ON THE PLANARITY OF THE REGULAR DIGRAPH OF IDEALS OF COMMUTATIVE RINGS

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ABSTRACT. Let R be a commutative ring. The regular digraph of ideals of R, denoted by $\mathcal{R}(R)$, is a digraph whose vertex-set is the set of all non-trivial ideals of R and, for every two distinct vertices I and J, there is an arc from I to J, whenever I contains a non-zero divisor on J. In this paper, we investigate the planarity of $\mathcal{R}(R)$. We also completely characterize the rings R such that $\mathcal{R}(R)$ is a ring graph, and the situations under which the genus of $\mathcal{R}(R)$ is finite. Moreover, we study the independence number and the girth of $\mathcal{R}(R)$, and also we find all cases that $\mathcal{R}(R)$ is bipartite.

1. INTRODUCTION

The investigation of graphs associated to algebraic structures is very important. Many fundamental papers devoted to graphs assigned to a ring and other algebraic structures have appeared recently, see for example [2], [3], [4], [5], [7], [9], [11], [12], [13], [14], [15], [17], [20], [21], [23], [24] and [27].

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The concept of the regular digraph of ideals of R, denoted by $\overrightarrow{\Gamma_{\text{reg}}}(R)$, was first introduced in [19]. $\overrightarrow{\Gamma_{\text{reg}}}(R)$ is a digraph whose vertex set is the set of all non-trivial ideals of R, and, for every two distinct vertices I and J, there is an arc from I to J, denoted by $I \longrightarrow J$, whenever I contains an element x such that $xy \neq 0$ for all $y \in J$. In other words, I contains a J-regular element. For simplicity of notations, we denote this graph by $\mathcal{R}(R)$. Recently in [1], the present authors studied some more properties of this graph such as connectedness and diameter of the graph.

In the second section of this paper, we characterize all rings R with planar, outerplanar and ring graph regular digraph $\mathcal{R}(R)$. Also we find the situations under which the genus of $\mathcal{R}(R)$ is finite. In Section 3, we study the independence number of $\mathcal{R}(R)$. We find some lower and upper bounds for the independence number of $\mathcal{R}(R)$, and we determine the case that the independence number is finite. In the final section, we completly investigate the girth of $\mathcal{R}(R)$. Moreover, we characterize all rings R such that $\mathcal{R}(R)$ is a bipartite graph.

Now we recall some definitions and notations on graphs. We use the standard terminology of graphs following [6]. Let $G = (V, E)$ be a simple graph, where V is the set of vertices and E is the set of edges. The graph $H = (V_0, E_0)$ is a subgraph of G if $V_0 \subseteq V$ and $E_0 \subseteq E$. Moreover, H is called an *induced subgraph* by V_0, denoted by $G[V_0]$, if $V_0 \subseteq V$ and $E_0 = \{\{u,v\} \in E \mid u, v \in V_0\}$. The *girth* of G, denoted by $\text{girth}(G)$, is the length of the shortest cycle in G if G has a cycle; otherwise, $\text{girth}(G) = \infty$. A graph G is called *triangle free* if it doesn't contain any cycles of length three. An *independent set* of G is a subset of the vertices of G such that no two vertices in the subset represent an edge of G. The *independence number* of G, denoted by $\alpha(G)$, is the cardinality of the largest independent set. For a positive integer r, an *r-partite graph* is one whose vertex set can be partitioned into r subsets, so that no edges has both ends in any one subset. A *complete r-partite graph* is one in which each vertex is joined to every vertex that is not in the same part. A graph G is said to be *planar* if it can be drawn in the plane, so that its edges intersect only at their ends. A *subdivision* of a graph is any graph that can be obtained from the original graph by replacing edges by paths. A remarkable characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$, where K_n is the complete graph with n vertices and $K_{m,n}$ is the complete bipartite graph, for positive integers m and n. Also, the *union* of graphs G_1 and G_2, which is denoted by $G_1 + G_2$, where G_1 and G_2 are two vertex-disjoint graphs, is a graph with $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2)$. We denote $G + G$ by $2G$.

Let $\mathcal{R}$ be a digraph. An arc from a vertex x to another vertex y of $\mathcal{R}$ is denoted by $x \to y$. Also, the *in-degree* (*out-degree*) of a vertex x in a digraph $\mathcal{R}$ is the number of arcs to (away from) x which is denoted by $d^+(x)$ ($d^-(x)$).

During the paper, unless we state, by investigating the properties of the regular digraph $\mathcal{R}(R)$, we mean the mentioned property of the undirected underlying simple graph obtained by replacing all directed edges of $\mathcal{R}(R)$ with undirected edges.

Throughout this paper, all rings are assumed to be commutative with identity. By $\mathrm{Max}(R)$, $\mathrm{Nil}(R)$ and $\mathrm{Ass}(R)$, we denote the set of all maximal ideals, the set of all nilpotent elements of R, and the set of all associated prime ideals of R, respectively. A ring R is called *reduced* if $\mathrm{Nil}(R) = \{0\}$. Also, the set of all zero-divisors of an R-module M, which is denoted by $Z(M)$, is the set

$$Z(M) = \{r \in R \mid rx = 0 \text{ for some non} - \text{zero element } x \text{ in } M\}.$$

An element $r \in R$ is called *M-regular* if $r \notin Z(M)$. An *R-sequence* is a d-tuple $r_1, \ldots, r_d$ in R such that, for every $i \leqslant d$, r_i is $R/(r_1, r_2, \ldots, r_{i-1})$-regular. We say that $\mathrm{depth}(R) = 0$, whenever every non-unit element of R is a zero-divisor. Also, two sets A and B are comparable if $A \subseteq B$ or $B \subseteq A$. For convenience, we denote the set $\{1, 2, \ldots, n\}$ by $[n]$, such that n is a natural number. Also for a real number x, we use $\lfloor x \rfloor$ to denote the greatest integer that is less than or equal to x, and we use $\lceil x \rceil$ to denote the least integer that is greater than or equal to x.

2. Planarity of $\mathcal{R}(R)$

In this section, first we provide a characterization of commutative Noetherian rings R such that $\mathcal{R}(R)$ is planar.

Clearly if R has a regular element, then $\mathcal{R}(R)$ contains an infinite clique, and so it is not planar. Therefore for the rest of this section, we assume that $\mathrm{depth}(R) = 0$.

Remark 2.1. *Suppose that I and J are ideals of R such that $I \cap J = 0$. Then for $x \in \mathrm{Z}(I + J)$, there exist elements $a \in I$ and $b \in J$ such that $x(a + b) = 0$ and $a + b \neq 0$. Hence $xa = -xb \in I \cap J = 0$. This implies that $x \in \mathrm{Z}(I) \cup \mathrm{Z}(J)$, and so $\mathrm{Z}(I + J) = \mathrm{Z}(I) \cup \mathrm{Z}(J)$.*

Lemma 2.2. *If $|\mathrm{Max}(R)| \geq 5$, then $\mathcal{R}(R)$ is not planar.*

Proof. Suppose that $\mathfrak{m}_1, \ldots, \mathfrak{m}_5$ are distinct maximal ideals of R. By [1, Corollary 2.3 (i)], there exist arcs $\mathfrak{m}_i \longrightarrow \mathrm{Ann}(\mathfrak{m}_j)$ for $i = 1, 2, 3$ and $j = 4, 5$. We will show that $\mathcal{R}(R)$ contains a subdivision of $K_{3,3}$ with parts $\{\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3\}$ and $\{\mathrm{Ann}(\mathfrak{m}_4) + \mathrm{Ann}(\mathfrak{m}_5), \mathrm{Ann}(\mathfrak{m}_4), \mathrm{Ann}(\mathfrak{m}_5)\}$. To achieve this we consider the following steps.

Step 1. Since $\mathfrak{m}_i \in \mathrm{Ass}(R)$ for $i = 1, \ldots, 5$, we have that $\mathrm{Ann}(\mathfrak{m}_i) \neq 0$. Also for $1 \leq i, j \leq 5$ with $i \neq j$, $\mathrm{Ann}(\mathfrak{m}_i) \cap \mathrm{Ann}(\mathfrak{m}_j) = \mathrm{Ann}(\mathfrak{m}_i + \mathfrak{m}_j) = 0$. Thus $\mathrm{Ann}(\mathfrak{m}_i) \neq \mathrm{Ann}(\mathfrak{m}_j)$ for all $1 \leq i, j \leq 5$ with $i \neq j$. Moreover for $1 \leq l \leq 5$ with $l \neq j$ and $l \neq i$, $\mathfrak{m}_i\mathrm{Ann}(\mathfrak{m}_i) = 0 \subseteq \mathfrak{m}_l$, and so $\mathrm{Ann}(\mathfrak{m}_i) \subseteq \mathfrak{m}_l$. Similarly $\mathrm{Ann}(\mathfrak{m}_j) \subseteq \mathfrak{m}_l$. These imply that $\mathrm{Ann}(\mathfrak{m}_i) + \mathrm{Ann}(\mathfrak{m}_j) \neq R$.

Step 2. If $\mathfrak{m}_i \subseteq \mathrm{Z}(\mathrm{Ann}(\mathfrak{m}_4) + \mathrm{Ann}(\mathfrak{m}_5))$ for some $i = 1, 2, 3$, then, by Remark 2.1, $\mathfrak{m}_i \subseteq \mathrm{Z}(\mathrm{Ann}(\mathfrak{m}_4)) \cup \mathrm{Z}(\mathrm{Ann}(\mathfrak{m}_5))$, and so, in view of [1, Lemma 2.2], $\mathfrak{m}_i \subseteq \mathfrak{m}_4 \cup \mathfrak{m}_5$ which is impossible. Hence there exists an arc $\mathfrak{m}_i \longrightarrow \mathrm{Ann}(\mathfrak{m}_4) + \mathrm{Ann}(\mathfrak{m}_5)$ in $\mathcal{R}(R)$ for $i = 1, 2, 3$.

Now $\mathcal{R}(R)$ contains a subdivision of $K_{3,3}$, and so $\mathcal{R}(R)$ is not planar. □

We say that R is a *decomposable ring* if we have that $R \cong R_1 \times R_2$, where R_1 and R_2 are non-zero rings. Otherwise R is called *indecomposable.* Clearly, all local rings are indecomposable.

Theorem 2.3. *Let R be an indecomposable ring such that $\mathcal{R}(R)$ is planar. Then R is an Artinian local ring.*

Proof. Since R is indecomposable, we need only to show that R is an Artinian ring. Assume in contrary that R is not Artinian. We claim that $\mathrm{J}(R)$ is not nilpotent. Since $\mathrm{depth}(R) = 0$, in view of [1, Remark 2.1], there are only finite number of maximal ideals in R, say $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_t$, for $t \geq 1$. If there exists an integer k such that $\mathrm{J}(R)^k = 0$, then $\mathrm{J}(R)^k = \mathfrak{m}_1^k \cap \mathfrak{m}_2^k \cap \cdots \cap \mathfrak{m}_t^k = 0$. On the other hand, we have that $\mathfrak{m}_i^k + \mathfrak{m}_j^k = R$, for $i, j = 1, \ldots, t$ and $i \neq j$. These imply that $R = R/\mathfrak{m}_1^k \times \cdots \times R/\mathfrak{m}_t^k$. Moreover $R/\mathfrak{m}_i^k$ is an Artinian local ring, for $i = 1, \ldots, t$, and so R is an Artinian ring which it is impossible, because we assumed that R is not Artinian. Thus $\mathrm{J}(R)$ is not nilpotent.

Now since $\mathrm{J}(R)$ is not nilpotent, by [1, Proposition 3.1], $\mathrm{J}(R)$ is not an isolated vertex in $\mathcal{R}(R)$. For every maximal ideal $\mathfrak{m}$ of R, we show that $\mathrm{Ann}(\mathfrak{m}) \subseteq \mathfrak{m}$. Suppose to the contrary that $\mathrm{Ann}(\mathfrak{m}) \not\subseteq \mathfrak{m}$. Hence $\mathrm{Ann}(\mathfrak{m}) + \mathfrak{m} = R$, and so $\mathfrak{m} \cap \mathrm{Ann}(\mathfrak{m}) = \mathfrak{m}\mathrm{Ann}(\mathfrak{m}) = 0$. These imply that $R \cong R/\mathfrak{m} \times R/\mathrm{Ann}(\mathfrak{m})$, which is impossible. Hence $\mathrm{Ann}(\mathfrak{m}) \subseteq \mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of R. Hence, by [1, Lemma 2.4(ii)], $\mathrm{Z}(\mathrm{Nil}(R)) = \mathrm{Z}(R)$, and so $\mathrm{Z}(\mathrm{J}(R)) = \mathrm{Z}(R)$. Now, since $\mathrm{J}(R)$ is not an isolated vertex in $\mathcal{R}(R)$, there exists an ideal I of R such that $\mathrm{J}(R) \longrightarrow I$ is an arc in $\mathcal{R}(R)$. Thus $\mathrm{J}(R)^i \neq 0$ for all $i \geq 0$. Therefore, by Nakayama's Lemma, $\mathrm{J}(R)^i \neq I\mathrm{J}(R)^j$ for all $i, j \geq 0$ with $i \neq j$. Hence clearly $\mathrm{J}(R)^i \longrightarrow I\mathrm{J}(R)^j$ for all $i, j \geq 0$. Hence $\mathcal{R}(R)$ contains a subdivision of $K_{3,3}$ with parts $\{I, I\mathrm{J}(R), I\mathrm{J}(R)^2\}$ and $\{\mathrm{J}(R), \mathrm{J}(R)^2, \mathrm{J}(R)^3\}$, and so $\mathcal{R}(R)$ is not planar which is the required contradiction. □

Corollary 2.4. *Let R be an indecomposable ring. Then $\mathcal{R}(R)$ is planar if and only if R is an Artinian local ring.*

Proof. It follows from [1, Theorem 2.5] and Theorem 2.3. □

Lemma 2.5. *Suppose that R and R' are non-trivial rings such that $\mathcal{R}(R)$ is not planar. Then $\mathcal{R}(R \times R')$ is also not planar.*

Proof. Put $C := \{I \times R' \mid I \text{ is a non-trivial ideal of } R\}$, and consider the induced subgraph $\mathcal{R}(R\times R')[C]$ of $\mathcal{R}(R\times R')$ with the vertex set C. Now, it is easy to see that the map $\varphi : \mathcal{R}(R\times R')[C] \longrightarrow \mathcal{R}(R)$ given by $\varphi(I\times R') = I$ provides an isomorphism between graphs $\mathcal{R}(R\times R')[C]$ and $\mathcal{R}(R)$. Hence $\mathcal{R}(R \times R')[C]$ is not planar, and so $\mathcal{R}(R \times R')$ is not planar. □

Corollary 2.6. *Suppose that $\mathcal{R}(R)$ is planar. Then R is an Artinian ring.*

Proof. We consider the following two cases.

Case 1. R is an indecomposable ring. Hence, by Theorem 2.3, R is an Artinian local ring.

Case 2. R is decomposable. Since $\operatorname{depth}(R) = 0$, we can see that $\operatorname{Max}(R) \subseteq \operatorname{Ass}(R)$. Hence R has only finite number of maximal ideals, and so $R \cong R_1 \times \cdots \times R_n$ for some indecomposable ring R_i, for $i = 1, \ldots, n$. Now, by Lemma 2.5, R_i is an Artinian local ring, and so R is Artinian. □

Theorem 2.7. *Suppose that R is a reduced ring. Then $\mathcal{R}(R)$ is planar if and only if R is a direct product of at most four fields.*

Proof. Suppose that $\mathcal{R}(R)$ is planar. Since R is reduced, by [1, Lemma 2.4 (ii)], $R \cong F_1 \times \cdots \times F_n$, where F_i is a field for all $i = 1, \ldots, n$. Also, by Lemma 2.2, we may assume that $n \leq 4$. So we have the following cases.

Case 1. If $n = 1$ or $n = 2$, then $\mathcal{R}(R)$ is the empty graph, and so it is planar.

Case 2. If $n = 3$, then $\mathcal{R}(R)$ is a cycle of length six which is again planar.

Case 3. Finally whenever $n = 4$, in view of Figure 1, the graph $\mathcal{R}(R)$ is planar.

The converse statement is clear. □

Theorem 2.8. *Suppose that R is not a reduced ring. If $\mathcal{R}(R)$ is planar, then R has at most three maximal ideals.*

Proof. By Corollary 2.6, R is an Artinian ring, and so $R \cong R_1 \times \cdots \times R_n$, where R_i is an Artinian local ring for $i = 1, \ldots, n$. Now, assume to the contrary that R has more than three maximal ideals, and so $n \geq 4$. Since R is not reduced, there exists $1 \leq i \leq n$ such that R_i is not a field. Hence we may assume that $R \cong R_1 \times R_2 \times R_3 \times R'$, where R' is a ring which is not a field. Figure 2 shows a subgraph of $\mathcal{R}(R)$ which is a subdivision of K_5. Hence $\mathcal{R}(R)$ is not planar which is the required contradiction. □

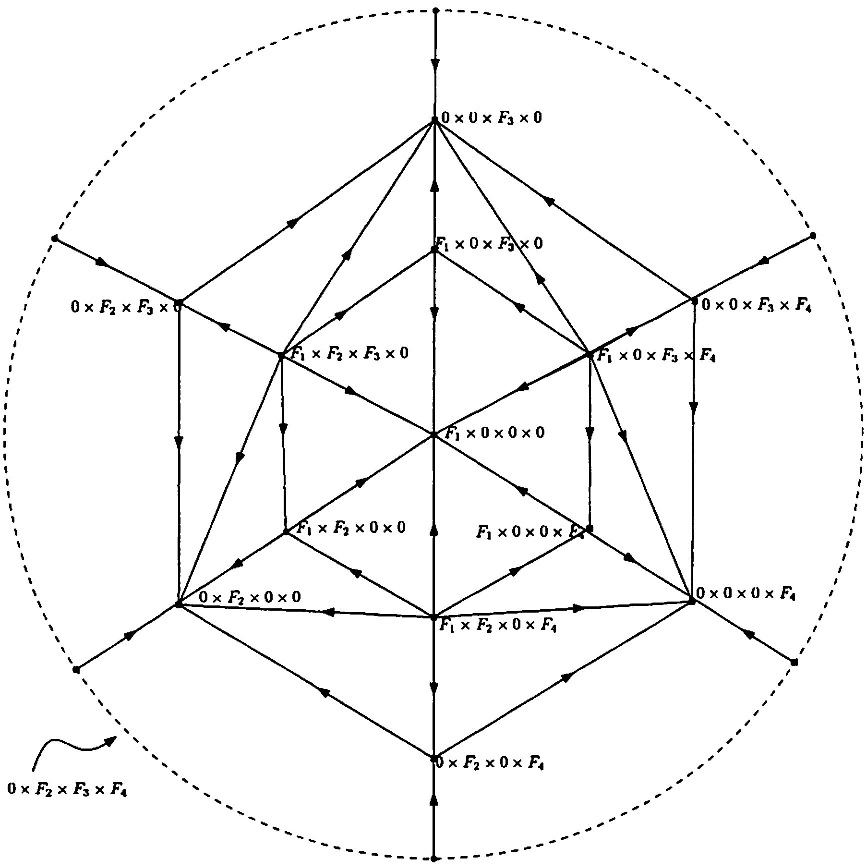

FIGURE 1. $\mathcal{R}(F_1 \times F_2 \times F_3 \times F_4)$

In the light of Theorem 2.8, we need only to study the case that R is a non-reduced ring with at most three maximal ideals. So suppose that $R \cong R_1 \times \cdots \times R_n$, where R_i is a non-trivial ring for $i = 1, \ldots, n$ and $n \leq 3$. Also suppose that α_i denotes the number of non-trivial ideals of R_i for $i = 1, 2, 3$. Whenever $n = 1$, $\mathcal{R}(R)$ is the empty graph, which is planar. So we assume that $2 \leq n \leq 3$.

Theorem 2.9. *Suppose that R is a non-reduced ring with exactly two maximal ideals. Then $\mathcal{R}(R)$ is planar if and only if $R \cong R_1 \times R_2$, where R_1 and R_2 are Artinian local rings that satisfy one of the following conditions.*

(i) *If $\alpha_1 \geq 3$, then $\alpha_2 \leq 1$.*
(ii) *If $\alpha_1 = 2$, then $\alpha_2 \leq 2$.*
(iii) *If $\alpha_1 \leq 1$, then $\alpha_2 \neq 0$.*

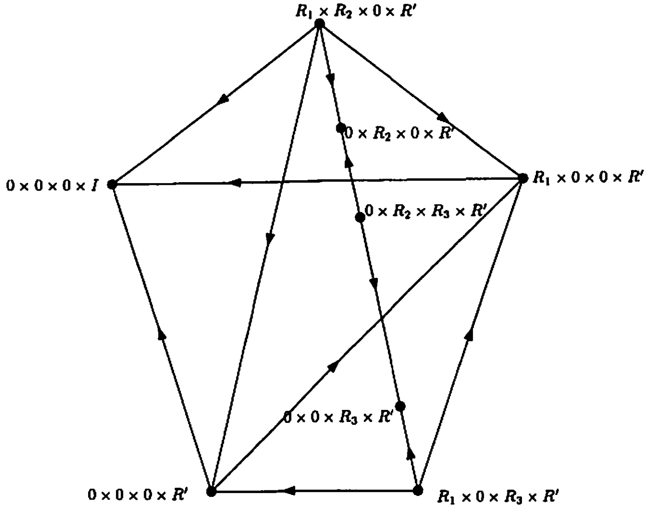

FIGURE 2. A subgraph of $\mathcal{R}(R_1 \times R_2 \times R_3 \times R')$

Proof. Assume that $\mathcal{R}(R)$ is planar and that $\alpha_1 \geq 3$. So there exist non-trivial ideals I_1, I_2 and I_3 of R_1. If $\alpha_2 \geq 2$, then there exist ideals J_1 and J_2 of R_2, and so there are arcs $R_1 \times 0 \longrightarrow I_i \times 0$, $R_1 \times J_j \longrightarrow I_i \times 0$ in $\mathcal{R}(R)$ for $i = 1, 2, 3$ and $j = 1, 2$. These imply that $\mathcal{R}(R)$ has a subgraph isomorphic to $K_{3,3}$, and so it is not planar. Hence $\alpha_2 \leq 1$.

Now, if $\alpha_1 = 2$, then, by using a method similar to that we used in the first paragraph of this proof, one can conclude that $\alpha_2 \leq 2$.

Finally, whenever $\alpha_1 = 1$ or $\alpha_1 = 0$, it is not hard to see that $\mathcal{R}(R)$ is isomorphic to the planar graph $2K_{1,1,\alpha_2} + \overline{K_{\alpha_2}}$ or $2K_{1,\alpha_2}$, respectively. Hence in this situation α_2 is an arbitrary non-negative integer.

The converse implication is clear. □

Theorem 2.10. *Suppose that R is a non-reduced ring with three maximal ideals. Then $\mathcal{R}(R)$ is planar if and only if $\alpha_1 = \alpha_2 = 0$ and $1 \leq \alpha_3 \leq 2$.*

Proof. First suppose that $\mathcal{R}(R)$ is planar. By Corollary 2.6, R is an Artinian ring. Since R has three maximal ideals, $R \cong R_1 \times R_2 \times R_3$ for some local rings R_1, R_2 and R_3. Now we consider the following cases.

Case 1. $\alpha_2 \geq 1$ and $\alpha_3 \geq 1$. Assume that I and I' are non-trivial ideals of R_2 and R_3, respectively. Then Figure 3, shows that the graph $\mathcal{R}(R)$ has a subgraph isomorphic to a subdivision of $K_{3,3}$ which is impossible.

Case 2. $\alpha_1 \geq 1$ and $\alpha_3 \geq 1$. By using a method similar to that we used in Case 1, one can easily show that this case is again impossible.

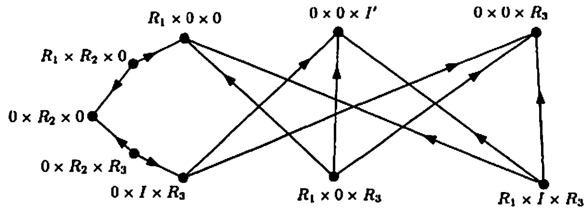

FIGURE 3. A subgraph of $\mathcal{R}(R_1 \times R_2 \times R_3)$, where I and I' are non-trivial ideals of R_2 and R_3, respectively.

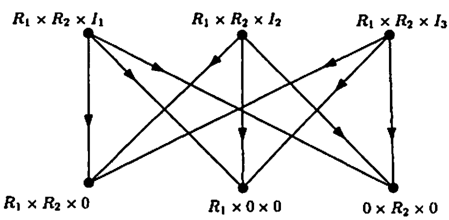

FIGURE 4. A subgraph of $\mathcal{R}(R_1 \times R_2 \times R_3)$, where I_1, I_2 and I_3 are non-trivial ideals of R_3.

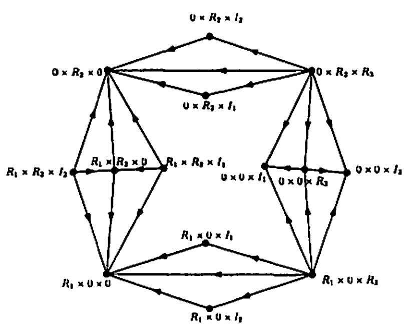

FIGURE 5. $\mathcal{R}(R_1 \times R_2 \times R_3)$, where $\alpha_1 = \alpha_2 = 0, \alpha_3 = 2$

Thus, by using the above cases, without loss of generality we may assume that $\alpha_2 = \alpha_1 = 0$ and $\alpha_3 \geq 0$. Since R is not reduced, $\alpha_3 \geq 1$. Also, if $\alpha_3 \geq 3$, then Figure 4 shows that $\mathcal{R}(R)$ contains a subgraph isomorphic to a subdivision of $K_{3,3}$. Therefore $1 \leq \alpha_3 \leq 2$.

The converse implication is clear by Figures 5 and 6. □

Let G be a graph with n vertices and q edges. We recall that a *chord* is any edge of G joining two non-adjacent vertices in a cycle of G. Let C be a cycle of G. We say that C is a *primitive cycle* if it has no chords. Also, a graph G has the *primitive cycle property* (PCP) if any two primitive cycles

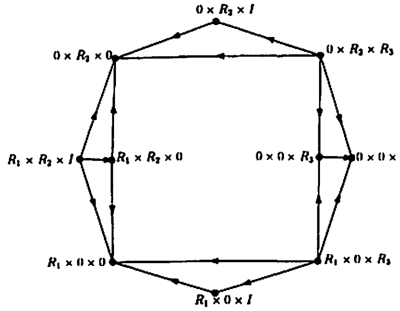

FIGURE 6. $\mathcal{R}(R_1 \times R_2 \times R_3)$, where $\alpha_1 = \alpha_2 = 0, \alpha_3 = 1$

intersect in at most one edge. The number frank(G) is called *free rank* of G and is the number of primitive cycles of G. Also the number rank$(G) = q - n + r$ is called the *cycle rank* of G, where r is the number of connected components of G. By [10, Proposition 2.2], we have rank$(G) \leq$ frank(G). A graph G is called a *ring graph* if it satisfies in one of the following equivalent conditions (see [10]).

(i) frank(G) = rank(G).
(ii) G satisfies the PCP and G does not contain a subdivision of K_4 as a subgraph.

Also, an undirected graph is an *outerplanar graph* if it can be drawn in the plane without crossing in such a way that all of the vertices belong to the unbounded face of the drawing. There is a characterization for outerplanar graphs that says a graph is outerplanar if and only if it does not contain a subdivision of K_4 or $K_{2,3}$. Clearly, every outerplanar graph is a ring graph and every ring graph is a planar graph.

Now we characterize those planar graphs which are outerplanar or ring graph. Since the graph of an Artinian local ring and the graph of direct product of two fields are empty, we ignore these two situations. Also, as we mentioned in the above paragraph, we may assume that the graph $\mathcal{R}(R)$ is planar, so by Corollary 2.6, R is an Artinian ring. Thus there exist Artinian local rings $R_1, \ldots, R_n$ such that $R \cong R_1 \times \cdots \times R_n$.

Theorem 2.11. *The graph $\mathcal{R}(R)$ is a ring graph if and only if R is isomorphic to one of the following rings.*

(i) $F_1 \times F_2 \times F_3$, *where* F_1, F_2 *and* F_3 *are fields.*
(ii) $R_1 \times R_2$, *where* $\alpha_1 \leq 1$ *and* $\alpha_2 \neq 0$.

Proof. Suppose that $\mathcal{R}(R)$ is a ring graph. Since every ring graph is planar, it is enough to consider the rings in Theorems 2.7, 2.9 and 2.10.

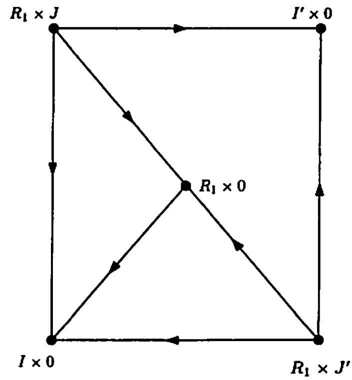

FIGURE 7. A subgraph of $\mathcal{R}(R_1 \times R_2)$, where $\alpha_1 = \alpha_2 = 2$. Assume that I, I' and J, J' are non-trivial ideals of R_1 and R_2, respectively.

First assume that R is reduced. Then, in view of Theorem 2.7, we have that $R \cong F_1 \times F_2 \times \cdots \times F_n$, where $F_1, \ldots, F_n$ are fields and $n \leq 4$. Moreover, Figure 1 shows that $\mathcal{R}(R)$ contains a subdivision of K_4 when $n = 4$. Thus in this situation $\mathcal{R}(R)$ is not a ring graph. On the other hand if $n = 3$, then $\mathcal{R}(R)$ is a cycle of length 6, and so it is a ring graph. Note that $n = 2$ implies that $\mathcal{R}(R)$ is an empty graph.

Now, suppose that R is not reduced. First, assume that R contains two maximal ideals. Then, in view of Theorem 2.9, $\mathcal{R}(R)$ is isomorphic to the planar graphs $2K_{2,2,1} + \overline{K_4}$, $2K_{1,1,\alpha_2} + \overline{K\alpha_2}$ or $2K_{1,\alpha_2}$ with $\alpha_2 \neq 0$. Figure 7 shows that the graph $2K_{2,2,1} + \overline{K_4}$, which corresponds to $\alpha_1 = \alpha_2 = 2$ in Theorem 2.9 (ii), contains a subgraph isomorphic to a subdivision of K_4. Thus in this situation $\mathcal{R}(R)$ is not a ring graph. On the other hand, both graphs $2K_{1,1,\alpha_2} + \overline{K_{\alpha_2}}$ and $2K_{1,\alpha_2}$ are ring graphs for $\alpha_2 \neq 0$. These imply that if $\alpha_1 \leq 1$, then α_2 is an arbitrary positive integer.

Now assume that R contains three maximal ideals. Since $\mathcal{R}(R)$ is planar, Theorem 2.10 implies that $(\alpha_1, \alpha_2, \alpha_3) \leq (0, 0, 2)$ with $\alpha_3 \neq 0$. Now, Figure 9, shows that $\mathcal{R}(R)$ admits a subdivision of K_4, so in this situation $\mathcal{R}(R)$ is not a ring graph. □

Corollary 2.12. *Let R be a ring. The graph $\mathcal{R}(R)$ is outerplanar if and only if $R \cong F_1 \times F_2 \times F_3$, where F_1, F_2 and F_3 are fields, or $R \cong R_1 \times R_2$ such that one of the following statements hold.*

(i) *If $\alpha_1 = 0$, then α_2 is arbitrary.*
(ii) *If $\alpha_1 = 1$, then $\alpha_2 \leq 2$.*

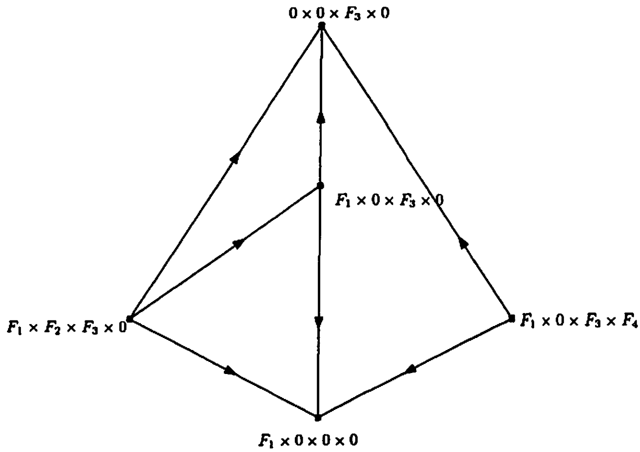

FIGURE 8. A subgraph of $\mathcal{R}(F_1 \times F_2 \times F_3 \times F_4)$

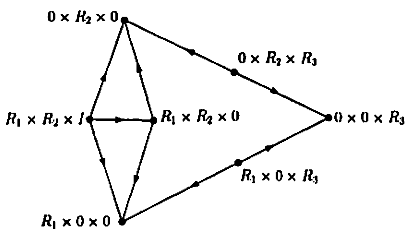

FIGURE 9. A subgraph of $\mathcal{R}(R_1 \times R_2 \times R_3)$ with $\alpha_3 \geq 1$.

Proof. Since in the case that $R \cong R_1 \times R_2$, where $\alpha_1 = 1$ and $\alpha_2 \geq 3$, $\mathcal{R}(R)$ is isomorphic to the graph $2K_{1,1,\alpha_2} + \overline{K\alpha_2}$, which is not outerplanar, the result follows from Theorem 2.11. □

Recall that the *genus* of a graph G which is denoted by $\gamma(G)$, is the minimal integer t such that the graph can be drawn without crossing itself on a sphere with t handles (i.e. an oriented surface of genus t). Thus a planar graph has genus zero, because it can be drawn on a sphere without self-crossing. A genus one graph is called a *toroidal graph.* In other words, a graph G is toroidal if it can be embedded on the torus; This means that the graph's vertices can be placed on a torus such that no edges cross. Usually, it is assumed that G is also non-planar.

In the rest of this section, for a given ring R, we find conditions under which $\gamma(\mathcal{R}(R))$ is finite. Also, if R contains a regular element, then clearly $\mathcal{R}(R)$ has an infinite clique. So we only consider the case that $\text{depth}(R) = 0$.

In the following theorem, we bring some well-known formulas for the genus of a graph (see [26] and [25]).

Theorem 2.13. (i) *For every integer* $n \geq 3$, $\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$.

(ii) *For every two integers* $r, s \geq 2$, $\gamma(K_{r,s}) = \left\lceil \frac{(r-2)(s-4)}{4} \right\rceil$.

Theorem 2.14. *Let R be a Noetherian ring. If $\gamma(\mathcal{R}(R))$ is finite, then R is an Artinian ring.*

Proof. Let $\gamma(\mathcal{R}(R))$ be finite. We have the following cases.

Case 1. $\text{Ann}(\mathfrak{m}) \subseteq \mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of R. Assume in contrary that R is not Artinian. Then, by a similar method that we used in the proof of Theorem 2.3, there exists an ideal I of R such that $\text{J}(R)^i \longrightarrow I\text{J}(R)^j$ are arcs in $\mathcal{R}(R)$ for all integers i and j. This means that $\mathcal{R}(R)$ has a $K_{r,s}$ for all integers $r, s \geq 2$. Thus $\gamma(\mathcal{R}(R))$ is infinite which is a contradiction, and so R is Artinian.

Case 2. There exists a maximal ideal $\mathfrak{m}$ of R such that $\text{Ann}(\mathfrak{m}) \nsubseteq \mathfrak{m}$. Since $\text{Ann}(\mathfrak{m}) + \mathfrak{m} = R$, we have $\text{Ann}(\mathfrak{m}) \cap \mathfrak{m} = 0$. Therefore $R \cong R/\mathfrak{m} \times R/\text{Ann}(\mathfrak{m})$, and so $|\text{Max}(R/\text{Ann}(\mathfrak{m}))| = |\text{Max}(R)| - 1$. Now, by using induction on the number of maximal ideals of R in the light of Case 1, one can shows that R is an Artinian ring as desired. □

Theorem 2.15. *Let R be an Artinian ring. Then $\gamma(\mathcal{R}(R))$ is finite if and only if one of the following statements hold.*

(i) *R is a local ring.*
(ii) *R has finite number ideals.*
(iii) *$R \cong R_1 \times R_2$, where R_1 is a local ring with infinite many ideals and R_2 is a local ring with at most one non-trivial ideal.*

Proof. First, suppose that $\gamma(\mathcal{R}(R))$ is finite. Then, in view of Theorem 2.14, $R \cong R_1 \times \cdots \times R_n$, where R_i is an Artinian local ring for $i = 1, \ldots, n$. If $n = 1$ or R has finitely many ideals, there is nothing to prove. Hence we may assume that R is not a local ring and that it has infinitely many ideals. Without loss of generality, suppose that R_1 has infinitely many ideals, say $\{I_i\}_{i \in \mathbb{N}}$. If the ring $R_2 \times \cdots \times R_n$ has non-trivial ideals I and J, then for every integer $r \geq 2$, there exists a subgraph of $\mathcal{R}(R)$ isomorphic to $K_{r,3}$ which is impossible (see Figure 10). This implies that we may have $n = 2$ with $\alpha_2 \leq 1$.

Conversely, if each of the statements (i) or (iii) holds, then $\mathcal{R}(R)$ is a planar graph, and so $\gamma(\mathcal{R}(R)) = 0$. If (ii) holds, then $\mathcal{R}(R)$ is a finite graph, consequently $\gamma(\mathcal{R}(R))$ is finite. □

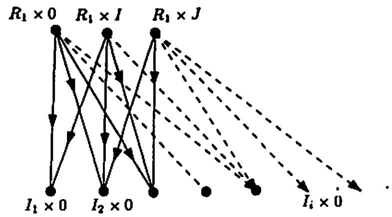

FIGURE 10. A subgraph of $\mathcal{R}(R_1 \times R_2 \times \cdots \times R_n)$, where R_1 has infinitely many ideals and I, J are non-trivial ideals of $R_2 \times \cdots \times R_n$.

3. Independence number of $\mathcal{R}(R)$

Remark 3.1. *Suppose that R and R' are non-trivial rings. Then in view of the proof of Lemma 2.5, $\mathcal{R}(R)$ is isomorphic to an induced subgraph of $\mathcal{R}(R \times R')$. Hence, if $\alpha(\mathcal{R}(R \times R'))$ is finite, then $\alpha(\mathcal{R}(R))$ is finite.*

Lemma 3.2. *Let $(R, \mathfrak{m})$ be a local ring with* $\operatorname{depth}(R) = 0$ *and maximal ideal $\mathfrak{m}$. Then $\alpha(\mathcal{R}(R)) < \infty$ if and only if R is an Artinian ring with finite number of ideals.*

Proof. Suppose that $\alpha(\mathcal{R}(R))$ is finite. In view of [1, Lemma 2.4], we may assume that $\operatorname{Nil}(R) \neq 0$, and hence $\mathrm{Z}(R) = \mathrm{Z}(\operatorname{Nil}(R))$. Thus $\mathrm{Z}(R) = \mathrm{Z}(I)$, for any ideal I containing $\operatorname{Nil}(R)$. These ideals form an independent set of vertices in $\mathcal{R}(R)$. Therefore $R/\operatorname{Nil}(R)$ admits only finite number of ideals, so $R/\operatorname{Nil}(R)$ is a reduced Artinian local ring. This implies that $\mathfrak{m} = \operatorname{Nil}(R)$. Hence R is an Artinian ring. Thus, by [1, Theorem 2.5], $\mathcal{R}(R)$ is an empty graph. Now since $\alpha(\mathcal{R}(R))$ is finite, R has only finite number of ideals.

The converse assertion is clear. □

Theorem 3.3. *Let R be a non-trivial ring with* $\operatorname{depth}(R) = 0$. *Then $\alpha(\mathcal{R}(R)) < \infty$ if and only if R is an Artinian ring with finite number of ideals.*

Proof. Suppose that $\alpha(\mathcal{R}(R))$ is finite. Since $\operatorname{depth}(R) = 0$, by [1, Remark 2.1], $|\operatorname{Max}(R)|$ is finite. Put $n := |\operatorname{Max}(R)|$. We proceed by induction on n. The case $n = 1$ follows from Lemma 3.2. Now, assume inductively that $n \geq 2$ and the result holds for smaller values of n. We consider the following cases.

Case 1. $\operatorname{Ann}(\mathfrak{m}) \subseteq \mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of R. Then, by a method similar to that we used in the proof of Lemma 3.2, $\operatorname{Nil}(R) = \mathrm{J}(R)$, and so $J(R)$ is nilpotent, which implies that R is an Artinian ring (see the proof of Theorem 2.3). Hence for some Artinian local rings $R_1, \ldots, R_n$, we have that $R \cong R_1 \times \cdots \times R_{n-1} \times R_n$. Now the result follows from induction hypothesis and Remark 3.1.

Case 2. $\mathrm{Ann}(\mathfrak{m}) \nsubseteq \mathfrak{m}$ for some maximal ideals $\mathfrak{m}$ of R. Then we have $R \cong R/\mathfrak{m} \times R/\mathrm{Ann}(\mathfrak{m})$. Furthermore $|\mathrm{Max}(R/\mathrm{Ann}(\mathfrak{m}))| = |\mathrm{Max}(R)| - 1$. Again, by induction hypothesis, $R/\mathrm{Ann}(\mathfrak{m})$ has a finite number of ideals, so R has finite number of ideals.

The converse implication is trivial. □

Theorem 3.4. *Let $(R, \mathfrak{m})$ be a local ring with* $\mathrm{depth}(R) \neq 0$ *and a unique maximal ideal* $\mathfrak{m}$. *If $\alpha(\mathcal{R}(R))$ is finite, then R is a reduced ring.*

Proof. Suppose that $\alpha(\mathcal{R}(R))$ is finite and assume in contrary that $\mathrm{Nil}(R) \neq 0$. Therefore, in view of [1, Lemma 2.6], the set of all ideals of R contained in $\mathrm{Nil}(R)$ forms an independent set in $\mathcal{R}(R)$. Thus the finiteness of $\alpha(\mathcal{R}(R))$ implies that $\mathrm{Nil}(R)$ contains only finite number of ideals of R. Hence there exists an ideal, say I, which is a minimal ideal among those ideals contained in $\mathrm{Nil}(R)$. Clearly, I is a minimal ideal of R, so $I = Rx$ for some element x in R. On the other hand, we have $R/\mathrm{Ann}(Rx) \cong Rx$. This implies that $\mathrm{Ann}(Rx)$ is a maximal ideal of R, so $\mathrm{Ann}(Rx) = \mathfrak{m}$. Consequently, $\mathrm{depth}(R) = 0$, which is a contradiction, so R is a reduced ring as desired. □

In the following theorem, we find a lower bound for the independence number of $\mathcal{R}(R)$, where R is a reduced ring. Note that Noetherian rings admit only finite number of minimal prime ideals.

Theorem 3.5. *Suppose that R is a reduced ring with* $\mathrm{Min}(R) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. *Then*

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} \leq \alpha(\mathcal{R}(R)).$$

Proof. First, note that $\mathrm{Ass}(R) = \mathrm{Min}(R)$ because R is a reduced ring. Now, we give our proof in two steps.

Step 1. For an arbitrary integer k with $1 \leq k < n$, we claim that

$$\mathrm{Ass}(\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k) = \{\mathfrak{p}_{k+1}, \ldots, \mathfrak{p}_n\}.$$

To achieve this, assume that $\mathfrak{p}_i$ is an arbitrary element in the set $\{\mathfrak{p}_{k+1}, \ldots, \mathfrak{p}_n\}$. Then there exists an element $x \in R$ such that $\mathfrak{p}_i = \mathrm{Ann}(x)$. Now $\mathfrak{p}_i x = 0$ implies that $x \in \mathfrak{p}_j$ for all $j = 1, \ldots, k$, so $\mathfrak{p}_i \in \mathrm{Ass}(\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k)$. Therefore

$$\{\mathfrak{p}_{k+1}, \ldots, \mathfrak{p}_n\} \subseteq \mathrm{Ass}(\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k). \tag{1}$$

Now assume that $\mathfrak{q} \in \mathrm{Ass}(\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k)$. Since $\mathfrak{q} \in \mathrm{Ass}(R) = \mathrm{Min}(R)$, we only need to show that $\mathfrak{q} \notin \{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$. Assume in contrary that $\mathfrak{q} = \mathfrak{p}_i$ for some $i = 1, \ldots, k$. Since $\mathfrak{q} \in \mathrm{Ass}(\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k)$ and $\mathrm{Ass}(\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k) \subseteq \mathrm{Ass}(\mathfrak{q})$, there exists a non-zero element $y \in \mathfrak{q}$ such that $\mathfrak{q} = \mathrm{Ann}(y)$. Then $\mathfrak{q}Ry = 0$ implies that $y \in \cap_{i=1}^{n} \mathfrak{p}_i$, so $y = 0$ since R is reduced. This is the required contradiction. Thus $\mathfrak{q} \in \{\mathfrak{p}_{k+1}, \ldots, \mathfrak{p}_n\}$, consequently

$$\mathrm{Ass}(\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k) \subseteq \{\mathfrak{p}_{k+1}, \ldots, \mathfrak{p}_n\}. \tag{2}$$

Now the claim follows from (1) and (2).

Step 2. For every $C \subset [n]$ we define the ideal $I_C := \bigcap_{i \in C} \mathfrak{p}_i$. By using a method similar to that we used in Step 1, one can shows that I_C satisfies in the following statements.

(i) $\mathrm{Ass}(I_C) = \{\mathfrak{p}_i \mid i \notin C\}$.
(ii) $\mathrm{Z}(I_C) = \cup_{i \notin C} \mathfrak{p}_i$, and so $I_C \not\subseteq \mathrm{Z}(I_C)$.
(iii) If $C' \subset [n]$, then $C \subset C'$ if and only if $I_{C'} \subset I_C$.
(iv) If $C' \subset [n]$, then $C \subset C'$ if and only if $I_C \longrightarrow I_{C'}$ is an arc in $\mathcal{R}(R)$.

By part (iv), for any two incomparable subsets C and C' of $[n]$, the vertices $I_{C'}$ and I_C are non-adjacent in $\mathcal{R}(R)$. This means that, for every integer k with $1 \leq k < n$, the set $\{I_C \mid C \subseteq [n] \text{ with } |C| = k\}$ is an independent set of vertices in $\mathcal{R}(R)$. Therefore, by means of any one of k-subset of $[n]$, we can form an independent set in $\mathcal{R}(R)$. This implies that $\binom{n}{\lfloor \frac{n}{2} \rfloor} \leq \alpha(\mathcal{R}(R))$. □

Theorem 3.6. *Suppose that $F_1, \ldots, F_n$ are fields. Then*

$$\alpha(\mathcal{R}(F_1 \times F_2 \times \cdots \times F_n)) = \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Proof. In view of Theorem 3.5, we only need to show that

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} \geq \alpha(\mathcal{R}(F_1 \times F_2 \times \cdots \times F_n)).$$

By using the notations that we used in the proof of [1, Lemma 3.5], we claim that $\mathrm{Supp}(J) \subset \mathrm{Supp}(I)$ if and only if $I \longrightarrow J$ is an arc in $\mathcal{R}(F_1 \times F_2 \times \cdots \times F_n)$, where I and J are distinct ideals of $F_1 \times F_2 \times \cdots \times F_n$.

It is not hard to see that $\mathrm{Supp}(J) \subset \mathrm{Supp}(I)$ implies that $I \longrightarrow J$. Hence suppose that $\mathrm{Supp}(J) \not\subseteq \mathrm{Supp}(I)$, and so there exists an element $t \in \mathrm{Supp}(J) \setminus \mathrm{Supp}(I)$. Now consider the element $\mathbf{y} := (y_1, \ldots, y_n) \in J$, where

$$y_i = \begin{cases} 1 & \text{if } i = t \\ 0 & \text{otherwise} . \end{cases}$$

For any arbitrary non-zero element $\mathbf{x} := (x_1, \ldots, x_n)$ of I we have $x_t = 0$. This implies that $\mathbf{xy} = 0$, and so $I \subseteq \mathrm{Z}(J)$. Hence $I \not\rightarrow J$, as desired.

Now, suppose that Θ is a maximal independent set in $\mathcal{R}(F_1 \times F_2 \times \cdots \times F_n)$. Put

$$\Theta' := \{\mathrm{Supp}(I) \mid I \in \Theta\}.$$

By our claim, the elements of Θ' are incomparable subsets of $[n]$. On the other hand, the Sperner's Lemma (cf. [16]), says that the size of a maximal set of incomparable subsets of $[n]$ is equal to $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. Hence we have that $\binom{n}{\lfloor \frac{n}{2} \rfloor} \geq |\Theta'|$. Now the result follows from the fact that $|\Theta'| = |\Theta|$. □

Corollary 3.7. *Suppose that* $R_1, \dots, R_n$ *are rings. Then*

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} \leq \alpha(\mathcal{R}(R_1 \times \cdots \times R_n)).$$

The equality holds if R_i *is an integral domain for* $i = 1, \dots, n$.

Proof. Suppose that $F_1, \dots, F_n$ are fields. Since $\mathcal{R}(F_1 \times F_2 \times \cdots \times F_n)$ is isomorphic to the subgraph of $\mathcal{R}(R_1 \times R_2 \times \cdots \times R_n)$ induced by the vertices in the set

$$\{I_1 \times \cdots \times I_n \mid I_i = 0 \text{ or } I_i = R_i\},$$

the results follows from Theorem 3.5.

To prove the second assertion, assume that R_i is an integral domain for $i = 1, \dots, n$. We need only to show that

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} \geq \alpha(\mathcal{R}(R_1 \times \cdots \times R_n)).$$

For an ideal I of $R_1 \times \cdots \times R_n$, we set:

$$\mathrm{Supp}(I) := \{i \in [n] \mid \pi_i(I) \neq 0\}.$$

Clearly, for distinct non-trivial ideals I and J of $R_1 \times \cdots \times R_n$, if $\mathrm{Supp}(J) \subseteq \mathrm{Supp}(I)$, then I and J are adjacent in $\mathcal{R}(R_1 \times R_2 \times \cdots \times R_n)$. This means that the supports of any two non-adjacent vertices in $\mathcal{R}(R_1 \times R_2 \times \cdots \times R_n)$ are incomparable.

Now, suppose that Θ is a maximal independent set in $\mathcal{R}(R_1 \times \cdots \times R_n)$. Put

$$\Theta' := \{\mathrm{Supp}(I) \mid I \in \Theta\}.$$

Thus $\binom{n}{\lfloor \frac{n}{2} \rfloor} \geq |\Theta'|$, and since $|\Theta'| = |\Theta|$, we have that $\binom{n}{\lfloor \frac{n}{2} \rfloor} \geq \alpha(\mathcal{R}(R_1 \times \cdots \times R_n))$, as desired. □

4. Girth of $\mathcal{R}(R)$

Remark 4.1. *If* $\mathrm{depth}(R) \neq 0$, *then one can easily see that* $\mathcal{R}(R)$ *admits an infinite clique. Thus in this situation* $\mathrm{girth}(\mathcal{R}(R)) = 3$. *For the rest of the paper we assume that* $\mathrm{depth}(R) = 0$.

Theorem 4.2. *Let* R *be a ring. Then the following statements hold.*

(i) *If* R *is a decomposable ring, then* $\mathrm{girth}(\mathcal{R}(R)) \in \{3, 6, \infty\}$.

(ii) *If* R *is an indecomposable ring, then* $\mathrm{girth}(\mathcal{R}(R)) \in \{3, 4, \infty\}$.

Proof. (i) Suppose that $R \cong R_1 \times R_2$ for some non-trivial rings R_1 and R_2. We have the following cases.

Case 1. R_1 and R_2 are not fields. Suppose that I_1 and I_2 are non-trivial ideals of R_1 and R_2, respectively. Then we have the following triangle.

$$R_1 \times 0 \longrightarrow I_1 \times 0 \longleftarrow R_1 \times I_2 \longrightarrow R_1 \times 0$$

Thus $\mathrm{girth}(\mathcal{R}(R)) = 3$.

Case 2. R_1 is a field and $\mathcal{R}(R_2)$ is a non-empty graph. Then there exists an arc $J' \longrightarrow J$ in $\mathcal{R}(R_2)$, and so there is a triangle as

$$R_1 \times 0 \longleftarrow R_1 \times J \longleftarrow R_1 \times J' \longrightarrow R_1 \times 0$$

in $\mathcal{R}(R_1 \times R_2)$ which implies that $\mathrm{girth}(\mathcal{R}(R)) = 3$.

Case 3. R_1 is a field and $\mathcal{R}(R_2)$ is an empty graph. In this situation, by [1, Theorem 2.5], R_2 is an Artinian local ring or a direct product of two fields. Hence $\mathcal{R}(R)$ is a disjoint union of two star graphs which implies that $\mathrm{girth}(\mathcal{R}(R)) = \infty$, or $\mathcal{R}(R)$ is isomorphic to C_6.

(ii) Suppose that R is an indecomposable ring. We consider two cases.

Case 1′. R is a local ring with maximal ideal $\mathfrak{m}$. If R is an Artinian ring, then $\mathcal{R}(R)$ is empty, so $\mathrm{girth}(\mathcal{R}(R)) = \infty$. If R is not Artinian, then, since $\mathcal{R}(R)$ is not empty, for a given arc $\mathfrak{m} \longrightarrow I$ in $\mathcal{R}(R)$, there exists a cycle of length at most 4 as follows.

$$\mathfrak{m} \longrightarrow I \longleftarrow \mathfrak{m}^2 \longrightarrow I\mathfrak{m}^2 \longleftarrow \mathfrak{m}$$

Thus $\mathrm{girth}(\mathcal{R}(R)) \leq 4$.

Case 2′. R is not a local ring. Since R is indecomposable, we have that R is not Artinian and, for all maximal ideals $\mathfrak{m}$ of R, $\mathrm{Ann}(\mathfrak{m}) \subseteq \mathfrak{m}$. Thus, in view of [1, Lemma 2.4], $\mathrm{Z}(\mathrm{J}(R)) = \mathrm{Z}(R)$. Since $\mathrm{J}(R)$ is not nilpotent, by [1, Lemma 3.1], $\mathrm{J}(R)$ is not an isolated vertex in $\mathcal{R}(R)$. This means that there is an arc $\mathrm{J}(R) \longrightarrow I$. Now, suppose that $\mathfrak{m}$ is a maximal ideal of R. Then we have a cycle of length at most 4 as follows.

$$\mathfrak{m} \longrightarrow I \longleftarrow \mathrm{J}(R) \longrightarrow I\mathrm{J}(R) \longleftarrow \mathfrak{m}$$

Thus $\mathrm{girth}(\mathcal{R}(R)) \leq 4$. □

Theorem 4.3. *Suppose that* $\mathrm{girth}(\mathcal{R}(R)) = 4$ *and* I *is a vertex of* $\mathcal{R}(R)$. *Then the following statements hold.*

(i) $\mathrm{d}^-(I) = 0$ *if and only if* I *is nilpotent.*

(ii) $\mathrm{d}^+(I) = 0$ *if and only if* $\mathrm{Z}(I) = \mathrm{Z}(R)$.

Proof. Note that, in view of Theorem 4.2, we may assume that R is indecomposable.

(i) Suppose that $\mathrm{d}^-(I) = 0$. Assume to the contrary that I is not a nilpotent ideal. Thus there exists a minimal prime ideal $\mathfrak{p}$ of R such that $I \not\subseteq \mathfrak{p}$. On the other hand, there is an ideal J_0 of R such that $\mathfrak{p} = \mathrm{Ann}(J_0)$, because $\mathfrak{p} \in \mathrm{Ass}(R)$. Now, since $R/\mathfrak{p} \cong J_0$, we have that $\mathfrak{p} = \mathrm{Z}(J_0)$. Also, since there is not any arc from I, $I \subseteq \mathrm{Z}(J_0) = \mathfrak{p}$, which is impossible. This means that $I = J_0$.

Now, we claim that I is a minimal ideal of R. To achieve this, suppose that $I' \subset I$ for some ideal I' of R. Thus we have $\mathrm{Z}(I') \subseteq \mathrm{Z}(I) = \mathrm{Z}(J_0) = \mathfrak{p}$. Moreover $I \not\subseteq \mathfrak{p}$ implies that $I \not\subseteq \mathrm{Z}(I')$. This shows that $I \longrightarrow I'$, which is impossible. Consequently I is a minimal ideal of R, so $\mathfrak{p}$ is a maximal ideal of R, because $R/\mathfrak{p} \cong I$. The maximality of $\mathfrak{p}$ together with the fact that $I \not\subseteq \mathfrak{p}$ and $I\mathfrak{p} = 0$ imply that $R \cong R/\mathfrak{p} \times R/I$, which is a contradiction, since R is indecomposable.

The converse implication follows from [1, Lemma 2.6].

(ii) Suppose that $\mathrm{d}^+(I) = 0$. So $J \subseteq \mathrm{Z}(I)$ for every ideal J of R with $J \neq I$. We consider the following two cases.

Case 1. Suppose that I is a maximal ideal of R. Since R is indecomposable, for all maximal ideals $\mathfrak{m}$ of R, we have $\mathrm{Ann}(\mathfrak{m}) \subseteq \mathfrak{m}$. Now, in view of [1, Lemma 2.4], $\mathrm{Z}(\mathrm{Nil}(R)) = \mathrm{Z}(R)$, so $\mathrm{Z}(I) = \mathrm{Z}(R)$.

Case 2. Suppose that I is not a maximal ideal. Since $\mathrm{d}^+(I) = 0$, one can see that $\mathfrak{m} \subseteq \mathrm{Z}(I)$ for all maximal ideals $\mathfrak{m}$ of R. This implies that $\mathrm{Z}(I) = \mathrm{Z}(R)$.

The converse is clear. □

Remark 4.4. *Suppose that $I \longrightarrow J$ and $J \longrightarrow K$ are arcs in $\mathcal{R}(R)$. Then it is not hard to see that $I \longrightarrow K$ is an arc in $\mathcal{R}(R)$.*

Lemma 4.5. *Assume that R is not an Artinian ring. Then* $\mathrm{girth}(\mathcal{R}(R)) \neq \infty$.

Proof. If R is a decomposable ring, then in view of Cases 1 and 2 in the proof of Theorem 4.2, one can see that $\mathrm{girth}(\mathcal{R}(R)) = 3$. Now suppose that R is an indecomposable ring. We consider the following two cases.

Case 1. R is a local ring with the maximal ideal $\mathfrak{m}$. Since $\mathcal{R}(R)$ is not empty, we have an arc $\mathfrak{m} \longrightarrow I$ in $\mathcal{R}(R)$. Now, one can easily see that the cycle

$$\mathfrak{m} \longrightarrow I \longleftarrow \mathfrak{m}^2 \longrightarrow I\mathfrak{m}^2 \longleftarrow \mathfrak{m}$$

is a cycle of length at most 4, and so $\mathrm{girth}(\mathcal{R}(R)) \neq \infty$.

Case 2. R is not a local ring. Since R is indecomposable, we have $\mathrm{Ann}(\mathfrak{m}) \subseteq \mathfrak{m}$, for all maximal ideals $\mathfrak{m}$ of R. Thus, in view of [1, Lemma 2.4], $\mathrm{Z}(\mathrm{J}(R)) = \mathrm{Z}(R)$. On the other hand, Since $\mathrm{J}(R)$ is not nilpotent, by [1, Lemma 3.1], $\mathrm{J}(R)$ is not an isolated vertex in $\mathcal{R}(R)$. This means that there is an arc $\mathrm{J}(R) \longrightarrow I$. Now, we can easily check that the cycle

$$\mathfrak{m} \longrightarrow I \longleftarrow \mathrm{J}(R) \longrightarrow I\mathrm{J}(R) \longleftarrow \mathfrak{m},$$

where $\mathfrak{m}$ is a maximal ideal of R, is a cycle of length at most 4, thus $\mathrm{girth}(\mathcal{R}(R)) \neq \infty$.

□

Theorem 4.6. *Suppose that R is not reduced and that $\mathcal{R}(R)$ is not empty. Then the following statements are equivalent.*

(i) $\mathrm{girth}(\mathcal{R}(R)) = 4$.
(ii) $\mathrm{Z}(I) = \mathrm{Z}(R)$ *for all non-nilpotent ideals I of R.*

Proof. (*ii*) $\Rightarrow$ (*i*) Assume in contrary that girth($\mathcal{R}(R)$) $\neq 4$. Then, in view of Theorem 4.2, girth($\mathcal{R}(R)$) $= 3, 6, \infty$.

If girth($\mathcal{R}(R)$) $= \infty$, by Lemma 4.5, R is an Artinian ring. Since $\mathcal{R}(R)$ is a forest, by [19, Corollary 3.1(i)], we have $R \cong F \times R'$, where F is a field and R' is an Artinian local ring which is not a field. Now one can easily see that $Z(R) = (0 \times R') \cup (F \times \mathfrak{m})$, where $\mathfrak{m}$ is the maximal ideal of R'. On the other hand, the ideal $0 \times R'$ is not nilpotent. Also we have $(0,1) \notin Z(0 \times R')$. This implies that $Z(0 \times R') \neq Z(R) = (0 \times R') \cup (F \times \mathfrak{m})$, which violates our assumption in statement (ii). Consequently girth($\mathcal{R}(R)$) $\neq \infty$.

If girth($\mathcal{R}(R)$) $= 6$, then, in view of Case 3 in the proof of Theorem 4.2, R is a direct product of three fields. Consequently R is a reduced ring which is a contradiction.

If girth($\mathcal{R}(R)$) $= 3$, then there exists a triangle as

$$I_1 \longrightarrow I_2 \longrightarrow I_3 \longleftarrow I_1.$$

Then, in view of [1, Lemma 2.6], I_2 is not a nilpotent ideal of R. Thus, by our assumption in the statement (ii), $Z(I_2) = Z(R)$. On the other hand, $I_1 \longrightarrow I_2$ implies that $I_1 \not\subseteq Z(I_2)$, and so $I_1 \not\subseteq Z(R)$. This is a contradiction with depth(R) $= 0$. Hence, by the above discussions, we have that girth($\mathcal{R}(R)$) $= 4$.

(*i*) $\Rightarrow$ (*ii*) Assume to the contrary that there exists an ideal $\mathfrak{m}$ of R such that it is not nilpotent and $Z(\mathfrak{m}) \subset Z(R)$. Then, by Theorem 4.3, there exist ideals I and J of R such that $I \longrightarrow \mathfrak{m} \longrightarrow J$. We have the following cases.

Case 1. $I \neq J$. Then, by Remark 4.4, we have girth($\mathcal{R}(R)$) $= 3$, which is impossible.

Case 2. $I = J$. Suppose that $\mathfrak{m} \subset \mathfrak{m}'$ for some ideal $\mathfrak{m}'$ of R. Then $\mathfrak{m}' \longrightarrow I$ is an arc in $\mathcal{R}(R)$. Thus, by Remark 4.4, girth($\mathcal{R}(R)$) $= 3$, which is impossible. This means that $\mathfrak{m}$ is a maximal ideal of R. In view of Theorem 4.2, R is an indecomposable ring, so by [1, Lemma 2.4 (i)], $Z(J(R)) = Z(R)$. Therefore we have $Z(\mathfrak{m}) = Z(R)$, which is a contradiction. □

Now we can provide a characterization of bipartite graphs.

Lemma 4.7. *$\mathcal{R}(R)$ is triangle free if and only if $\mathcal{R}(R)$ is bipartite.*

Proof. Suppose that $\mathcal{R}(R)$ is triangle free. If $\mathcal{R}(R)$ contains an odd cycle, say C, then one can find a path $I \longrightarrow J \longrightarrow K$ in C. Now, in view of Remark 4.4, $\mathcal{R}(R)$ admits a triangle as $I \longrightarrow J \longrightarrow K \longleftarrow I$, which is impossible.

The converse implication follows from the fact that a graph is bipartite if and only if it does not contain any odd cycle. □

Theorem 4.8. *$\mathcal{R}(R)$ is bipartite if and only if one of the following statements holds.*

(i) $R \cong F_1 \times F_2 \times F_3$, *where* F_1, F_2 *and* F_3 *are fields.*
(ii) R *is not a reduced ring with* $Z(I) = Z(R)$ *for all non-nilpotent ideals* I.
(iii) $R \cong F \times R'$ *for some field* F *and an Artinian local ring* R'.

Proof. Suppose that R is isomorphic to $F_1 \times F_2 \times F_3$, where F_1, F_2 and F_3 are fields. Then $\mathcal{R}(R)$ is a cycle of length 6. Thus $\mathcal{R}(R)$ is bipartite.

Assume that (ii) holds. Then, by Theorem 4.6, we have $\text{girth}(\mathcal{R}(R)) = 4$. This implies that $\mathcal{R}(R)$ is a triangle free graph. Now, by Lemma 4.7, $\mathcal{R}(R)$ is bipartite.

If $R \cong F \times R'$ for some field F and an Artinian local ring R', then $\mathcal{R}(R)$ is a forest.

Conversely, assume that $\mathcal{R}(R)$ is a bipartite graph, which this implies $\text{girth}(\mathcal{R}(R)) \neq 3$. Assume that $\text{girth}(\mathcal{R}(R)) = 4$. Thus R is not reduced, so in view of Theorem 4.6, the statement (ii) holds. If $\text{girth}(\mathcal{R}(R)) = 6$ then Theorem 4.2 implies that the statements (i) holds, and if $\text{girth}(\mathcal{R}(R)) = \infty$, Lemma 4.5 implies that R is Artinian ring. Now [19, Corollary 3.1 (i)], we have $R \cong F \times R'$, where F is a field and R' is an Artinian local ring. □

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