A class of degree-magic join graphs

Tao WANG[1], Deming LI[2]

1. *Depart. of Foundation, North China Institute of Science and Technology* 065201, *P. R. China;*

2. *Depart. of Math., Capital Normal University,* 100048, *P. R. China*

Abstract A graph is called degree-magic if it admits a labelling of the edges by integers $\{1, 2, ..., |E(G)|\}$ such that the sum of the labels of the edges incident with any vertex v is equal to $(1+|E(G)|)/2 \deg(v)$. In this paper, we show that a class of join graphs are degree-magic.

Keywords degree-magic graphs, join graphs, supermagic graphs.

1. Introduction

Graphs considered in this paper are finite, undirected and loopless. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, respectively. If $E(G) = \emptyset$, then G is an empty graph.

Let G be a graph and f be a mapping from $E(G)$ into positive integers. The index mapping of f^* is the mapping from $V(G)$ into positive integers defined by $f^*(v) = \sum_{e \in E_v} f(e)$ for every $v \in V(G)$, where E_v is the set of edges incident with v in G. An injective mapping f from $E(G)$ into positive integers is called a magic labelling of G for an index c if its index mapping f^* satisfies $f^*(v) = c$ for all $v \in V(G)$.

A magic labelling f of G is called a supermagic labelling if the set $\{f(e) : e \in E(G)\}$ consists of consecutive positive integers. We say that a graph G is supermagic (magic) whenever there exists a supermagic (magic) labelling of G. A bijective mapping f from $E(G)$ into $\{1, 2, ..., |E(G)|\}$ is called a degree-magic labelling (or only d-magic labelling) of a graph G if its index mapping f^* satisfies $f^*(v)=(1+|E(G)|)/2 \deg(v)$ for every $v \in V(G)$. A d-magic labelling f of G is called balanced if for every vertex $v \in V(G)$,

v is incident with the same number of edges with labels at most $\lfloor |E(G)|/2 \rfloor$ as that of edges with labels exceeding $\lfloor |E(G)|/2 \rfloor$. We say that a graph G is degree-magic (balanced degree-magic, or only d-magic) when there exists a d-magic (balanced-magic) labelling of G. A magic (m,n)-rectangle $R = (r_{i,j})$ is an $m \times n$ matrix in which the first mn positive integers are placed such that the sum over each row of R is a constant and the sum over each column of R is another constant. Magic rectangles are a natural generalization of the magic squares. Magic rectangles were first studied by Harmuth, who presented in [5,6] the necessary and sufficient conditions for the existence. Evidently, a mapping f from $E(K_{m,n})$ into positive integers given by $f(u_iv_j) = r_{i,j}$ is a d-magic labelling of $K_{m,n}$ if and only if R is a magic (m,n)-rectangle.

The concept of magic graphs was introduced by Sedláček [8]. Supermagic graphs were introduced by M. B. Stewart [9]. There is by now a considerable number of papers published on magic and supermagic graphs. The concept of degree-magic graphs was introduced in [1] as some extension of supermagic regular graphs. Some properties of degree-magic graphs and characterizations of some classes of degree-magic and balanced degree-magic graphs were described in [1-4].

Throughout this paper, we use the following symbols. The set $\{x \in Z | a \leq x \leq b\}$ is denoted by $[a,b]$ if $a, b \in Z$. For a number set A, $\sum_{x \in A} x$ is denoted by $\sum A$. Let $E(G) = E_1 \cup E_2$, where E_1 and E_2 are two disjoint subsets. Let f_i be mappings from E_i to the set of integers for $i = 1, 2$. Then $(f_1 + f_2)$ is a mapping from E and $(f_1 + f_2)^*(v) = f_1^*(v) + f_2^*(v)$ for $v \in V(G)$, where $f^*(v)$ is the sum of the labels of edges incident with vertex v. The m-vertex empty graph is denoted by $\overline{K_m}$. Given two graphs G and H, the join of G and H, denoted by $G \vee H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{gh : g \in V(G), h \in V(H)\}$. In this paper, we show that a class of join graphs are d-magic.

2. Some lemmas

Lemma 2.1[1] *For $m, n \geq 2$, the complete bipartite graph $K_{m,n}$ is d-magic if and only if $m \equiv n \pmod 2$ and $(m,n) \neq (2,2)$.*

Lemma 2.2[2] *The complete bipartite graph $K_{m,n}$ is balanced d-magic if and only if the following statements hold:*

(i) $m \equiv n \equiv 0 \pmod 2$,

(ii) if $m \equiv n \equiv 2 \pmod 4$, *then* $min(m,n) \geq 6$.

Lemma 2.3[1] *Let G be a regular graph. Then G is supermagic if and only if it is degree-magic.*

Lemma 2.4[7] *Let $S = \{a+1,\ a+2,\ \dots,\ a+rn\}$ and H be a $2r$-regular graph with $V(H) = \{x_1,\ x_2,\ \dots,\ x_n\}$. Let $f : V(H) \to \{b+1,\ b+2,\dots,\ b+n\}$ be a bijective mapping. Then there is a bijective mapping $g : E(H) \to S$ such that $\{(f+g^*)(x_i) : 1 \leq i \leq n\} = \{r(2a+rn+1)+b+i : 1 \leq i \leq n\}$.*

Lemma 2.5 *Let $R = (r_{i,j})_{m\times n}$ be a magic (m,n)-rectangle and $A_l = [a+(l-1)p+1, a+lp]$ for $1 \leq l \leq mn$. Let $B = (b_{i,j})_{m\times n}$ be a matrix defined by $b_{i,j} = \sum A_{r_{i,j}} = \sum_{t=1}^{p}(a+p(r_{i,j}-1)+t)$. Then the sum over each row of B is a constant $npa+np(pnm+1)/2$ and the sum over each column of B is another constant $mpa+mp(pnm+1)/2$.*

Proof Let $B_{i,\cdot} = \bigcup_{j=1}^{n} b_{i,j}$ and $B_{\cdot,j} = \bigcup_{i=1}^{m} b_{i,j}$. For a magic (m,n)-rectangle $R = (r_{i,j})_{m\times n}$, we have $\sum_{j=1}^{n} r_{i,j} = n(mn+1)/2$, for $i = 1,\ 2,\ \dots,\ m$, and $\sum_{i=1}^{m} r_{i,j} = m(mn+1)/2$, for $j = 1,\ 2,\ \dots,\ n$.

Since $b_{i,j} = \sum_{t=1}^{p}(a+p(r_{i,j}-1)+t)$, $b_{i,j} = \sum_{t=1}^{p}(a+p(r_{i,j}-1)+t) = p^2 r_{i,j} + p(2a-p+1)/2$. So, we have that the sum over each row of matrix B equal to

$$\begin{aligned}\sum B_{i,\cdot} &= \textstyle\sum_{j=1}^{n} b_{i,j} = \sum_{j=1}^{n}(p^2 r_{i,j} + p(2a-p+1)/2)\\ &= np(2a-p+1)/2 + p^2\textstyle\sum_{j=1}^{n} r_{i,j} = npa + np(pnm+1)/2,\end{aligned}$$

and similarly, the sum over each column of matrix B equal to

$$\begin{aligned}\sum B_{\cdot,j} &= \textstyle\sum_{i=1}^{m} b_{i,j} = \sum_{i=1}^{m}(p^2 r_{i,j} + p(2a-p+1)/2)\\ &= np(2a-p+1)/2 + p^2\textstyle\sum_{i=1}^{m} r_{i,j} = mpa + mp(pnm+1)/2.\square\end{aligned}$$

Lemma 2.6 Suppose that $A = [a+1, a+2t+1]$, $B = [b+1, b+2t+1]$, then there exists a $2\times(2t+1)$-matrix M, such that the entries of the first row are the elements of A, the entries of the second row are the elements of B, and $C(j) = a+b+t+1+j$ for $j = 1, 2, ..., 2t+1$, where $C(j)$ denote the sum of the elements in column j.

Proof Let M be the following matrix

$$\begin{pmatrix} a+t+1 & a+1 & a+t+2 & a+2 & \cdots & a+t & a+2t+1 \\ b+1 & b+t+2 & b+2 & b+t+3 & \cdots & b+2t+1 & b+t+1 \end{pmatrix}.$$

That is $m_{1,j} = a+t+1+(j-1)/2$, $m_{2,i} = b+1+(j-1)/2$ for $j = 1, 3, ..., 2t+1$, $m_{1,j} = a+j/2$, $m_{2,j} = b+t+1+j/2$ for $j = 2, 4, ..., 2t$.

It is easy to know that $C(j) = m_{1,j} + m_{2,j} = a+b+t+1+j$ for $j = 1, 2, ..., 2t+1$. □

3. Main Theorems

Theorem 3.1 Let G_i be a p-vertex $4r_i$-regular graph for every $i \in \{1, 2, ..., n\}$.

(1) Let m, n and p be positive integers such that $m \equiv n \equiv 0 \pmod 2$ and $(m, n) \neq (2, 2)$. Then the graph $(\bigcup_{i=1}^{n} G_i) \vee \overline{K_m}$ is d-magic;

(2) Let m, n and p be odd integers greater than 2. Then the graph $(\bigcup_{i=1}^{n} G_i) \vee \overline{K_m}$ is d-magic.

Proof Let $V(G_i) = \{y_{i,1}, y_{i,2}, ..., y_{i,p}\}$, where $i = 1, 2, ..., n$, $V(\overline{K_m}) = \{x_1, x_2, ..., x_m\}$ and $G = (\cup_{i=1}^{n} G_i) \vee \overline{K_m}$. Then $e = |E(G)| = mnp + 2p(r_1 + r_2 + \cdots + r_n)$. Let $I_i = [a_i, b_i] \bigcup [c_i, d_i]$ and $A_j = [p(r_1+r_2+\cdots+r_n)+(j-1)p+1, p(r_1+r_2+\cdots+r_n)+jp]$, where $a_1 = 1$, $b_i - a_i = pr_i - 1$, $a_{i+1} = b_i + 1$, $d_1 = e$, $d_i - c_i = pr_i - 1$, $d_{i+1} = c_i - 1$, for $1 \le i \le n, 1 \le j \le mn$. We have $(\bigcup_{i=1}^{n} I_i) \bigcup (\bigcup_{j=1}^{mn} A_j) = [1, e]$.

For odd integers m, n greater than 2 and $m \equiv n \equiv 0 \pmod 2$, $(m, n) \neq (2, 2)$, by Lemma 2.1, there is a magic rectangle $R = (r_{i,j})_{m\times n}$. Let $B_{i,\cdot} = \bigcup_{j=1}^{n} A_{r_{i,j}}$ and $B_{\cdot,j} = \bigcup_{i=1}^{m} A_{r_{i,j}}$. We label the edges incident with $V(\overline{K_m})$ and $V(G_j)$ by the elements of $B_{\cdot,j}$, for $1 \le j \le n$.

We construct a d-magic mapping of G by three steps.

Step 1. For $1 \le i \le m-1$, let $A_{r_{i,j}} = [p(r_1+r_2+\cdots+r_n)+(r_{i,j}-1)p+1, p(r_1+r_2+...+r_n)+r_{i,j}p]$. We label the edges incident with x_i and $V(G_j)$

by the elements of $A_{r_{i,j}}$, and denote the labeling by f_1. We split this step by two cases.

Case 1. For $m \equiv n \equiv 0 \pmod 2$ and $(m,n) \neq (2,2)$, we let $f_1(x_i y_{j,k}) = p(r_1 + r_2 + ... + r_n) + (r_{i,j} - 1)p + k$, for $k = 1, 2, ..., p$ and $i = 1, 3, ..., m-1$, $f_1(x_i y_{j,k}) = p(r_1 + r_2 + ... + r_n) + r_{i,j}p + 1 - k$, for $k = 1, 2, ..., p$ and $i = 2, 4, ..., m-2$.

Case 2. For odd integers $m, n, p (\geq 3)$, we let $A = A_{r_{1,j}}$, $B = A_{r_{2,j}}$, and $p = 2t+1$. By Lemma 2.6, there is a $2 \times (2t+1)$-matrix M, which is

$$\begin{pmatrix} a+t+1 & a+1 & a+t+2 & a+2 & \cdots & a+t & a+2t+1 \\ b+1 & b+t+2 & b+2 & b+t+3 & \cdots & b+2t+1 & b+t+1 \end{pmatrix},$$

where $a = p(r_1+r_2+...+r_n)+(r_{1,j}-1)p$, $b = p(r_1+r_2+...+r_n)+(r_{2,j}-1)p$.

We define the labeling as what follows.

$f_1(x_1 y_{j,k}) = m_{1,k}$, $f_j(x_2 y_{j,k}) = m_{2,k}$, for, $k = 1, 2, ..., p$. $f_1(x_i y_{j,k}) = p(r_1 + r_2 + \cdots + r_n) + r_{i,j}p + 1 - k$, for $k = 1, 2, ..., p$ and $i = 3, 5, ..., m-2$. $f_1(x_i y_{j,k}) = p(r_1 + r_2 + \cdots + r_n) + (r_{i,j} - 1)p + k$, for $k = 1, 2, ..., p$ and $i = 4, 6, ..., m-1$.

For the above two cases, we have $f_1^*(y_{j,k}) = f_1^*(y_{j,1}) - 1 + k$, for $k = 1, 2, ..., p$ and $j = 1, 2, ..., n$.

Step 2. For $I_j = [a_j, b_j] \bigcup [c_j, d_j]$, we label edges of G_j by the elements of I_j. We decompose the $4r_j$-regular graph G_j into two $2r_j$-factors $H_{j,1}$ and $H_{j,2}$.

We label the edges of $H_{j,1}$ by the elements of $[a_j, b_j]$ as doing in Lemma 2.4, and we denote the labeling by f_2^1. There is an m_j, such that $\{(f_1 + f_2^1)^*(y_{j,k}), 1 \leq k \leq p\} = \{m_j + k, 1 \leq k \leq p\}$. We label the edges of $H_{j,2}$ by the elements of $[c_j, d_j]$, as doing in Lemma 2.4, and we denote the labeling by f_2^2. There is a w_j, such that $\{(f_1 + f_2)^*(y_{j,k}) = (f_1 + f_2^1 + f_2^2)^*(y_{j,k})$, $1 \leq k \leq p\} = \{w_j + k, 1 \leq k \leq p\}$. Without loss of generality, we can rename the vertices of G_j such that $\{(f_1 + f_2)^*(v_{jk}) = w_j + k, 1 \leq k \leq p\}$.

Step 3. For $A_{r_{m,j}} = [p(r_1 + r_2 + \cdots + r_n) + (r_{m,j} - 1)p + 1, p(r_1 + r_2 + ... + r_n) + r_{m,j}p]$, we label the edges incident with x_m and $V(G_j)$ by the elements of $A_{r_{m,j}}$, and denote the labeling by f_3. Let $f_3(x_m v_{jk}) = p(r_1 + r_2 + \cdots + r_n) + r_{m,j}p - k, 1 \leq k \leq p$. So, we have $\{f^*(v_{jk}) = (f_1 + f_2 + f_3)^*(v_{jk}) = w_j + p(r_1 + r_2 + \cdots + r_n) + r_{m,j}p, 1 \leq k \leq p\}$.

For fixed j, $f_j^*(v_{jk})$ is a constant for every $k = 1, 2, ..., p$, we denote the constant by c_j. By Lemma 2.5, we have $c_j = \frac{2(\sum I_j)+\sum B_{\cdot,j}}{p} = \frac{(4r_j+m)(e+1)}{2}$.

We label the edges incident with x_i by the elements of $B_{i,\cdot}$ as in the above proof, for $1 \le i \le m$. By Lemma 2.5, we have

$$f^*(x_i) = \sum B_{i,\cdot} = np^2(r_1 + r_2 + \cdots + r_n) + \frac{np(pnm+1)}{2} = \frac{np(e+1)}{2}.$$

Thus, $f^*(v) = \frac{1+|E(G)|}{2} \cdot \deg(v)$ for every $v \in V(G)$, and $(\bigcup_{i=1}^n G_i) \vee \overline{K_m}$ is d-magic. □

Corollary 3.2 Let G_i be a $p(\ge 3)$-vertex $4r_i$-regular graph for every $i \in \{1, 2, ..., n\}$. If the complete bipartite graph $K_{m,n}$ is balanced d-magic, then the graph $(\bigcup_{i=1}^n G_i) \vee \overline{K_m}$ is balanced d-magic.

Proof If the complete bipartite graph $K_{m,n}$ is balanced d-magic, by Lemma 2.2, there is magic rectangle matrix $R = (r_{i,j})_{m\times n}$ such that there are $n/2$ elements more than $mn/2$ in each column of R and $m/2$ elements not more than $mn/2$ in each column of R. It is easy to know that $p(r_1 + r_2 + \cdots + r_n) + (r_{i,j} - 1)p + 1 > e/2$ if and only if $r_{i,j} > mn/2$. Since $A_{r_{i,j}} = [p(r_1 + r_2 + \cdots + r_n) + (r_{i,j} - 1)p + 1, p(r_1 + r_2 + \cdots + r_n) + r_{i,j}p]$, thus, there are $n/2$ sets with elements more than $e/2$ in $B_{i,\cdot}$ and $m/2$ sets with elements not more than $e/2$ in $B_{\cdot,j}$. We label edges of G_j with I_j twice. The first time, every element is not more than $e/2$, the second time, every element is more than $e/2$. Therefore, every $v \in V(G_j)$ is incident with the $2r_j + m/2$ edges with labels more than $e/2$, each vertex x_i is incident with the $np/2$ edges with labels not more than $e/2$. And so, $(\bigcup_{i=1}^n G_i) \vee \overline{K_m}$ is balanced d-magic. □

By Lemmas 2.1 and 2.3, we have the following corollary.

Corollary 3.3 Let G_i be a p-vertex $4r$-regular graph for every $i \in \{1, 2, ..., n\}$.

(1) If $m \equiv n \equiv 0 \pmod 2$, $(m, n) \ne (2, 2)$ and $pn = 4r + m$, then the regular graph $(\bigcup_{i=1}^n G_i) \vee \overline{K_m}$ is supermagic;

(2) If all of $m, n, p(\ge 3)$ are odd integers and $pn = 4r + m$, then the regular graph $(\bigcup_{i=1}^n G_i) \vee \overline{K_m}$ is supermagic. □

Theorem 3.4 Let G_i be a p-vertex $4r_i$-regular graphs for every $i \in \{1, 2, ..., n-1\}$, and G_n be a p-vertex $2r_n$-regular graph. If the complete bipartite graph $K_{m,n}$ is balanced d-magic, then the graph $(\bigcup_{i=1}^{n} G_i) \vee \overline{K_m}$ is d-magic.

Proof By Theorem 3.1, it is sufficient to consider that r_n is odd.

Let $G = (\bigcup_{i=1}^{n} G_i) \vee \overline{K_m}$. Then $e = |E(G)| = mnp + 2p(r_1 + r_2 + \ldots + r_{n-1}) + pr_n$. Let $I_i = [a_i, b_i] \bigcup [c_i, d_i]$, for $i = 1, 2, \ldots, n-1$ and $I_n = [mnp/2 + p(r_1 + r_2 + \cdots + r_{n-1}) + 1, mnp/2 + p(r_1 + r_2 + \cdots + r_{n-1}) + pr_n]$, where $a_1 = mnp/2 + 1$, $b_i - a_i = pr_i - 1$, $a_{i+1} = b_i + 1$, $d_1 = mnp/2 + p(r_1 + r_2 + \cdots + r_{n-1}) + pr_n + 1$, $d_i - c_i = pr_i - 1$, $d_{i+1} = c_i - 1$, for $1 \leq i \leq n-1$. Let $A_j = [(j-1)p + 1, jp]$ for $1 \leq j \leq mn/2$, and $A_j = [2p(r_1 + r_2 + \cdots + r_{n-1}) + (j-1)p + 1, 2p(r_1 + r_2 + \cdots + r_{n-1}) + jp$ for $mn/2 + 1 \leq j \leq mn$.

It is easy to check that $(\bigcup_{i=1}^{n} I_i) \bigcup (\bigcup_{j=1}^{mn} A_j) = [1, e]$.

For the complete bipartite graph $K_{m,n}$ is balanced d-magic, by Lemma 2.2, there is magic rectangle matrix $R = (r_{i,j})_{m \times n}$, such that it has $n/2$ elements more than $mn/2$ in each column of R and $m/2$ elements not more than $mn/2$ in each column of R. Let $B = (\sum A_{r_{i,j}})_{m \times n}$, $B_{i,\cdot} = \bigcup_{j=1}^{n} A_{r_{i,j}}$ and $B_{\cdot,j} = \bigcup_{i=1}^{m} A_{r_{i,j}}$. By Lemma 2.5, we obtain

$$\sum B_{i,\cdot} = np(mnp+1)/2 + np^2(2r_1 + 2r_2 + \cdots + 2r_{n-1} + r_n)/2 = np(1+e)/2$$

$$\sum B_{\cdot,j} = mp(mnp+1)/2 + mp^2(2r_1 + 2r_2 + \cdots + 2r_{n-1} + r_n)/2 = mp(1+e)/2$$

Similar to Theorem 3.1, we label the edges incident with $V(\overline{K_m})$ and $V(G_j)$ by the elements of $B_{\cdot,j}$. By Lemma 2.5, we label edges of G_j by the elements of I_j, for $1 \leq j \leq n$. For $j = 1, 2, \ldots, n-1$, we have

$$f^*(v_j) = (2(\sum I_j) + \sum B_{\cdot,j})/p = (4r_j + m)(e+1)/2, v_j \in V(G_j)$$

$$f^*(v_n) = (2(\sum I_n) + \sum B_{\cdot,j})/p = (2r_n + m)(e+1)/2, v_n \in V(G_n).$$

For $i = 1, 2, \ldots, m$, we have

$$f^*(x_i) = \sum B_{i,\cdot} = np^2(r_1 + r_2 + \cdots + r_n) + np(pnm + 1)/2 = np(e+1)/2.$$

Thus, $f^*(v) = (1 + |E(G)|)/2 \deg(v)$ for every $v \in V(G)$, and $(\bigcup_{i=1}^{n} G_i) \vee \overline{K_m}$ is d-magic. □

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