-set. If $k \geq 2$, then $\{x_1, x_2, \ldots, x_k\}$ is a $\tilde{\gamma}_2$-set since every dynamical 2-dominating set contains at least one vertex of each G_2^i.

c) It can be easily seen that $\tilde{\gamma}_2(P_3 \times P_3) = 3$, so the bound in part (d) of Theorem 2.7 is sharp.
d) The case $\tilde{\gamma}_2(P_m \Box P_n)$ is completely solved by P. Dorbec et al. [8] using a very nice invariant argument and is showed that $\tilde{\gamma}_2(P_m \Box P_n) = \lceil \frac{m+n}{2} \rceil$.

3 Graphs Reaching the Extremal Values of $\tilde{\gamma}_2$

The inequality $2 \le \tilde{\gamma}_2(G) \le n$ holds for every non-trivial graph G. The aim of this section is to study the conditions under which the extreme cases hold. Firstly, the graphs for which $\tilde{\gamma}_2(G) = n$ or $\tilde{\gamma}_2(G) = n - 1$ are characterized, and next a sufficient condition for $\tilde{\gamma}_2(G) = 2$ is presented.

Theorem 3.1. *For every graph G of order n,*
(a) $\tilde{\gamma}_2(G) = n$ if and only if $\Delta(G) \le 1$.
(b) $\tilde{\gamma}_2(G) = n - 1$ if and only if $G = tK_1 \bigcup sK_2 \bigcup H$ where tK_1 (sK_2) is the disjoint union of t copies (s copies) of K_1 (K_2), and $H \in \{K_3, P_4, K_{1,m}\}$ for some positive integers s, t and $m \ge 2$.

Proof. (a) If $\Delta(G) \le 1$, all the connected components of G are K_1 and K_2. Hence $\tilde{\gamma}_2(G) = n$, by Theorem 2.7(a). By contraposition, assume that there exists a vertex $x \in V(G)$ such that x is of degree at least 2, then clearly $x \in cl(V(G) \setminus \{x\})$ and hence $\tilde{\gamma}_2(G) \le n - 1$.

(b) Let $G = tK_1 \bigcup sK_2 \bigcup H$ where $H \in \{K_3, P_4, K_{1,m}\}$. By applying Example 2.4 and Theorem 2.7, $\tilde{\gamma}_2(G) = n - 1$. Conversely, let $G_1, G_2, \cdots, G_k$ be the connected components of G, D be a $\tilde{\gamma}_2$-set of G and $|D| = n - 1$. Hence for exactly one $1 \le i \le k$, say $i = 1$, we have $V(G_1) \cap D \ne V(G_1)$ and for $i \ge 2$, $V(G_i) \cap D = V(G_i)$. So by part (a), every G_i is K_1 or K_2 for $i \ge 2$. Now we determine the structure of the connected graph $G_1 = H$. Let $D_1 = V(H) \cap D$ and $\{x\} = V(H) \setminus D_1$ which means $H =< D_1 \cup \{x\} >$ and D_1 is a $\tilde{\gamma}_2$-set of H. Now by Proposition 2.3(a), $< D_1 >= pK_1 \bigcup qK_2$. We claim that $q \le 1$. If $q \ge 2$, then we can consider two parallel edges $e_1 = u_1v_1$ and $e_2 = u_2v_2$ in $< D_1 >$. Since H is connected, then x is adjacent to at least one end-point of each of the edges e_1 and e_2. Suppose that x is adjacent to u_1 and u_2. It follows that $(D_1 \setminus \{u_1, u_2\}) \cup \{x\}$ is a $\tilde{\gamma}_2$-set of H, which is a contradiction. Hence $q \le 1$. Now we consider two cases as follows:

(i) $q = 0$. In this case $< D_1 >$ is an independent set and x has to be adjacent to every vertex of D_1. Therefore H is $K_{1,m}$.

(ii) $q = 1$. We claim that $p \le 1$. Suppose that $e = uv$ is the only edge of $< D_1 >$ hence x is adjacent to at least one of the vertices u or v since H is connected. Suppose x is adjacent to u. If $p \ge 2$, then $V(H \setminus \{u, x\})$ is a $\tilde{\gamma}_2$-set of H, which is a contradiction. Therefore $p \le 1$. Now if $p = 0$, then x must be adjacent to v and we have $H = K_3$. If $p = 1$, then x is not adjacent to v and H is the path P_4. □

Theorem 3.2. *For every graph G of order n, if $\delta(G) \geq \frac{n}{2}$, then $\tilde{\gamma}_2(G) = 2$.*

Proof. If G is a complete graph there is nothing to prove. Let x and y be two non-adjacent vertices of G. It suffices to show that all vertices of G belong to $D = cl(\{x, y\})$. Since $deg(x) \geq \frac{n}{2}$, $deg(y) \geq \frac{n}{2}$ and x and y are not adjacent, there exists a vertex $z \in N(x) \cap N(y)$. So $z \in D$ and $k = |D| \geq 3$. Now we claim $k = n$. Assume otherwise and let $k < n$. Every vertex a in $\bar{D} = V(G) \setminus D$ has at most one neighbor in D and therefore $|N(a) \cap \bar{D}| \geq \frac{n}{2} - 1$ and thus $|\bar{D}| \geq \frac{n}{2}$, which means $k \leq \frac{n}{2}$. On the other hand, let u be an arbitrary vertex in D. We have

$$|N(u) \cap \bar{D}| \geq deg(u) - (k-1) \geq \frac{n}{2} - k + 1$$

Since for each $u, v \in D$ we have $N(u) \cap N(v) \cap \bar{D} = \emptyset$. So

$$|\bigcup_{u \in D} (N(u) \cap \bar{D})| \geq k(\frac{n}{2} - k + 1)$$

$$n - k \geq k(\frac{n}{2} - k + 1)$$

$$(k - \frac{n}{2})(k - 2) \geq 0$$

Since $k \geq 3$, the inequality above implies $k \geq \frac{n}{2}$. Therefore $k = \frac{n}{2}$. In this case all the inequalities in the formula above are equalities. This means that for each $u \in D$, $|N(u) \cap \bar{D}| = 1$. This implies u has $k - 1$ neighbors in D and hence $\langle D \rangle$ is a clique which is a contradiction because x and y are non-adjacent. □

Remark 3.3. Note that there are infinitely many graphs with $\delta(G) < \frac{n}{2}$ and $\tilde{\gamma}_2(G) = 2$, for example $\tilde{\gamma}_2(K_{2,n}) = 2$ for any $n \geq 2$. On the other hand there exists a graph G with $\delta(G) < \frac{n}{2} - 1$ and $\tilde{\gamma}_2(G) \neq 2$, for example consider the graph C_6.

4 Some Bounds

In this section some lower and upper bounds for $\tilde{\gamma}_2(G)$ are presented. As $\tilde{\gamma}_2(G) \leq \gamma_2(G)$, every upper bound for $\gamma_2(G)$ is an upper bound for $\tilde{\gamma}_2(G)$.

Theorem 4.1. *Let G be a graph and e be an edge of G then $\tilde{\gamma}_2(G) \leq \tilde{\gamma}_2(G \setminus e) \leq \tilde{\gamma}_2(G) + 1$.*

Proof. Let $e = ab$. It is clear that every dynamical 2-dominating set of $G \setminus e$ is a dynamical 2-dominating set of G. So the left inequality holds. Now let S be a dynamical 2-dominating set of G. Let $k \geq 1$ be the smallest integer such that at least one of the endpoints of e, say a, belongs to $c^k(S)$. If $b \in c^k(S)$ then e

does not affect the process of forming the closure of S and so S is a dynamical 2-dominating set of $G \setminus e$. Otherwise there exists an integer $k' \geq k$ such that $a \in c^{k'}(S)$, $b \notin c^{k'}(S)$, and $b \in c^{k'+1}(S)$. Since the edge e can only be used to force b to belong to $cl(S)$, so if we add b to S, the process of forming the closure of $S \cup \{b\}$ in G is also ends with $V(G)$ without using the edge e. This means that $S \cup \{b\}$ is a dynamical 2-dominating set of $G \setminus e$ and so, $\tilde{\gamma}_2(G \setminus e) \leq \tilde{\gamma}_2(G)+1$. □

In the following Lemma we denote by m_X the number of edges of an induced subgraph $< X >$.

Lemma 4.2. *Suppose that D is a dynamical 2-dominating set of a graph G of order n and size m. Then $|D| \geq n - \frac{m-m_D}{2}$.*

Proof. Consider the sequence of sets $D = V_0 \subseteq V_1 \subseteq V_2 \cdots \subseteq V_k = V$ such that for each i, V_{i-1} is a 2-dominating set of $< V_i >$. Since every vertex of $V_{i+1} \setminus V_i$ is adjacent to at least two vertices of V_i we conclude that $m_{V_{i+1}} - m_{V_i} \geq 2|V_{i+1} \setminus V_i|$. Hence $m - m_D \geq 2(|V(G)| - |D|)$. □

Let H be a graph and G a subgraph of H. The graph H is called an *ear decomposition started from* G, if there exists a sequence of graphs $G = G_0, G_1, \cdots, G_l = H$ such that G_{i+1} is obtained by adding a path P_{j_i} to G_i such that the endpoints of P_{j_i} are the only common vertices of P_{j_i} and G_i. The graph H is called a *P_k-ear decomposition started from G* if H is an ear decomposition started from G such that each path P_{j_i} is of order k. Every P_3-ear decomposition started from G, can be denoted by $(G; v_1, v_2, \cdots, v_l)$, since every path P_{j_i} has only one additional vertex compared to G_i.

Corollary 4.3. *For every graph G of order n and size m, we have $\tilde{\gamma}_2(G) \geq n - \frac{m}{2}$ and equality holds if and only if m is even and G is a P_3-ear decomposition started from $\overline{K_{n-\frac{m}{2}}}$.*

Proof. Lemma 4.2 trivially yields $\tilde{\gamma}_2(G) \geq n - \frac{m}{2}$. Now let G be a P_3-ear decomposition denoted by $(\overline{K_{n-\frac{m}{2}}}; v_1, v_2, \cdots, v_{\frac{m}{2}})$. Let S be the vertex set of $\overline{K_{n-\frac{m}{2}}}$. Obviously $cl(S) = V(G)$. So $\tilde{\gamma}_2(G) \leq n - \frac{m}{2}$, and by the first part of the corollary equality holds.

Now let G be a graph with $\tilde{\gamma}_2(G) = n - \frac{m}{2}$ and S be a $\tilde{\gamma}_2$-set of G. By Lemma 4.2, $m_S = 0$ and thus S is an independent set. Now let $v_1, v_2, \cdots, v_{n-|S|}$ be a sequence of the vertices of G which could be forced to be in $cl(S)$ in the process of forming the closure. Now we show that v_{i+1} has exactly two neighbors in $S \cup \{v_1, v_2, \cdots, v_i\}$, which imply that G is a P_3-ear decomposition started from $\overline{K_{n-\frac{m}{2}}}$. Conversely, if v_{i+1} has more than two neighbors in $S \cup \{v_1, v_2, \cdots, v_i\}$ then $m > 2(n - |S|)$ and $|S| > n - \frac{m}{2}$, which is a contradiction. □

The following corollary is an immediate consequence of the corollary above.

Corollary 4.4. *For every 3-regular graph G of order n, $\tilde{\gamma}_2(G) > \frac{n}{4}$.*

Proof. Let G be a 3-regular graph. By Corollary 4.3, $\tilde{\gamma}_2(G) \geq n - \frac{3n}{4}$. Since there are no vertices of degree 2, G cannot be a P_3-ear decomposition started from a trivial graph and so the equality does not hold. □

By the definition of dynamical 2-dominating sets one can see that if H is a spanning subgraph of G, then $\tilde{\gamma}_2(G) \leq \tilde{\gamma}_2(H)$. Hence if G is a connected graph and T is a spanning tree of G, $\tilde{\gamma}_2(G) \leq \tilde{\gamma}_2(T)$. So it is natural to study the dynamical 2-domination number of trees. In the next two theorems some bounds for $\tilde{\gamma}_2(T)$ will be given when T is a tree.

In [2] the authors obtained a lower bound for the 2-domination number of a tree with respect to its independence number, $\alpha(T)$. In the following theorem we also show an upper bound for dynamical 2-domination number of a tree using its independence number. From now on for a tree T, set $D_i(T) = \{x \in V(T) : deg_T(v) = i\}$ and $d_i(T) = |D_i(T)|$.

Theorem 4.5. *For every tree T, $\tilde{\gamma}_2(T) \leq \alpha(T) + 1$.*

Proof. Clearly, the result holds for K_1 and stars (i.e. $K_{1,m}$). We prove the assertion for other trees by induction on n, the order of T. Let $n \geq 4$ and for every tree T' of order $n' < n$ we have $\tilde{\gamma}_2(T') \leq \alpha(T') + 1$. Suppose T is a tree of order n, T is not a star, and u is a leaf in $T \setminus D_1(T)$. Let L_u be the set of all leaves in T which are adjacent to u. Obviously, $deg_T(u) \geq 2$ and $|L_u| \geq 1$.

Suppose that I is an independent set of T and $u \in I$. By definition of u, $I' = (I \setminus u) \cup L_u$ is also an independent set of T, and $|I'| \geq |I|$. Therefore there exists a maximum independent set I of T which dose not include u. Now if $T' = T \setminus (L_u \cup u)$ then $\alpha(T') = \alpha(T) - |L_u|$. Let S' be a $\tilde{\gamma}_2$-set of T' and $S = S' \cup L_u$. Then S is a dynamical 2-dominating set of T. So $\tilde{\gamma}_2(T) \leq \tilde{\gamma}_2(T') + |L_u|$. Now by induction we have $\tilde{\gamma}_2(T) \leq \alpha(T') + 1 + |L_u| = \alpha(T) + 1$ and the proof is complete. □

The following example shows that the difference between $\alpha(T)$ and $\tilde{\gamma}_2(T)$ could be arbitrarily large. In other words, for any $k \geq 2$ there exists a tree T such that $\tilde{\gamma}_2(T) < \alpha(T) - k$.

Example. Let P_{2k+1} be the path with vertex set $\{x_1, x_2, \ldots, x_{2k+1}\}$ in which x_{i+1} is adjacent to x_i. For each $1 \leq l \leq k+1$, consider $t \geq 2$ new vertices and join each of them to x_{2l-1}. The resulting graph is a tree with $\alpha(T) = k + kt + t$. On the other hand $D_1(T)$ is a $\tilde{\gamma}_2$-set in T. So $\tilde{\gamma}_2(T) \leq kt + t$ and by Proposition 2.3 we have $\tilde{\gamma}_2(T) = kt + t$. Therefore $\tilde{\gamma}_2(T) < \alpha(T) - k$.

By Theorem 4.3, $\tilde{\gamma}_2(T) \geq \frac{n+1}{2}$ for every tree T. Also Blidia *et al.* [3] proved that $\gamma_2(T) \leq \frac{n+d_1(T)}{2}$. So the following proposition is verified.

Corollary 4.6. *If T is a tree of order $n \geq 2$, then*

$$\frac{n+1}{2} \leq \tilde{\gamma}_2(T) \leq \frac{n+d_1(T)}{2}.$$

An interesting property of trees is that the set of all vertices of degree one and two is always a dynamical 2-dominating set. We prove a stronger result in the next theorem.

Theorem 4.7. *Let T be a tree of order $n \geq 2$. If $D_2(T) = \emptyset$, then $D_1(T)$ is the unique $\tilde{\gamma}_2$-set of T, and if $D_2(T) \neq \emptyset$, then for every $x \in D_2(T)$, $D_1(T) \cup D_2(T) \setminus \{x\}$ is a dynamical 2-dominating set of T.*

Proof. The assertion is proved by induction on order of the tree T. Assume that $d_2(T) = 0$. For $n \leq 3$ there is nothing to prove. Suppose $n \geq 4$ and the assertion holds for all trees of order less than n and $x_1, x_2, \ldots, x_m$ is one longest path in T. Also let $L = N(x_{m-1}) \cap D_1(T)$. Since T has no vertex of degree 2, $|L| \geq 2$. $T' = T \setminus L$ is a tree with no vertex of degree 2 and so by induction, the set of all leaves of T', say $D_1(T')$, is a $\tilde{\gamma}_2$-set in T'. It is obvious that $D_1(T) = D_1(T') \cup L \setminus \{x_{m-1}\}$ and $x_{m-1} \in c(L)$ which implies that $D_1(T)$ is a dynamical 2-dominating set of T. By Proposition 2.3(b) any dynamical 2-dominating set includes $D_1(T)$, hence $D_1(T)$ is the unique $\tilde{\gamma}_2$-set of T.

Now suppose that $d_2(T) \geq 1$. Let x be an arbitrary vertex of degree 2 in T and $N(x) = \{a, b\}$. Suppose T_a and T_b are connected components of $T \setminus \{x\}$ containing a and b, respectively. Now we define the set D_a. If T_a is trivial, then $D_a = \{a\}$. Otherwise D_a is defined as follows:

$$D_a = \begin{cases} D_1(T_a) \cup D_2(T_a) \setminus \{a\} & \text{if } deg_{T_a}(a) = 2, \\ D_1(T_a) \cup D_2(T_a) & \text{otherwise.} \end{cases}$$

The set D_b is defined similarly. By induction, D_a (D_b) is a dynamical 2-dominating set of T_a (T_b).

Since $deg_T(x) = 2$ so $x \in cl\{a, b\}$. Therefore $D_a \cup D_b$ is a dynamical 2-dominating set of T. On the other hand $D_a \cup D_b = D_1(T) \cup D_2(T) \setminus \{x\}$ which completes the proof. □

Corollary 4.8. *Let T be a tree of order $n \geq 2$. If $d_2 = 0$, then $\tilde{\gamma}_2(T) = d_1(T)$ otherwise $d_1(T) \leq \tilde{\gamma}_2(T) \leq d_1(T) + d_2(T) - 1$.*

Note Added in Proof. While this paper was in the process of minor corrections, the authors were informed by M. Zaker that a concept similar to dynamical k-domination studied in the paper, has been previously explored in graph theory literature as *dynamic monopoly*. The authors' investigation shows that the routine results of Proposition 2.3 and Example 2.4, the immediate consequence of the definition, are not new. Moreover Theorem 2.7 which seems to have been known before, c.f. [1] and [6]. Furthermore, corollary 4.4 has been proved in [12] by another argument. But we believe that our other results are new. The authors believe that the main concept of this paper is well situated in domination theory better than any other area of graph theory, and that this concept and all of its generalizations could be developed further and grown more organically in domination theory. That is why the authors have decided to keep the original form of the paper and not to change it to an article on dynamic monopoly.

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