Yang Hui type magic squares with t-powered sum

Yong Zhang[1,2], Wen Li[1], Ruizhong Wei[2*]

1. School of Mathematics and Statistics, Yancheng Teachers University, Jiangsu 224002, PR China

2. Department of Computer Science, Lakehead University, Thunder Bay, ON P7B 5E1, Canada

Abstract

In this paper, the existence of Yang Hui type magic squares of order n with t-powered sum (YMS(n,t)) for general t is investigated. Some constructions of YMS(n,t) are obtained by using strongly symmetric self-orthogonal diagonal Latin squares and magic rectangles. Applying these constructions, it is proved that for an integer $t > 1$ there exist both a symmetric elementary YMS$(2^t, 2t-2)$ and a symmetric elementary YMS$(2^t \cdot k, 2t)$ for odd $k > 1$, which improves the known result on YMSs.

Keywords: Magic square, Yang Hui type, self-orthogonal Latin square, strongly symmetric

*Corresponding author: Ruizhong Wei(rwei@lakeheadu.ca)

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1 Introduction

An $n \times n$ matrix consisting of n^2 consecutive integers is a *magic square of order* n, denoted by MS(n), if the sum of elements in each row, each column, and each of the two main diagonals is the same.

The study of magic squares probably dates back to prehistoric times (see [9]). Examples for recent research papers on magic squares are [1–3, 6, 9]. Chinese mathematician Yang Hui (Song dynasty, 1238-1298) listed magic squares of orders from 3 to 10 in [14]. As pointed out by Chikaraishi et al in [8], the MS(8), $Y = (y_{i,j})$, given by Yang has an additional property that for each $e = 2, 3, 4$, $\sum_{i=0}^{3}\sum_{j=0}^{7} y_{i,j}^e = \sum_{i=4}^{7}\sum_{j=0}^{7} y_{i,j}^e$ and $\sum_{i=0}^{7}\sum_{j=0}^{3} y_{i,j}^e = \sum_{i=0}^{7}\sum_{j=4}^{7} y_{i,j}^e$. Such an MS(8) is called a Yang Hui type magic square in [4, 15].

Generally, for an even integer n and an integer $t \geq 2$, an MS(n) $A = (a_{i,j})$ is called a *Yang Hui type magic square of order* n *with* t*-powered sum*, denoted by YMS(n, t), if the sum of the elements of the first $\frac{n}{2}$ rows of A^{*e} is the same as that of the last $\frac{n}{2}$ rows, and the sum of the elements of the left $\frac{n}{2}$ columns of A^{*e} is the same as that of the right $\frac{n}{2}$ columns, where $A^{*e} = (a_{i,j}^e)$ and $e = 2, 3, \cdots, t$. A YMS(n, t) is also called a magic square with t-powered sum in [8]. A YMS(n, t) is a kind of weak form of a t-multimagic square. As for t-multimagic squares, we refer readers to [7, 10, 16].

Chikaraishi et al in [8] gave a family of Yang Hui type magic squares of order 2^t with $(2t-2)$-powered sum. Recently, Cao et al [4] proved that the necessary condition of a YMS($n, 2$) to exist is also sufficient except for $n = 2$. Zhang et al [15] proved that there is a YMS($n, 4$) if and only if $n \equiv 0 \pmod 4$ and $n \neq 4$. The known results for the existence of YMSs are summarized as follows.

Lemma 1.1. ([4, 8, 15]) (1) *There exists a YMS*($2^t, 2t-2$) *for all integers* $t \geq 2$.

(2) *There exists a YMS*($n, 2$) *if and only if* n *is even and* $n \neq 2$.

(3) *There exists a YMS*($n, 4$) *if and only if* $n \equiv 0 \pmod 4$ *and* $n \neq 4$.

Strongly symmetric self-orthogonal diagonal Latin squares were used to construct Yang Hui type magic squares [15]. Let $I_n = \{0, 1, \cdots, n-1\}$. A *Latin square* of order n, denoted by LS(n), is an $n \times n$ array such that every row and every column is a permutation of I_n. A *diagonal* Latin square is a Latin square with the additional property that the main diagonal and back diagonal are both permutations of I_n. Two LS(n)s are *orthogonal* if each symbol in the first square meets each symbol in the second square exactly once when they are superposed. A Latin square is *self-orthogonal* if it is orthogonal to its transpose. An LS(n) L over I_n is *strongly symmetric* if $l_{i,j} + l_{n-1-i,n-1-j} = n-1$ for all $i, j \in I_n$. A strongly symmetric self-orthogonal diagonal LS(n) is denoted by SSSODLS(n). It is easy to see that if A is an SSSODLS(n), then $nA + A^T$ is an MS(n).

Zhang et al [15] introduced a special SSSODLS(n) of even n and it is denoted by *SSSODLS(n). They proved that if A is a *SSSODLS(n) then $nA+A^T$ is a YMS($n, 4$). In this note, we will use some special SSSODLS(n) of even n, that is a generalization of *SSSODLS(n), to investigate the existence of YMS(n, t) for general t. Let n be an even integer, and let $t \geq 2$. Suppose that A is an SSSODLS(n). Then A is an *Yang Hui type SSSODLS*(n) *with t-powered sum*, denoted by YSSSODLS(n, t), if $nA + A^T$ is a YMS(n, t). We should point out that a *SSSODLS(n) is necessarily a YSSSODLS($n, 4$), but a YSSSODLS($n, 4$) is not always a *SSSODLS(n).

In this paper, the existence of YSSSODLS(n, t) is investigated and several constructions are given. The main results about YSSSODLS are as follows.

Theorem 1.2. *Let $t \geq 2$ and k be odd. Then there exists a YSSSODLS$(2^t \cdot k, 2t-2l)$ for $l=1$ if $k=1$, and $l=0$ if $k>1$ except possibly for $(t,k)=(3,3)$.*

Let C be an MS(n). C is *symmetric* if $c_{i,j}+c_{n-1-i,n-1-j}=n^2-1$, $i,j \in I_n$; C is *elementary* if $C=nA+B$ and both A and B are Latin squares. For more details about magic squares and Latin squares, we refer readers to [1, 9]. So, if A is a YSSSODLS(n,t) then $nA+A^T$ is a symmetric elementary YMS(n,t). Consequently, the following two families of symmetric elementary Yang Hui type magic squares are followed after some work that fixed the exception.

Theorem 1.3. (1) *There is a symmetric elementary YMS$(2^t, 2t-2)$ for $t>1$.*

(2) *There is a symmetric elementary YMS$(2^t \cdot k, 2t)$ for $t>1$ and odd $k>1$.*

The above new results not only added some interesting properties to the known YMSs of order 2^t, but also constructed YMSs for general order. The rest of this paper are arranged as follows. Constructions of Yang Hui type magic squares based on YSSSODLS are presented in Section 2 and the proofs of main results are presented in Section 3.

2 Constructions of YSSSODLSs

In this section, constructions of Yang Hui type magic squares are investigated by making use of Yang Hui type strongly symmetric self-orthogonal diagonal Latin squares. We start with row t-multimagic rectangles.

Let C be an $m \times n$ matrix consisting of $0,1,\cdots,mn-1$ with the property that for each $u, 0 \leq u \leq t$, the sum of u-th power of all elements of each row is the same. Such a C is called a *row t-multimagic rectangle*, denoted by RMR$(m, n; t)$.

Let $H = (h_{i,j})_{2\times 2m}$ over I_{4m}. Then H is *vertically symmetric* if $h_{1,j} = 4m - 1 - h_{0,j}$, for all $j \in I_{2m}$. H is *horizontally symmetric* if $h_{i,j} = 4m - 1 - h_{i,2m-1-j}$, for all $j \in I_m, i = 0, 1$.

Suppose that H is an RMR$(2, 2m; t)$. If H is vertically symmetric or horizontally symmetric, and it holds that $\sum_{j=0}^{m-1} h_{i,j}^e = \sum_{j=0}^{m-1} h_{1-i,j+m}^e$ for each $e < t$, $i = 0, 1$, then such an RMR$(2, 2m; t)$ is denoted by *RMR$(2, 2m; t)$. It will be used to construct a YSSSODLS$(4m, t)$.

In this section, $\sum_{S_1}, \sum_{S_2}, \sum_{S_3}, \sum_{S_4}$ will be used to denote $\sum_{i=0}^{\frac{n}{2}-1}\sum_{j=0}^{\frac{n}{2}-1}$, $\sum_{i=0}^{\frac{n}{2}-1}\sum_{j=\frac{n}{2}}^{n-1}$, $\sum_{i=\frac{n}{2}}^{n-1}\sum_{j=0}^{\frac{n}{2}-1}$, $\sum_{i=\frac{n}{2}}^{n-1}\sum_{j=\frac{n}{2}}^{n-1}$, respectively. The following is essential in our construction.

Lemma 2.1. *Let n be an even integer, and $t \geq 2$. Suppose that A is an SSSODLS(n). If*

$$\sum_{S_1} a_{i,j}^p a_{j,i}^q = \sum_{S_4} a_{i,j}^p a_{j,i}^q, \qquad \sum_{S_2} a_{i,j}^p a_{j,i}^q = \sum_{S_3} a_{i,j}^p a_{j,i}^q,$$

where $p + q \leq t$, then A is a YSSSODLS(n, t).

Proof. Let $A = (a_{i,j})$ be an SSSODLS(n) satisfying the given conditions, and let

$$C = (c_{i,j}), c_{i,j} = na_{i,j} + a_{j,i}, i, j \in I_n.$$

It is readily verified that C is an MS(n). For each $e \in \{2, 3, \cdots, t\}$, we have

$$\sum_{S_1} c_{i,j}^e = \sum_{S_1} (na_{i,j} + a_{j,i})^e = \sum_{k=0}^{e} \binom{e}{k} n^{e-k} \sum_{S_1} a_{i,j}^{e-k} a_{j,i}^k.$$

By the assumption we have $\sum_{S_1} a_{i,j}^{e-k} a_{j,i}^k = \sum_{S_4} a_{i,j}^{e-k} a_{j,i}^k$. So,

$$\sum_{S_1} c_{i,j}^e = \sum_{k=0}^{e} \binom{e}{k} n^{e-k} \sum_{S_4} a_{i,j}^{e-k} a_{j,i}^k = \sum_{S_4} (na_{i,j} + a_{j,i})^e = \sum_{S_4} c_{i,j}^e.$$

Similarly, we can prove that $\sum_{S_2} c_{i,j}^e = \sum_{S_3} c_{i,j}^e$. Thus

$$\sum_{S_1} c^e_{i,j} + \sum_{S_2} c^e_{i,j} = \sum_{S_3} c^e_{i,j} + \sum_{S_4} c^e_{i,j}, \quad \sum_{S_1} c^e_{i,j} + \sum_{S_3} c^e_{i,j} = \sum_{S_2} c^e_{i,j} + \sum_{S_4} c^e_{i,j}.$$

Therefore, C is a YMS(n,t) and A is a YSSSODLS(n,t). □

Now we use SSSODLS and *RMR to give a construction of YSSSODLS.

Lemma 2.2. *If there exist both an SSSODLS(m) and a *RMR$(2,2m;t)$, then there exists a YSSSODLS$(4m,2t)$.*

Proof. Let $n = 4m$, and let

$$U = \begin{bmatrix} 0 & 1 & 3 & 2 \\ 3 & 2 & 0 & 1 \\ 2 & 3 & 1 & 0 \\ 1 & 0 & 2 & 3 \end{bmatrix}.$$

Let V be an SSSODLS(m). We define a matrix W with the entries

$$w_{i,j} = mu_{r,s} + v_{x,y}, \text{ where } i = mr + x, j = ms + y, \; r,s \in I_4, \; x,y \in I_m.$$

It is readily checked that W is an SSSODLS$(4m)$.

Let $H = \begin{bmatrix} H_0 & H_1 \\ H_2 & H_3 \end{bmatrix}$ be a *RMR$(2,2m;t)$, where $H_i = (h_{i,0}, \cdots, h_{i,m-1}), i \in I_4$. Let $L = (L_0, L_1, L_2, L_3) = (l_0, l_1, \cdots, l_{4m-1})$ constructed as follows.

(1) If H is vertically symmetric, then $L = (H_0, \vec{H}_1, H_3, \vec{H}_2)$, where $\vec{H}_i = (h_{i,m-1}, \cdots, h_{i,0})$.

(2) If H is horizontally symmetric, then $L = (H_0, H_2, H_3, H_1)$.

Clearly, both in Case (1) and Case (2) we have $l_j + l_{4m-1-j} = 4m-1$, for all $j \in I_{4m}$. Let

$$A = (a_{i,j}), \; a_{i,j} = l_{w_{i,j}}, \; i,j \in I_{4m}.$$

Then A is also an SSSODLS$(4m)$. We shall prove that A is a YSSSODLS$(4m, 2t)$.

For $i,j \in I_{4m}$, there are $r,s \in I_4$ and $x,y \in I_m$ such that

$$a_{i,j} = l_{w_{i,j}} = l_{w_{mr+x,ms+y}} = l_{mu_{r,s}+v_{x,y}}, \; i = mr + x, j = ms + y.$$

Since V is self-orthogonal, $\{(v_{x,y}, v_{y,x}) | x, y \in I_m\} = \{(x, y) | x, y \in I_m\}$. It follows that for any r, s and for any p, q, we have

$$\begin{aligned}\sum_x \sum_y l^p_{mu_{r,s}+v_{x,y}} l^q_{mu_{s,r}+v_{y,x}} &= \sum_x \sum_y l^p_{mu_{r,s}+x} l^q_{mu_{s,r}+y} \\ &= \sum_x l^p_{mu_{r,s}+x} \sum_y l^q_{mu_{s,r}+y},\end{aligned}$$

where $\sum_z$ denotes the sum $\sum_{z=0}^{m-1}$ for convenience, where z can be x, y, etc.

Superposing U and U^T we get

$$D = \left[\begin{array}{cc|cc} (0,0) & (1,3) & (3,2) & (2,1) \\ (3,1) & (2,2) & (0,3) & (1,0) \\ \hline (2,3) & (3,0) & (1,1) & (0,2) \\ (1,2) & (0,1) & (2,0) & (3,3) \end{array}\right].$$

Let $p+q \leq 2t$. We consider the sum $\sum_{S_1} a^p_{i,j} a^q_{j,i}$. It is $\sum_{r=0}^{1} \sum_{s=0}^{1} (\sum_x l^p_{mu_{r,s}+x})(\sum_y l^q_{mu_{s,r}+y})$, which corresponds to the upper-left corner of D. Since $\{(u_{r,s}, u_{s,r}) | r = 0, 1, s = 0, 1\} = \{(0,0), (1,3), (3,1), (2,2)\}$, we have

$$\sum_{S_1} a^p_{i,j} a^q_{j,i} = \sum_x l^p_x \sum_y l^q_y + \sum_x l^p_{m+x} \sum_y l^q_{3m+y} + \sum_x l^p_{3m+x} \sum_y l^q_{m+y} + \sum_x l^p_{2m+x} \sum_y l^q_{2m+y}. \quad \text{(i)}$$

Similarly, we have

$$\sum_{S_2} a^p_{i,j} a^q_{j,i} = \sum_x l^p_{3m+x} \sum_y l^q_{2m+y} + \sum_x l^p_{2m+x} \sum_y l^q_{m+y} + \sum_x l^p_x \sum_y l^q_{3m+y} + \sum_x l^p_{m+x} \sum_y l^q_y, \quad \text{(ii)}$$

$$\sum_{S_3} a^p_{i,j} a^q_{j,i} = \sum_x l^p_{2m+x} \sum_y l^q_{3m+y} + \sum_x l^p_{3m+x} \sum_y l^q_y + \sum_x l^p_{m+x} \sum_y l^q_{2m+y} + \sum_x l^p_x \sum_y l^q_{m+y}, \quad \text{(iii)}$$

$$\sum_{S_4} a^p_{i,j} a^q_{j,i} = \sum_x l^p_{m+x} \sum_y l^q_{m+y} + \sum_x l^p_x \sum_y l^q_{2m+y} + \sum_x l^p_{2m+x} \sum_y l^q_y + \sum_x l^p_{3m+x} \sum_y l^q_{3m+y}. \quad \text{(iv)}$$

The following two cases need to be considered.

Case 1: H is vertically symmetric

Since $L = (L_0, L_1, L_2, L_3) = (H_0, \vec{H}_1, H_3, \vec{H}_2)$, for any integer e we have

$$\sum_z l_z^e = \sum_z h_{0,z}^e, \sum_z l_{m+z}^e = \sum_z h_{1,z}^e, \sum_z l_{2m+z}^e = \sum_z h_{3,z}^e, \sum_z l_{3m+z}^e = \sum_z h_{2,z}^e.$$

Since H is a *RMR$(2, 2m; t)$, $\sum_z h_{0,z}^e = \sum_z h_{3,z}^e$, $\sum_z h_{2,z}^e = \sum_z h_{1,z}^e$, $e < t$. Thus,

$$\sum_z l_z^e = \sum_z l_{2m+z}^e, \sum_z l_{3m+z}^e = \sum_z l_{m+z}^e, \ e < t.$$

Now we will prove $\sum_{S_1} a_{i,j}^p a_{j,i}^q = \sum_{S_4} a_{i,j}^p a_{j,i}^q$, $p + q \leq 2t$.

For $p > q$, by (i) and (iv) we get

$$\sum_{S_1} a_{i,j}^p a_{j,i}^q = (\sum_x l_x^p + \sum_x l_{2m+x}^p) \sum_{y=0}^{m-1} l_y^q + (\sum_x l_{m+x}^p + \sum_x l_{3m+x}^p) \sum_y l_{m+y}^q,$$

$$\sum_{S_4} a_{i,j}^p a_{j,i}^q = (\sum_x l_{m+x}^p + \sum_x l_{3m+x}^p) \sum_y l_{m+y}^q + (\sum_x l_x^p + \sum_x l_{2m+x}^p) \sum_y l_y^q.$$

Clearly, $\sum_{S_1} a_{i,j}^p a_{j,i}^q = \sum_{S_4} a_{i,j}^p a_{j,i}^q$.

For $p < q$, we have

$$\sum_{S_1} a_{i,j}^p a_{j,i}^q = \sum_x l_x^p (\sum_y l_y^q + \sum_y l_{2m+y}^q) + \sum_x l_{m+x}^p (\sum_y l_{3m+y}^q + \sum_y l_{m+y}^q),$$

$$\sum_{S_4} a_{i,j}^p a_{j,i}^q = \sum_x l_{m+x}^p (\sum_y l_{m+y}^q + \sum_y l_{3m+y}^q) + \sum_x l_x^p (\sum_y l_{2m+y}^q + \sum_y l_y^q).$$

It also leads to $\sum_{S_1} a_{i,j}^p a_{j,i}^q = \sum_{S_4} a_{i,j}^p a_{j,i}^q$.

For $p = q = t$, since $\sum_z h_{0,z}^t + \sum_z h_{1,z}^t = \sum_z h_{2,z}^t + \sum_z h_{3,z}^t$, we have

$$\sum_z l_z^t + \sum_z l_{m+z}^t = \sum_z l_{3m+z}^t + \sum_z l_{2m+z}^t.$$

By (i) and (ii) we get

$$\sum_{S_1} a_{i,j}^t a_{j,i}^t + \sum_{S_2} a_{i,j}^t a_{j,i}^t$$

$$= \sum_x l_x^t \sum_y l_y^t + \sum_x l_{m+x}^t \sum_y l_{3m+y}^t + \sum_x l_{3m+x}^t \sum_y l_{m+y}^t + \sum_x l_{2m+x}^t \sum_y l_{2m+y}^t$$

$$+ \sum_x l_{3m+x}^t \sum_y l_{2m+y}^t + \sum_x l_{2m+x}^t \sum_y l_{m+y}^t + \sum_x l_x^t \sum_y l_{3m+y}^t + \sum_x l_{m+x}^t \sum_y l_y^t$$

$$= (\sum_x l_x^t + \sum_x l_{m+x}^t)(\sum_y l_y^t + \sum_y l_{3m+y}^t) + (\sum_x l_{3m+x}^t + \sum_x l_{2m+x}^t)(\sum_y l_{m+y}^t + \sum_y l_{2m+y}^t)$$

$$= (\sum_x l_x^t + \sum_x l_{m+x}^t)(\sum_y l_y^t + \sum_y l_{3m+y}^t + \sum_y l_{m+y}^t + \sum_y l_{2m+y}^t).$$

By (ii) and (iv) we get

$$\sum_{S_2} a_{i,j}^t a_{j,i}^t + \sum_{S_4} a_{i,j}^t a_{j,i}^t$$

$$= \sum_x l_{3m+x}^t \sum_y l_{2m+y}^t + \sum_x l_{2m+x}^t \sum_y l_{m+y}^t + \sum_x l_x^t \sum_y l_{3m+y}^t + \sum_x l_{m+x}^t \sum_y l_y^t$$

$$+ \sum_x l_{m+x}^t \sum_y l_{m+y}^t + \sum_x l_x^t \sum_y l_{2m+y}^t + \sum_x l_{2m+x}^t \sum_y l_y^t + \sum_x l_{3m+x}^t \sum_y l_{3m+y}^t$$

$$= (\sum_x l_x^t + \sum_x l_{3m+x}^t + \sum_x l_{m+x}^t + \sum_x l_{2m+x}^t)(\sum_y l_y^t + \sum_y l_{m+y}^t).$$

Thus $\sum_{S_1} a_{i,j}^t a_{j,i}^t + \sum_{S_2} a_{i,j}^t a_{j,i}^t = \sum_{S_2} a_{i,j}^t a_{j,i}^t + \sum_{S_4} a_{i,j}^t a_{j,i}^t$, and we have $\sum_{S_1} a_{i,j}^t a_{j,i}^t = \sum_{S_4} a_{i,j}^t a_{j,i}^t$.

So far we have proved that $\sum_{S_1} a_{i,j}^p a_{j,i}^q = \sum_{S_4} a_{i,j}^p a_{j,i}^q, p + q \leq 2t$.

By using (i),(ii) and (iii) one can prove $\sum_{S_2} a_{i,j}^p a_{j,i}^q = \sum_{S_3} a_{i,j}^p a_{j,i}^q, p + q \leq 2t$, in a similar way. Thus A is a YSSSODLS$(4m, 2t)$ by Lemma 2.1.

Case 2: H is horizontally symmetric

Since $L = (L_0, L_1, L_2, L_3) = (H_0, H_2, H_3, H_1)$, for any integer e we have

$$\sum_z l_z^e = \sum_x h_{0,z}^e, \sum_z l_{m+z}^e = \sum_x h_{2,z}^e, \sum_z l_{2m+z}^e = \sum_x h_{3,z}^e, \sum_z l_{3m+z}^e = \sum_x h_{1,z}^e.$$

Since $\sum_z h_{0,z}^e = \sum_z h_{3,z}^e$, $\sum_z h_{2,z}^e = \sum_z h_{1,z}^e$, $e < t$, we have

$$\sum_z l_z^e = \sum_z l_{2m+z}^e, \sum_z l_{m+z}^e = \sum_z l_{3m+z}^e, e < t,$$

which is exactly the same as that in the proof of Case 1. Thus by a same argument as in the proof of Case 1, we get

$$\sum_{S_1} a_{i,j}^p a_{j,i}^q = \sum_{S_4} a_{i,j}^p a_{j,i}^q, \sum_{S_2} a_{i,j}^p a_{j,i}^q = \sum_{S_3} a_{i,j}^p a_{j,i}^q, p + q \leq 2t, p \neq q.$$

For $p = q = t$, since $\sum_x h^t_{0,x} + \sum_x h^t_{1,x} = \sum_x h^t_{2,x} + \sum_x h^t_{3,x}$, we have

$$\sum_z l^t_z + \sum_z l^t_{3m+z} = \sum_z l^t_{m+z} + \sum_z l^t_{2m+z}.$$

By (i) and (ii) we get

$$\sum_{S_1} a^t_{i,j}a^t_{j,i} + \sum_{S_2} a^t_{i,j}a^t_{j,i}$$

$= (\sum_x l^t_x + \sum_x l^t_{m+x})(\sum_y l^t_y + \sum_y l^t_{3m+y}) + (\sum_x l^t_{3m+x} + \sum_x l^t_{2m+x})(\sum_y l^t_{m+y} + \sum_y l^t_{2m+y}).$

$= (\sum_x l^t_x + \sum_x l^t_{m+x} + \sum_x l^t_{3m+x} + \sum_x l^t_{2m+x})(\sum_y l^t_y + \sum_y l^t_{3m+y}).$

By (ii) and (iv) we get

$$\sum_{S_2} a^t_{i,j}a^t_{j,i} + \sum_{S_4} a^t_{i,j}a^t_{j,i}$$

$= (\sum_x l^t_x + \sum_x l^t_{3m+x})(\sum_y l^t_{2m+y} + \sum_y l^t_{3m+y}) + (\sum_x l^t_{2m+x} + \sum_x l^t_{m+x})(\sum_y l^t_y + \sum_y l^t_{m+y})$

$= (\sum_x l^t_x + \sum_x l^t_{3m+x})(\sum_y l^t_{2m+y} + \sum_y l^t_{3m+y} + \sum_y l^t_y + \sum_y l^t_{m+y}).$

Thus $\sum_{S_1} a^t_{i,j}a^t_{j,i} + \sum_{S_2} a^t_{i,j}a^t_{j,i} = \sum_{S_2} a^t_{i,j}a^t_{j,i} + \sum_{S_4} a^t_{i,j}a^t_{j,i}$, and $\sum_{S_1} a^t_{i,j}a^t_{j,i} = \sum_{S_4} a^t_{i,j}a^t_{j,i}$.

So, we proved that $\sum_{S_1} a^p_{i,j}a^q_{j,i} = \sum_{S_4} a^p_{i,j}a^q_{j,i}, p + q \leq 2t$. By using (i),(ii) and (iii) one can also prove that $\sum_{S_2} a^p_{i,j}a^q_{j,i} = \sum_{S_3} a^p_{i,j}a^q_{j,i}, p + q \leq 2t$. Thus A is a YSSSODLS$(4m, 2t)$ by Lemma 2.1.

Combining Case 1 and Case 2 we have proved that A is a YSSSODLS $(4m, 2t)$. □

Next, we will use examples of YSSSODLS$(16, 6)$ and YSSSODLS$(32, 8)$ to illustrate the construction of Lemma 2.2.

Example 1. Let U be the SSSODLS(4) given in Lemma 2.2 and let $V = U$. Compounding U and V as in Lemma 2.2 we get an SSSODLS(16) as follows.

$$W = \left[\begin{array}{cccc|cccc|cccc|cccc}
0 & 1 & 3 & 2 & 4 & 5 & 7 & 6 & 12 & 13 & 15 & 14 & 8 & 9 & 11 & 10 \\
3 & 2 & 0 & 1 & 7 & 6 & 4 & 5 & 15 & 14 & 12 & 13 & 11 & 10 & 8 & 9 \\
2 & 3 & 1 & 0 & 6 & 7 & 5 & 4 & 14 & 15 & 13 & 12 & 10 & 11 & 9 & 8 \\
1 & 0 & 2 & 3 & 5 & 4 & 6 & 7 & 13 & 12 & 14 & 15 & 9 & 8 & 10 & 11 \\
\hline
12 & 13 & 15 & 14 & 8 & 9 & 11 & 10 & 0 & 1 & 3 & 2 & 4 & 5 & 7 & 6 \\
15 & 14 & 12 & 13 & 11 & 10 & 8 & 9 & 3 & 2 & 0 & 1 & 7 & 6 & 4 & 5 \\
14 & 15 & 13 & 12 & 10 & 11 & 9 & 8 & 2 & 3 & 1 & 0 & 6 & 7 & 5 & 4 \\
13 & 12 & 14 & 15 & 9 & 8 & 10 & 11 & 1 & 0 & 2 & 3 & 5 & 4 & 6 & 7 \\
\hline
8 & 9 & 11 & 10 & 12 & 13 & 15 & 14 & 4 & 5 & 7 & 6 & 0 & 1 & 3 & 2 \\
11 & 10 & 8 & 9 & 15 & 14 & 12 & 13 & 7 & 6 & 4 & 5 & 3 & 2 & 0 & 1 \\
10 & 11 & 9 & 8 & 14 & 15 & 13 & 12 & 6 & 7 & 5 & 4 & 2 & 3 & 1 & 0 \\
9 & 8 & 10 & 11 & 13 & 12 & 14 & 15 & 5 & 4 & 6 & 7 & 1 & 0 & 2 & 3 \\
\hline
4 & 5 & 7 & 6 & 0 & 1 & 3 & 2 & 8 & 9 & 11 & 10 & 12 & 13 & 15 & 14 \\
7 & 6 & 4 & 5 & 3 & 2 & 0 & 1 & 11 & 10 & 8 & 9 & 15 & 14 & 12 & 13 \\
6 & 7 & 5 & 4 & 2 & 3 & 1 & 0 & 10 & 11 & 9 & 8 & 14 & 15 & 13 & 12 \\
5 & 4 & 6 & 7 & 1 & 0 & 2 & 3 & 9 & 8 & 10 & 11 & 13 & 12 & 14 & 15
\end{array}\right].$$

We construct a *RMR(2, 8; 3) as follows.

$$H = \begin{bmatrix} H_0 & H_1 \\ H_2 & H_3 \end{bmatrix} = \left[\begin{array}{cccc|cccc} 0 & 3 & 6 & 5 & 10 & 9 & 12 & 15 \\ 13 & 14 & 11 & 8 & 7 & 4 & 1 & 2 \end{array}\right].$$

Note that H is horizontally symmetric. As in Lemma 2.2 we get a sequence L as follows.

$$L = (l_j) = (H_0, H_2, H_3, H_1) = [\,0\ 3\ 6\ 5 \,|\, 13\ 14\ 11\ 8 \,|\, 7\ 4\ 1\ 2 \,|\, 10\ 9\ 12\ 15\,].$$

Let $A = (a_{i,j}),\ a_{i,j} = l_{w_{i,j}},\ i, j \in I_{16}$. Then

$$A = \left[\begin{array}{cccc|cccc|cccc|cccc}
0 & 3 & 5 & 6 & 13 & 14 & 8 & 11 & 10 & 9 & 15 & 12 & 7 & 4 & 2 & 1 \\
5 & 6 & 0 & 3 & 8 & 11 & 13 & 14 & 15 & 12 & 10 & 9 & 2 & 1 & 7 & 4 \\
6 & 5 & 3 & 0 & 11 & 8 & 14 & 13 & 12 & 15 & 9 & 10 & 1 & 2 & 4 & 7 \\
3 & 0 & 6 & 5 & 14 & 13 & 11 & 8 & 9 & 10 & 12 & 15 & 4 & 7 & 1 & 2 \\
\hline
10 & 9 & 15 & 12 & 7 & 4 & 2 & 1 & 0 & 3 & 5 & 6 & 13 & 14 & 8 & 11 \\
15 & 12 & 10 & 9 & 2 & 1 & 7 & 4 & 5 & 6 & 0 & 3 & 8 & 11 & 13 & 14 \\
12 & 15 & 9 & 10 & 1 & 2 & 4 & 7 & 6 & 5 & 3 & 0 & 11 & 8 & 14 & 13 \\
9 & 10 & 12 & 15 & 4 & 7 & 1 & 2 & 3 & 0 & 6 & 5 & 14 & 13 & 11 & 8 \\
\hline
7 & 4 & 2 & 1 & 10 & 9 & 15 & 12 & 13 & 14 & 8 & 11 & 0 & 3 & 5 & 6 \\
2 & 1 & 7 & 4 & 15 & 12 & 10 & 9 & 8 & 11 & 13 & 14 & 5 & 6 & 0 & 3 \\
1 & 2 & 4 & 7 & 12 & 15 & 9 & 10 & 11 & 8 & 14 & 13 & 6 & 5 & 3 & 0 \\
4 & 7 & 1 & 2 & 9 & 10 & 12 & 15 & 14 & 13 & 11 & 8 & 3 & 0 & 6 & 5 \\
\hline
13 & 14 & 8 & 11 & 0 & 3 & 5 & 6 & 7 & 4 & 2 & 1 & 10 & 9 & 15 & 12 \\
8 & 11 & 13 & 14 & 5 & 6 & 0 & 3 & 2 & 1 & 7 & 4 & 15 & 12 & 10 & 9 \\
11 & 8 & 14 & 13 & 6 & 5 & 3 & 0 & 1 & 2 & 4 & 7 & 12 & 15 & 9 & 10 \\
14 & 13 & 11 & 8 & 3 & 0 & 6 & 5 & 4 & 7 & 1 & 2 & 9 & 10 & 12 & 15
\end{array}\right].$$

It is readily checked that A is a YSSSODLS(16, 6). □

Example 2. Let U be the SSSODLS(4) given in Lemma 2.2. An SSSODLS (8) is listed below (also see Lemma 4.2 in [15]).

$$V = \begin{bmatrix} 0 & 2 & 4 & 6 & 5 & 7 & 1 & 3 \\ 6 & 4 & 2 & 0 & 3 & 1 & 7 & 5 \\ 1 & 3 & 5 & 7 & 4 & 6 & 0 & 2 \\ 7 & 5 & 3 & 1 & 2 & 0 & 6 & 4 \\ 3 & 1 & 7 & 5 & 6 & 4 & 2 & 0 \\ 5 & 7 & 1 & 3 & 0 & 2 & 4 & 6 \\ 2 & 0 & 6 & 4 & 7 & 5 & 3 & 1 \\ 4 & 6 & 0 & 2 & 1 & 3 & 5 & 7 \end{bmatrix}.$$

Compounding U and V as in Lemma 2.2 an SSSODLS(32) W is followed. We use the *RMR(2, 8; 3) given in Example 1 to get a *RMR(2, 16; 4) as follows.

$$H = \begin{bmatrix} H_0 & H_1 \\ H_2 & H_3 \end{bmatrix} = \left[\begin{array}{cccccccc|cccccccc} 0 & 3 & 6 & 5 & 10 & 9 & 12 & 15 & 18 & 17 & 20 & 23 & 24 & 27 & 30 & 29 \\ 31 & 28 & 25 & 26 & 21 & 22 & 19 & 16 & 13 & 14 & 11 & 8 & 7 & 4 & 1 & 2 \end{array}\right].$$

Note that H is vertically symmetric. As in Lemma 2.2 we get a sequence L as follows.

$$L = (l_j) = (H_0, \vec{H}_1, H_3, \vec{H}_2)$$
$$= [0\,3\,6\,5\,10\,9\,12\,15 | 29\,30\,27\,24\,23\,20\,17\,18 | 13\,14\,11\,8\,7\,4\,1\,2 | 16\,19\,22\,21\,26\,25\,28\,31].$$

Let $A = (a_{i,j})$, $a_{i,j} = l_{w_{i,j}}$, $i, j \in I_{32}$. Then A is as displayed in Fig. 1 It is readily checked that A is a YSSSODLS(32, 8). □

The following construction gives YSSSODLS($8m$, 6) for odd m.

Lemma 2.3. *For odd $m > 1$, if there exists a *RMR$(2, 4m; 3)$, then there exists a YSSSODLS$(8m, 6)$.*

Proof. Let $n = 8m$, and let U be the SSSODLS(8) given in Example 2. Let V be an SSSODLS(m). We define a matrix $W = (w_{i,j})$ as

$$w_{i,j} = mu_{r,s} + v_{x,y}, \; i = mr + x, j = ms + y, \; r, s \in I_8, \; x, y \in I_m.$$

It is readily checked that W is an SSSODLS($8m$). Let

$$H = \begin{bmatrix} H_0 \, H_5 \, H_2 \, H_7 \\ H_3 \, H_6 \, H_1 \, H_4 \end{bmatrix}$$

be a *RMR$(2, 4m; 3)$, where $H_i = (h_{i,0}, \cdots, h_{i,m-1}), i \in I_8$. Let

$$L = (l_j)_{1 \times 8m} = (H_0, H_1, H_2, H_3, H_4, H_5, H_6, H_7),$$

$$A = \left[\begin{array}{cccccccc|cccccccc|cccccccc|cccccccc}
0 & 6 & 10 & 12 & 9 & 15 & 3 & 5 & 29 & 27 & 23 & 17 & 20 & 18 & 30 & 24 & 16 & 22 & 26 & 28 & 25 & 31 & 19 & 21 & 13 & 11 & 7 & 1 & 4 & 2 & 14 & 8 \\
12 & 10 & 6 & 0 & 5 & 3 & 15 & 9 & 17 & 23 & 27 & 29 & 24 & 30 & 18 & 20 & 28 & 26 & 22 & 16 & 21 & 19 & 31 & 25 & 1 & 7 & 11 & 13 & 8 & 14 & 2 & 4 \\
3 & 5 & 9 & 15 & 10 & 12 & 0 & 6 & 30 & 24 & 20 & 18 & 23 & 17 & 29 & 27 & 19 & 21 & 25 & 31 & 26 & 28 & 16 & 22 & 14 & 8 & 4 & 2 & 7 & 1 & 13 & 11 \\
15 & 9 & 5 & 3 & 6 & 0 & 12 & 10 & 18 & 20 & 24 & 30 & 27 & 29 & 17 & 23 & 31 & 25 & 21 & 19 & 22 & 16 & 28 & 26 & 2 & 4 & 8 & 14 & 11 & 13 & 1 & 7 \\
5 & 3 & 15 & 9 & 12 & 10 & 6 & 0 & 24 & 30 & 18 & 20 & 17 & 23 & 27 & 29 & 21 & 19 & 31 & 25 & 28 & 26 & 22 & 16 & 8 & 14 & 2 & 4 & 1 & 7 & 11 & 13 \\
9 & 15 & 3 & 5 & 0 & 6 & 10 & 12 & 20 & 18 & 30 & 24 & 29 & 27 & 23 & 17 & 25 & 31 & 19 & 21 & 16 & 22 & 26 & 28 & 4 & 2 & 14 & 8 & 13 & 11 & 7 & 1 \\
6 & 0 & 12 & 10 & 15 & 9 & 5 & 3 & 27 & 29 & 17 & 23 & 18 & 20 & 24 & 30 & 22 & 16 & 28 & 26 & 31 & 25 & 21 & 19 & 11 & 13 & 1 & 7 & 2 & 4 & 8 & 14 \\
10 & 12 & 0 & 6 & 3 & 5 & 9 & 15 & 23 & 17 & 29 & 27 & 30 & 24 & 20 & 18 & 26 & 28 & 16 & 22 & 19 & 21 & 25 & 31 & 7 & 1 & 13 & 11 & 14 & 8 & 4 & 2 \\
\hline
16 & 22 & 26 & 28 & 25 & 31 & 19 & 21 & 13 & 11 & 7 & 1 & 4 & 2 & 14 & 8 & 0 & 6 & 10 & 12 & 9 & 15 & 3 & 5 & 29 & 27 & 23 & 17 & 20 & 18 & 30 & 24 \\
28 & 26 & 22 & 16 & 21 & 19 & 31 & 25 & 1 & 7 & 11 & 13 & 8 & 14 & 2 & 4 & 12 & 10 & 6 & 0 & 5 & 3 & 15 & 9 & 17 & 23 & 27 & 29 & 24 & 30 & 18 & 20 \\
19 & 21 & 25 & 31 & 26 & 28 & 16 & 22 & 14 & 8 & 4 & 2 & 7 & 1 & 13 & 11 & 3 & 5 & 9 & 15 & 10 & 12 & 0 & 6 & 30 & 24 & 20 & 18 & 23 & 17 & 29 & 27 \\
31 & 25 & 21 & 19 & 22 & 16 & 28 & 26 & 2 & 4 & 8 & 14 & 11 & 13 & 1 & 7 & 15 & 9 & 5 & 3 & 6 & 0 & 12 & 10 & 18 & 20 & 24 & 30 & 27 & 29 & 17 & 23 \\
21 & 19 & 31 & 25 & 28 & 26 & 22 & 16 & 8 & 14 & 2 & 4 & 1 & 7 & 11 & 13 & 5 & 3 & 15 & 9 & 12 & 10 & 6 & 0 & 24 & 30 & 18 & 20 & 17 & 23 & 27 & 29 \\
25 & 31 & 19 & 21 & 16 & 22 & 26 & 28 & 4 & 2 & 14 & 8 & 13 & 11 & 7 & 1 & 9 & 15 & 3 & 5 & 0 & 6 & 10 & 12 & 20 & 18 & 30 & 24 & 29 & 27 & 23 & 17 \\
22 & 16 & 28 & 26 & 31 & 25 & 21 & 19 & 11 & 13 & 1 & 7 & 2 & 4 & 8 & 14 & 6 & 0 & 12 & 10 & 15 & 9 & 5 & 3 & 27 & 29 & 17 & 23 & 18 & 20 & 24 & 30 \\
26 & 28 & 16 & 22 & 19 & 21 & 25 & 31 & 7 & 1 & 13 & 11 & 14 & 8 & 4 & 2 & 10 & 12 & 0 & 6 & 3 & 5 & 9 & 15 & 23 & 17 & 29 & 27 & 30 & 24 & 20 & 18 \\
\hline
13 & 11 & 7 & 1 & 4 & 2 & 14 & 8 & 16 & 22 & 26 & 28 & 25 & 31 & 19 & 21 & 29 & 27 & 23 & 17 & 20 & 18 & 30 & 24 & 0 & 6 & 10 & 12 & 9 & 15 & 3 & 5 \\
1 & 7 & 11 & 13 & 8 & 14 & 2 & 4 & 28 & 26 & 22 & 16 & 21 & 19 & 31 & 25 & 17 & 23 & 27 & 29 & 24 & 30 & 18 & 20 & 12 & 10 & 6 & 0 & 5 & 3 & 15 & 9 \\
14 & 8 & 4 & 2 & 7 & 1 & 13 & 11 & 19 & 21 & 25 & 31 & 26 & 28 & 16 & 22 & 30 & 24 & 20 & 18 & 23 & 17 & 29 & 27 & 3 & 5 & 9 & 15 & 10 & 12 & 0 & 6 \\
2 & 4 & 8 & 14 & 11 & 13 & 1 & 7 & 31 & 25 & 21 & 19 & 22 & 16 & 28 & 26 & 18 & 20 & 24 & 30 & 27 & 29 & 17 & 23 & 15 & 9 & 5 & 3 & 6 & 0 & 12 & 10 \\
8 & 14 & 2 & 4 & 1 & 7 & 11 & 13 & 21 & 19 & 31 & 25 & 28 & 26 & 22 & 16 & 24 & 30 & 18 & 20 & 17 & 23 & 27 & 29 & 5 & 3 & 15 & 9 & 12 & 10 & 6 & 0 \\
4 & 2 & 14 & 8 & 13 & 11 & 7 & 1 & 25 & 31 & 19 & 21 & 16 & 22 & 26 & 28 & 20 & 18 & 30 & 24 & 29 & 27 & 23 & 17 & 9 & 15 & 3 & 5 & 0 & 6 & 10 & 12 \\
11 & 13 & 1 & 7 & 2 & 4 & 8 & 14 & 22 & 16 & 28 & 26 & 31 & 25 & 21 & 19 & 27 & 29 & 17 & 23 & 18 & 20 & 24 & 30 & 6 & 0 & 12 & 10 & 15 & 9 & 5 & 3 \\
7 & 1 & 13 & 11 & 14 & 8 & 4 & 2 & 26 & 28 & 16 & 22 & 19 & 21 & 25 & 31 & 23 & 17 & 29 & 27 & 30 & 24 & 20 & 18 & 10 & 12 & 0 & 6 & 3 & 5 & 9 & 15 \\
\hline
29 & 27 & 23 & 17 & 20 & 18 & 30 & 24 & 0 & 6 & 10 & 12 & 9 & 15 & 3 & 5 & 13 & 11 & 7 & 1 & 4 & 2 & 14 & 8 & 16 & 22 & 26 & 28 & 25 & 31 & 19 & 21 \\
17 & 23 & 27 & 29 & 24 & 30 & 18 & 20 & 12 & 10 & 6 & 0 & 5 & 3 & 15 & 9 & 1 & 7 & 11 & 13 & 8 & 14 & 2 & 4 & 28 & 26 & 22 & 16 & 21 & 19 & 31 & 25 \\
30 & 24 & 20 & 18 & 23 & 17 & 29 & 27 & 3 & 5 & 9 & 15 & 10 & 12 & 0 & 6 & 14 & 8 & 4 & 2 & 7 & 1 & 13 & 11 & 19 & 21 & 25 & 31 & 26 & 28 & 16 & 22 \\
18 & 20 & 24 & 30 & 27 & 29 & 17 & 23 & 15 & 9 & 5 & 3 & 6 & 0 & 12 & 10 & 2 & 4 & 8 & 14 & 11 & 13 & 1 & 7 & 31 & 25 & 21 & 19 & 22 & 16 & 28 & 26 \\
24 & 30 & 18 & 20 & 17 & 23 & 27 & 29 & 5 & 3 & 15 & 9 & 12 & 10 & 6 & 0 & 8 & 14 & 2 & 4 & 1 & 7 & 11 & 13 & 21 & 19 & 31 & 25 & 28 & 26 & 22 & 16 \\
20 & 18 & 30 & 24 & 29 & 27 & 23 & 17 & 9 & 15 & 3 & 5 & 0 & 6 & 10 & 12 & 4 & 2 & 14 & 8 & 13 & 11 & 7 & 1 & 25 & 31 & 19 & 21 & 16 & 22 & 26 & 28 \\
27 & 29 & 17 & 23 & 18 & 20 & 24 & 30 & 6 & 0 & 12 & 10 & 15 & 9 & 5 & 3 & 11 & 13 & 1 & 7 & 2 & 4 & 8 & 14 & 22 & 16 & 28 & 26 & 31 & 25 & 21 & 19 \\
23 & 17 & 29 & 27 & 30 & 24 & 20 & 18 & 10 & 12 & 0 & 6 & 3 & 5 & 9 & 15 & 7 & 1 & 13 & 11 & 14 & 8 & 4 & 2 & 26 & 28 & 16 & 22 & 19 & 21 & 25 & 31
\end{array}\right]$$

Fig. 1. YSSSODLS(32, 8)

and let

$$A=(a_{i,j}),\ a_{i,j}=l_{w_{i,j}},\ i,j\in I_{8m}.$$

Since H is horizontally symmetric, $l_j+l_{8m-1-j}=8m-1, j\in I_{4m}$. Thus A is also an SSSODLS($8m$). We claim that A is a YSSSODLS($8m,6$).

Superposing U and U^T we get

$$D=\left[\begin{array}{cccc|cccc}
00 & 26 & 41 & 67 & 53 & 75 & 12 & 34\\
62 & 44 & 23 & 05 & 31 & 17 & 70 & 56\\
14 & 32 & 55 & 73 & 47 & 61 & 06 & 20\\
76 & 50 & 37 & 11 & 25 & 03 & 64 & 42\\
\hline
35 & 13 & 74 & 52 & 66 & 40 & 27 & 01\\
57 & 71 & 16 & 30 & 04 & 22 & 45 & 63\\
21 & 07 & 60 & 46 & 72 & 54 & 33 & 15\\
43 & 65 & 02 & 24 & 10 & 36 & 51 & 77
\end{array}\right].$$

We still use $\sum\limits_z$ to denote the sum $\sum\limits_{z=0}^{m-1}$, and z can be x,y, etc. Note that

$$\sum_x\sum_y l^p_{mu_{r,s}+v_{x,y}}l^q_{mu_{s,r}+v_{y,x}}=\sum_x l^p_{mu_{r,s}+x}\sum_y l^q_{mu_{s,r}+y},$$

and $\{(u_{r,s},u_{s,r})|r,s\in I_4\}$ is the set of the elements of the upper-left part of D, we have

$$\sum_{S_1}a^p_{i,j}a^q_{j,i}=\sum_{r=0}^{3}\sum_{s=0}^{3}(\sum_x l^p_{mu_{r,s}+x})(\sum_y l^q_{mu_{s,r}+y})$$

$$=\sum_{r=0}^{3}\sum_{s=0}^{3}(\sum_x h^p_{u_{r,s},x})(\sum_y h^q_{u_{s,r},y})$$

$$=(\sum_x h^p_{0,x}+\sum_x h^p_{5,x})(\sum_y h^q_{0,y}+\sum_y h^q_{5,y})+(\sum_x h^p_{1,x}+\sum_x h^p_{4,x})(\sum_y h^q_{1,y}+\sum_y h^q_{4,y})+(\sum_x h^p_{2,x}+\sum_x h^p_{7,x})(\sum_y h^q_{3,y}+\sum_y h^q_{6,y})+(\sum_x h^p_{3,x}+\sum_x h^p_{6,x})(\sum_y h^q_{2,y}+\sum_y h^q_{7,y}).$$

Similarly, we have

$$\sum_{S_4}a^p_{i,j}a^q_{j,i}=\sum_{r=4}^{7}\sum_{s=4}^{7}(\sum_x l^p_{mu_{r,s}+x})(\sum_y l^q_{mu_{s,r}+y})$$

$$=\sum_{r=4}^{7}\sum_{s=4}^{7}(\sum_x h^p_{u_{r,s},x})(\sum_y h^q_{u_{s,r},y})$$

$$=(\sum_x h^p_{0,x}+\sum_x h^p_{5,x})(\sum_y h^q_{1,y}+\sum_y h^q_{4,y})+(\sum_x h^p_{1,x}+\sum_x h^p_{4,x})(\sum_y h^q_{0,y}+\sum_y h^q_{5,y})+(\sum_x h^p_{2,x}+\sum_x h^p_{7,x})(\sum_y h^q_{2,y}+\sum_y h^q_{7,y})+(\sum_x h^p_{3,x}+\sum_x h^p_{6,x})(\sum_y h^q_{3,y}+$$

$\sum_y h^q_{6,y})$.

Since H is a *RMR$(2, 4m; 3)$, for $u = 1, 2$ we have

$$\sum_z h^u_{0,z} + \sum_z h^u_{5,z} = \sum_z h^u_{1,z} + \sum_z h^u_{4,z},\ \sum_x h^u_{2,z} + \sum_z h^u_{7,z} = \sum_z h^u_{3,z} + \sum_z h^u_{6,z},$$

For the cases of $p < 3$ or $q < 3$, by checking the terms of $\sum_{S_1} a^p_{i,j}a^q_{j,i}$ and $\sum_{S_4} a^p_{i,j}a^q_{j,i}$ we have $\sum_{S_1} a^p_{i,j}a^q_{j,i} = \sum_{S_4} a^p_{i,j}a^q_{j,i}$. Similarly one can prove $\sum_{S_2} a^p_{i,j}a^q_{j,i} = \sum_{S_3} a^p_{i,j}a^q_{j,i}$.

For $p = q = 3$, we have

$$\sum_{S_2} a^3_{i,j}a^3_{j,i} = (\sum_x h^3_{0,x} + \sum_x h^3_{5,x})(\sum_y h^3_{3,y} + \sum_y h^3_{6,y}) + (\sum_x h^3_{1,x} + \sum_x h^3_{4,x}) (\sum_y h^3_{2,y} + \sum_y h^3_{7,y}) + (\sum_x h^3_{2,x} + \sum_x h^3_{7,x})(\sum_y h^3_{0,y} + \sum_y h^3_{5,y}) + (\sum_x h^3_{3,x} + \sum_x h^3_{6,x}) (\sum_y h^3_{1,y} + \sum_y h^3_{4,y}).$$

So,

$$\sum_{S_1} a^3_{i,j}a^3_{j,i} + \sum_{S_2} a^3_{i,j}a^3_{j,i}$$
$$= (\sum_x h^3_{0,x} + \sum_x h^3_{5,x} + \sum_x h^3_{2,x} + \sum_x h^3_{7,x})(\sum_y h^3_{0,y} + \sum_y h^3_{5,y} + \sum_y h^3_{3,y} + \sum_y h^3_{6,y}) + (\sum_x h^3_{1,x} + \sum_x h^3_{4,x} + \sum_x h^3_{3,x} + \sum_x h^3_{6,x})(\sum_y h^3_{1,y} + \sum_y h^3_{4,y} + \sum_y h^3_{2,y} + \sum_y h^3_{7,y}).$$

Since H is row 3-multimagic, we have

$$\sum_x h^3_{0,z} + \sum_z h^3_{5,z} + \sum_z h^3_{2,z} + \sum_z h^3_{7,z} = \sum_z h^3_{3,z} + \sum_z h^3_{6,z} + \sum_z h^3_{1,z} + \sum_z h^3_{4,z}.$$

Thus

$$\sum_{S_1} a^3_{i,j}a^3_{j,i} + \sum_{S_2} a^3_{i,j}a^3_{j,i} = (\sum_x h^3_{0,x} + \sum_x h^3_{5,x} + \sum_x h^3_{2,x} + \sum_x h^3_{7,x})(\sum_{w=0}^{7} \sum_y h^3_{w,y}).$$

Similarly we have

$$\sum_{S_2} a^3_{i,j}a^3_{j,i} + \sum_{S_4} a^3_{i,j}a^3_{j,i} = (\sum_{w=0}^{7} \sum_x h^3_{w,x})(\sum_y h^3_{1,y} + \sum_y h^3_{4,y} + \sum_y h^3_{3,y} + \sum_y h^3_{6,y}).$$

So,

$$\sum_{S_1} a_{i,j}^3 a_{j,i}^3 + \sum_{S_2} a_{i,j}^3 a_{j,i}^3 = \sum_{S_2} a_{i,j}^3 a_{j,i}^3 + \sum_{S_4} a_{i,j}^3 a_{j,i}^3.$$

Thus $\sum_{S_1} a_{i,j}^3 a_{j,i}^3 = \sum_{S_4} a_{i,j}^3 a_{j,i}^3$. Similarly one can prove $\sum_{S_2} a_{i,j}^3 a_{j,i}^3 = \sum_{S_3} a_{i,j}^3 a_{j,i}^3$.
Consequently, A is a YSSSODLS$(8m, 6)$. □

An example of a YSSSODLS(40, 6) is listed as follows.

Example 3. Let U be the SSSODLS(8) listed in Example 2, and let

$$V = \begin{bmatrix} 1 & 3 & 0 & 2 & 4 \\ 2 & 4 & 1 & 3 & 0 \\ 3 & 0 & 2 & 4 & 1 \\ 4 & 1 & 3 & 0 & 2 \\ 0 & 2 & 4 & 1 & 3 \end{bmatrix}.$$

By compounding U, V as in Lemma 2.3, we get an SSSODLS(40) W. An RMR(2, 20; 3) is listed below.

$$H = \left[\begin{array}{ccccc|ccccc|ccccc|ccccc} 4 & 6 & 10 & 11 & 12 & 14 & 0 & 18 & 17 & 3 & 36 & 22 & 21 & 39 & 25 & 27 & 28 & 29 & 33 & 35 \\ \hline 24 & 26 & 30 & 31 & 32 & 34 & 20 & 38 & 37 & 23 & 16 & 2 & 1 & 19 & 5 & 7 & 8 & 9 & 13 & 15 \end{array}\right].$$

H is horizontally symmetric. By using H we get a permutation L in the following.

$$L = \left[\begin{array}{ccccc|ccccc|ccccc|ccccc} 4 & 6 & 10 & 11 & 12 & 16 & 2 & 1 & 19 & 5 & 36 & 22 & 21 & 39 & 25 & 24 & 26 & 30 & 31 & 32 \\ 7 & 8 & 9 & 13 & 15 & 14 & 0 & 18 & 17 & 3 & 34 & 20 & 38 & 37 & 23 & 27 & 28 & 29 & 33 & 35 \end{array}\right]$$

Let $A = (a_{i,j})$, $a_{i,j} = l_{w_{i,j}}$, $i, j \in I_{40}$. The matrix is displayed in Fig 2. It is readily checked that A is a YSSSODLS(40, 6). □

3 Proofs of Theorem 1.2 and Theorem 1.3

We start with the concept of magic rectangle, which will be used to construct a *RMR$(2, 2m; t)$.

An $m \times n$ *magic rectangle*, denoted by MR(m, n), is an $m \times n$ matrix consisting of all the numbers of I_{mn} such that the sum of each row is the same, and the sum of each column is the same (the two constants differ if $m \neq n$). Harmuth [12, 13] proved the following.

$A =$

6	11	4	10	12	22	39	36	21	25	8	13	7	9	15	20	37	34	38	23	0	17	14	18	3	28	33	27	29	35	2	19	16	1	5	26	31	24	30	32
10	12	6	11	4	21	25	22	39	36	9	15	8	13	7	38	23	20	37	34	18	3	0	17	14	29	35	28	33	27	1	5	2	19	16	30	32	26	31	24
11	4	10	12	6	39	36	21	25	22	13	7	9	15	8	37	34	38	23	20	17	14	18	3	0	33	27	29	35	28	19	16	1	5	2	31	24	30	32	26
12	6	11	4	10	25	22	39	36	21	15	8	13	7	9	23	20	37	34	38	3	0	17	14	18	35	28	33	27	29	5	2	19	16	1	32	26	31	24	30
4	10	12	6	11	36	21	25	22	39	7	9	15	8	13	34	38	23	20	37	14	18	3	0	17	27	29	35	28	33	16	1	5	2	19	24	30	32	26	31
20	37	34	38	23	8	13	7	9	15	22	39	36	21	25	6	11	4	10	12	26	31	24	30	32	2	19	16	1	5	28	33	27	29	35	0	17	14	18	3
38	23	20	37	34	9	15	8	13	7	21	25	22	39	36	10	12	6	11	4	30	32	26	31	24	1	5	2	19	16	29	35	28	33	27	18	3	0	17	14
37	34	38	23	20	13	7	9	15	8	39	36	21	25	22	11	4	10	12	6	31	24	30	32	26	19	16	1	5	2	33	27	29	35	28	17	14	18	3	0
23	20	37	34	38	15	8	13	7	9	25	22	39	36	21	12	6	11	4	10	32	26	31	24	30	5	2	19	16	1	35	28	33	27	29	3	0	17	14	18
34	38	23	20	37	7	9	15	8	13	36	21	25	22	39	4	10	12	6	11	24	30	32	26	31	16	1	5	2	19	27	29	35	28	33	14	18	3	0	17
2	19	16	1	5	26	31	24	30	32	0	17	14	18	3	28	33	27	29	35	8	13	7	9	15	20	37	34	38	23	6	11	4	10	12	22	39	36	21	25
1	5	2	19	16	30	32	26	31	24	18	3	0	17	14	29	35	28	33	27	9	15	8	13	7	38	23	20	37	34	10	12	6	11	4	21	25	22	39	36
19	16	1	5	2	31	24	30	32	26	17	14	18	3	0	33	27	29	35	28	13	7	9	15	8	37	34	38	23	20	11	4	10	12	6	39	36	21	25	22
5	2	19	16	1	32	26	31	24	30	3	0	17	14	18	35	28	33	27	29	15	8	13	7	9	23	20	37	34	38	12	6	11	4	10	25	22	39	36	21
16	1	5	2	19	24	30	32	26	31	14	18	3	0	17	27	29	35	28	33	7	9	15	8	13	34	38	23	20	37	4	10	12	6	11	36	21	25	22	39
28	33	27	29	35	0	17	14	18	3	26	31	24	30	32	2	19	16	1	5	22	39	36	21	25	6	11	4	10	12	20	37	34	38	23	8	13	7	9	15
29	35	28	33	27	18	3	0	17	14	30	32	26	31	24	1	5	2	19	16	21	25	22	39	36	10	12	6	11	4	38	23	20	37	34	9	15	8	13	7
33	27	29	35	28	17	14	18	3	0	31	24	30	32	26	19	16	1	5	2	39	36	21	25	22	11	4	10	12	6	37	34	38	23	20	13	7	9	15	8
35	28	33	27	29	3	0	17	14	18	32	26	31	24	30	5	2	19	16	1	25	22	39	36	21	12	6	11	4	10	23	20	37	34	38	15	8	13	7	9
27	29	35	28	33	14	18	3	0	17	24	30	32	26	31	16	1	5	2	19	36	21	25	22	39	4	10	12	6	11	34	38	23	20	37	7	9	15	8	13
26	31	24	30	32	2	19	16	1	5	28	33	27	29	35	0	17	14	18	3	20	37	34	38	23	8	13	7	9	15	22	39	36	21	25	6	11	4	10	12
30	32	26	31	24	1	5	2	19	16	29	35	28	33	27	18	3	0	17	14	38	23	20	37	34	9	15	8	13	7	21	25	22	39	36	10	12	6	11	4
31	24	30	32	26	19	16	1	5	2	33	27	29	35	28	17	14	18	3	0	37	34	38	23	20	13	7	9	15	8	39	36	21	25	22	11	4	10	12	6
32	26	31	24	30	5	2	19	16	1	35	28	33	27	29	3	0	17	14	18	23	20	37	34	38	15	8	13	7	9	25	22	39	36	21	12	6	11	4	10
24	30	32	26	31	16	1	5	2	19	27	29	35	28	33	14	18	3	0	17	34	38	23	20	37	7	9	15	8	13	36	21	25	22	39	4	10	12	6	11
0	17	14	18	3	28	33	27	29	35	2	19	16	1	5	26	31	24	30	32	6	11	4	10	12	22	39	36	21	25	8	13	7	9	15	20	37	34	38	23
18	3	0	17	14	29	35	28	33	27	1	5	2	19	16	30	32	26	31	24	10	12	6	11	4	21	25	22	39	36	9	15	8	13	7	38	23	20	37	34
17	14	18	3	0	33	27	29	35	28	19	16	1	5	2	31	24	30	32	26	11	4	10	12	6	39	36	21	25	22	13	7	9	15	8	37	34	38	23	20
3	0	17	14	18	35	28	33	27	29	5	2	19	16	1	32	26	31	24	30	12	6	11	4	10	25	22	39	36	21	15	8	13	7	9	23	20	37	34	38
14	18	3	0	17	27	29	35	28	33	16	1	5	2	19	24	30	32	26	31	4	10	12	6	11	36	21	25	22	39	7	9	15	8	13	34	38	23	20	37
22	39	36	21	25	6	11	4	10	12	20	37	34	38	23	8	13	7	9	15	28	33	27	29	35	0	17	14	18	3	26	31	24	30	32	2	19	16	1	5
21	25	22	39	36	10	12	6	11	4	38	23	20	37	34	9	15	8	13	7	29	35	28	33	27	18	3	0	17	14	30	32	26	31	24	1	5	2	19	16
39	36	21	25	22	11	4	10	12	6	37	34	38	23	20	13	7	9	15	8	33	27	29	35	28	17	14	18	3	0	31	24	30	32	26	19	16	1	5	2
25	22	39	36	21	12	6	11	4	10	23	20	37	34	38	15	8	13	7	9	35	28	33	27	29	3	0	17	14	18	32	26	31	24	30	5	2	19	16	1
36	21	25	22	39	4	10	12	6	11	34	38	23	20	37	7	9	15	8	13	27	29	35	28	33	14	18	3	0	17	24	30	32	26	31	16	1	5	2	19
8	13	7	9	15	20	37	34	38	23	6	11	4	10	12	22	39	36	21	25	2	19	16	1	5	26	31	24	30	32	0	17	14	18	3	28	33	27	29	35
9	15	8	13	7	38	23	20	37	34	10	12	6	11	4	21	25	22	39	36	1	5	2	19	16	30	32	26	31	24	18	3	0	17	14	29	35	28	33	27
13	7	9	15	8	37	34	38	23	20	11	4	10	12	6	39	36	21	25	22	19	16	1	5	2	31	24	30	32	26	17	14	18	3	0	33	27	29	35	28
15	8	13	7	9	23	20	37	34	38	12	6	11	4	10	25	22	39	36	21	5	2	19	16	1	32	26	31	24	30	3	0	17	14	18	35	28	33	27	29
7	9	15	8	13	34	38	23	20	37	4	10	12	6	11	36	21	25	22	39	16	1	5	2	19	24	30	32	26	31	14	18	3	0	17	27	29	35	28	33

Fig. 2. YSSSODLS(40, 6)

Lemma 3.1. ([12, 13]) *For $m, n > 1$, there exists an $MR(m,n)$ if and only if $m \equiv n \pmod 2$ and $(m,n) \neq (2,2)$.*

We now consider the existence of *RMR$(2, 2m; t)$ for any positive integer m. Note that m can be written in the form $m = 2^s \cdot k, s \geq 0$, $gcd(2, k) = 1$.

Lemma 3.2. *For odd integer k and integer $s \geq 0$ there is a $^*RMR(2, 2 \cdot 2^s \cdot k; t)$, where $t = s+1$ if $k = 1$, and $t = s+2$ if $k \neq 1$.*

Proof. Let $m = 2^s \cdot k$. We give a proof by induction on s.

Let $s = 0$. If $k = 1$ then $\begin{bmatrix} 0\,3 \\ 1\,2 \end{bmatrix}$ is a *RMR$(2, 2; 1)$, which is horizontally symmetric. If $k > 1$, then there is an MR$(2, 2m; 1)$ coming from Lemma 3.1. One can prove that an MR$(2, 2m; 1)$ is also a *RMR$(2, 2m; 2)$, which is vertically symmetric. In this case $t = s + 2$. So the conclusion hold for the case $s = 0$.

Now we consider $s > 0$. Let $m_1 = 2^{s-1} \cdot k$, then $m = 2m_1$. Suppose that $F = \begin{bmatrix} F_0 \\ F_1 \end{bmatrix}$ is a *RMR$(2, 2m_1; t-1)$, where $F_i = (f_{i,0}, \cdots, f_{i,m-1}), i = 0, 1$. Let

$$H = \begin{cases} \begin{bmatrix} F_0 & 4m-1-F_1 \\ 4m-1-F_0 & F_1 \end{bmatrix}, & \text{if } F \text{ is horizontally symmetric,} \\ \begin{bmatrix} F_0 & 4m-1-\vec{F}_0 \\ 4m-1-\vec{F}_1 & F_1 \end{bmatrix}, & \text{if } F \text{ is vertically symmetric,} \end{cases}$$

where $\vec{F}_i = (f_{i,m-1}, \cdots, f_{i,0}), i = 0, 1$.

For example, if $s = 1$ and $k = 1$, then $H = \begin{bmatrix} 0\,3\,6\,5 \\ 7\,4\,1\,2 \end{bmatrix}$. It is a *RMR$(2, 4; 2)$, which is vertically symmetric. One may check that t is even if H is vertically symmetric, and t is odd if H is horizontally symmetric for any odd integer k. We claim that H is a *RMR$(2, 2m; t)$ for general s.

Clearly, H is either horizontally symmetric or vertically symmetric. Note that F is a *RMR$(2, 2m_1; t-1)$. For each $e < t$, $\sum_{j=0}^{m-1} f_{0,j}^e = \sum_{j=0}^{m-1} f_{1,j}^e$, hence $\sum_{j=0}^{m-1} h_{0,j}^e = \sum_{j=0}^{m-1} h_{1,j+m}^e$. On the other hand, one can get $\sum_{j=0}^{m-1}(4m - 1 - f_{0,j})^e = \sum_{j=0}^{m-1}(4m-1-f_{1,j})^e$ by expanding each side of the equality. Thus $\sum_{j=0}^{m-1} h_{0,j+m}^e = \sum_{j=0}^{m-1} h_{1,j}^e$. So we also get $\sum_{j=0}^{2m-1} h_{0,j}^e = \sum_{j=0}^{2m-1} h_{1,j}^e$, where $e < t$. It remains to verify that $\sum_{j=0}^{2m-1} h_{0,j}^t = \sum_{j=0}^{2m-1} h_{1,j}^t$.

If F is horizontally symmetric then H is vertically symmetric. Note that t is even. For each $i = 0, 1$, we have

$\sum_{j=0}^{2m-1} h_{i,j}^t = \sum_{j=0}^{m-1} f_{i,j}^t + \sum_{j=0}^{m-1}(4m-1-f_{1-i,j})^t = \sum_{j=0}^{m-1} f_{i,j}^t + \sum_{u=0}^{t} C_t^u(4m - 1)^{t-u}(-1)^u \sum_{j=0}^{m-1} f_{1-i,j}^u$.

$$= \sum_{j=0}^{m-1} f_{i,j}^t + \sum_{u=0}^{t-1} C_t^u(4m-1)^{t-u}(-1)^u \sum_{j=0}^{m-1} f_{1-i,j}^u + \sum_{j=0}^{m-1} f_{1-i,j}^t.$$

$$= \sum_{j=0}^{m-1} f_{0,j}^t + \sum_{j=0}^{m-1} f_{1,j}^t + \sum_{u=0}^{t-1} C_t^u(4m-1)^{t-u}(-1)^u \sum_{j=0}^{m-1} f_{1-i,j}^u.$$

Since $\sum_{j=0}^{m-1} f_{0,j}^u = \sum_{j=0}^{m-1} f_{1,j}^u$ for each $u < t$, by checking the terms we have $\sum_{j=0}^{2m-1} h_{0,j}^t = \sum_{j=0}^{2m-1} h_{1,j}^t$. Thus H is an RMR$(2, 2m; t)$ when F is horizontally symmetric.

If F is vertically symmetric then H is horizontally symmetric. Considering the fact that t is odd, for each $i = 0, 1$ we have

$\sum_{j=0}^{2m-1} h_{i,j}^t = \sum_{j=0}^{m-1} f_{i,j}^t + \sum_{u=0}^{t-1} C_{t-1}^u(4m-1)^{t-u}(-1)^u \sum_{j=0}^{m-1} f_{i,j}^u - \sum_{j=0}^{m-1} f_{i,j}^t$

$$= \sum_{u=0}^{t-1} C_{t-1}^u(4m-1)^{t-u}(-1)^u \sum_{j=0}^{m-1} f_{i,j}^u,$$

which is independent of i. So H is an RMR$(2, 2m; t)$ when F is vertically symmetric. Thus H is a *RMR$(2, 2m; t)$. □

Du and Cao [11], Cao and Li [5] investigated the existence of SS-

SODLSs, they proved the following.

Lemma 3.3. ([5, 11]) *There exists an SSSODLS(m) if and only if* $m \equiv 0, 1, 3 \pmod 4$ *and* $m > 3$.

Proof of Theorem 1.2 Let k be an odd integer and $s \geq 0$. Let $t = s + 2$.

For $k = 1$ and $s \geq 0$, by Lemma 3.2 there is a *RMR$(2, 2 \cdot 2^s; s + 1)$, so there is a YSSSODLS$(2^{s+2}, 2(s + 1))$ by Lemma 2.2. Thus there is a YSSSODLS$(2^t, 2t - 2)$ for $t \geq 2$.

For odd $k > 1$, and $s \neq 1$, since there is an SSSODLS$(2^s \cdot k)$ by Lemma 3.3 and there is a *RMR$(2, 2 \cdot 2^s \cdot k; s + 2)$ by Lemma 3.2, there is a YSSSODLS$(2^{s+2} \cdot k, 2(s + 2))$ by Lemma 2.2. For $k > 3$, and $s = 1$, since there is a *RMR$(2, 4 \cdot 2k; 3)$ by Lemma 3.2, there is a YSSSODLS$(2^3 \cdot k, 6)$ by Lemma 2.3. Thus there is a YSSSODLS$(2^t \cdot k, 2t)$ for odd $k > 1$ and $t \geq 2$ except possibly for $(t, k) = (3, 3)$. The proof is completed. □

Lemma 3.4. *There is a symmetric elementary YMS*(24, 6).

Proof. By using computer program we get a symmetric elementary YMS(24, 6) as in Fig. 3. □

Proof of Theorem 1.3 For $t \geq 2$ and odd k, $(t, k) \neq (3, 3)$, since there is a YSSSODLS$(2^t \cdot k, 2t - 2l)$ for $l = 1$ if $k = 1$ and $l = 0$ if $k > 1$ by Theorem 1.2, there is a YMS$(2^t \cdot k, 2t - 2l)$ by the definition of YSSSODLS. The resulting magic squares are both symmetric and elementary. A symmetric elementary YMS(24, 6) is given in Lemma 3.4. The results follows immediately. □

Remark We want to get a YMS$(2^t, t')$ for $t' > 2t - 2$. The YSSSODLS(8, 5) is tried by using computer searching. We found that there are 147456 SSSODLS(8)s, among which, there are 2688 YSSSODLS(8, 4)s. We checked all the 2688 YSSSODLS(8, 4)s, however, there is no YSSSODLS(8, 5).

54	0	146	188	214	259	337	435	292	479	545	501	389	323	369	519	421	568	106	80	31	228	282	134
144	50	6	262	187	212	291	340	433	497	477	551	371	393	317	565	520	423	32	103	82	138	230	276
2	150	48	211	260	190	436	289	339	549	503	473	321	365	395	424	567	517	79	34	104	278	132	234
466	536	487	348	450	302	173	203	249	63	13	160	217	267	124	119	89	45	510	408	554	404	334	379
488	463	538	306	350	444	251	177	197	157	64	15	123	220	265	41	117	95	552	506	414	382	403	332
535	490	464	446	300	354	201	245	179	16	159	61	268	121	219	93	47	113	410	558	504	331	380	406
108	90	38	226	272	127	399	325	376	509	419	561	359	449	309	457	531	484	68	22	163	174	192	242
42	110	84	128	223	274	373	400	327	563	513	413	305	357	455	483	460	529	166	67	20	240	170	198
86	36	114	271	130	224	328	375	397	417	557	515	453	311	353	532	481	459	19	164	70	194	246	168
524	430	571	390	312	362	239	281	141	97	75	28	183	205	256	53	11	153	468	546	494	346	440	295
574	523	428	360	386	318	137	237	287	27	100	73	253	184	207	155	57	5	498	470	540	296	343	442
427	572	526	314	366	384	285	143	233	76	25	99	208	255	181	9	149	59	542	492	474	439	298	344
231	277	136	101	83	33	516	426	566	394	320	367	476	550	499	342	432	290	191	209	261	49	3	148
133	232	279	35	105	77	570	518	420	368	391	322	502	475	548	288	338	438	257	189	215	147	52	1
280	135	229	81	29	107	422	564	522	319	370	392	547	500	478	434	294	336	213	263	185	4	145	51
407	329	381	505	411	556	116	94	43	222	264	122	60	18	158	178	200	247	351	445	304	461	539	489
377	405	335	555	508	409	46	115	92	120	218	270	162	62	12	248	175	202	301	352	447	491	465	533
333	383	401	412	553	507	91	44	118	266	126	216	14	156	66	199	250	176	448	303	349	537	485	467
169	195	244	71	17	165	462	528	482	356	454	307	514	416	559	396	330	374	221	275	129	111	85	40
243	172	193	161	69	23	480	458	534	310	355	452	560	511	418	378	398	324	131	225	269	37	112	87
196	241	171	21	167	65	530	486	456	451	308	358	415	562	512	326	372	402	273	125	227	88	39	109
341	443	297	471	541	496	58	8	151	180	210	254	102	72	26	236	286	139	385	315	364	527	425	573
299	345	437	493	472	543	152	55	10	258	182	204	24	98	78	142	235	284	363	388	313	569	525	431
441	293	347	544	495	469	7	154	56	206	252	186	74	30	96	283	140	238	316	361	387	429	575	521

Fig. 3. YMS(24,6)

Open problem: Find a YMS(8, 5) or prove that there doesn't exist any YMS(8, 5).

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