On the domination number of digraphs*

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Abstract: A vertex subset S of a digraph D is called a dominating set of D if every vertex not in S is adjacent from at least one vertex in S. The domination number of D, denoted by $\gamma(D)$, is the minimum cardinality of a dominating set of D. We characterize the rooted trees and connected contrafunctional digraphs D of order n satisfying $\gamma(D) = \lceil n/2 \rceil$. Moreover, we show that for every digraph D of order n with minimum in-degree at least one, $\gamma(D) \leq (k+1)n/(2k+1)$, where $2k+1$ is the length of a shortest odd directed cycle in D, and we characterize the corresponding digraphs achieving this upper bound. In particular, if D contains no odd directed cycles, then $\gamma(D) \leq n/2$.

Keywords: Domination number; Rooted tree; Contrafunctional digraph; Directed graph

1 Introduction

In recent years domination in graphs has been studied extensively since it has many applications. The literature on this subject has been surveyed and detailed in the two books [5, 6]. Compared to undirected graphs, domination in digraphs has not yet gained the same amount of attention, although it has several useful applications as well. For example, domination in digraphs has been used in the study of the routing problems in networks [12] and answering skyline query in database [7]. Our aim in this paper is to study the domination number of digraphs first introduced by Fu [2] and which is now well studied (see, for example, [1, 3, 8, 10, 11]).

Throughout this paper, $D = (V, A)$ denotes a digraph with neither loops nor multiple arcs (but pairs of opposite arcs are allowed). For two vertices $u, v \in$

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$V(D)$, we use (u, v) to denote the arc with direction from u to v, that is, u is adjacent to v, or equivalently, v is adjacent from u. For $v \in V(D)$, the *out-neighborhood* and *in-neighborhood* of v are $N_D^+(v) = \{u \in V(D) : (v, u) \in A(D)\}$ and $N_D^-(v) = \{u \in V(D) : (u, v) \in A(D)\}$, respectively, and let $N_D^+[v] = N_D^+(v) \cup \{v\}$. The *out-degree* and *in-degree* of a vertex v of D are defined by $d_D^+(v) = |N_D^+(v)|$ and $d_D^-(v) = |N_D^-(v)|$, respectively. The minimum in-degree among the vertices of D is denoted by $\delta^-(D)$. For two vertices u and v of D, the *distance* $d_D(u, v)$ *from* u *to* v is the length of a shortest u-v directed path in D. If D contains no u-v directed path, then $d_D(u, v) = \infty$. For a subdigraph H of D and $v \in V(D)$, the *distance from* H *to* v in D is defined by $d_D(H, v) = \min\{d_D(u, v) : u \in V(H)\}$.

A digraph D is *connected* if its underlying graph is connected. A *connected component* of a digraph D is the digraph induced by a connected component of the underlying graph of D. A *rooted tree* is a connected digraph with a vertex of in-degree 0, called the *root*, such that every vertex different from the root has in-degree 1. The *height* of a rooted tree T, denoted by $h(T)$, is $\max\{d_T(r, v) : v \in V(T)\}$, where r is the root of T. A rooted tree of order n with height 1 is called a (*directed*) *star* and is denoted by $\overrightarrow{S_n}$. A digraph D is *contrafunctional* if every vertex of D has in-degree 1.

Given two vertices u and v of a digraph D, we say u *dominates* v if $u = v$ or $(u, v) \in A(D)$. Clearly a vertex u dominates all vertices in $N_D^+[u]$. A vertex subset S of a digraph D is called a *dominating set* of D if every vertex not in S is dominated by at least one vertex in S. The *domination number* of a digraph D, denoted by $\gamma(D)$, is the minimum cardinality of a dominating set of D. A dominating set of D of cardinality $\gamma(D)$ is called a $\gamma(D)$-*set*.

2 Main results

Lemma 2.1. *Let x be a vertex of a digraph D such that $\{y \in N_D^+(x) : d_D^-(y) = 1$ and $d_D^+(y) = 0\} \neq \emptyset$ (see Figure 1). Then there exists a $\gamma(D)$-set $S(D)$ such that $x \in S(D)$.*

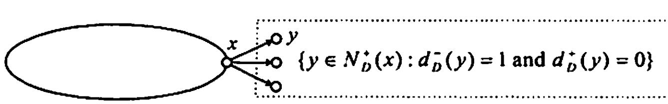

Figure 1: A digraph D and its local structure at a vertex x.

Proof. Let S be a $\gamma(D)$-set and let $A = \{y \in N_D^+(x) : d_D^-(y) = 1$ and $d_D^+(y) = 0\}$. If $x \in S$, then we choose S as $S(D)$ and hence the assertion is trivial. Assume next that $x \notin S$. Then $A \subseteq S$ since x is the unique vertex adjacent to each vertex in A. Since x dominates each vertex in A, $(S - A) \cup \{x\}$ is a dominating set of D and hence $|S| \leq |(S - A) \cup \{x\}|$. Moreover, clearly $|(S - A) \cup \{x\}| \leq |S|$. Hence it follows that $|(S-A)\cup\{x\}| = |S|$, implying that $(S-A)\cup\{x\}$ is also a $\gamma(D)$-set. Then we choose $(S-A)\cup\{x\}$ as $S(D)$ and hence the desired result follows. □

Lemma 2.2. *Let x be a vertex of a digraph D such that $d_D^-(x) = 1, d_D^+(x) \geq 1$ and $d_D^-(y) = 1, d_D^+(y) = 0$ for each $y \in N_D^+(x)$ (see Figure 2). Then*

$$\gamma(D) = \gamma(D - N_D^+[x]) + 1.$$

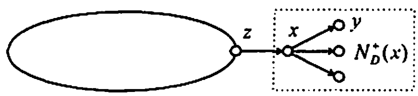

Figure 2: A digraph D and its local structure at a vertex x.

Proof. Let S be a $\gamma(D - N_D^+[x])$-set and by Lemma 2.1, let $S(D)$ be a $\gamma(D)$-set such that $x \in S(D)$. It is easy to observe that $S \cup \{x\}$ is a dominating set of D and hence $|S(D)| \leq |S \cup \{x\}| = |S|+1$. On the other hand, since x dominates no vertex in $V(D - N_D^+[x])$, $S(D) - \{x\}$ is a dominating set of $D - N_D^+[x]$ and hence $|S| \leq| S(D) - \{x\}| = |S(D)| - 1$. As a result, we have that $|S(D)| = |S| + 1$, implying that $\gamma(D) = \gamma(D - N_D^+[x]) + 1$. □

2.1 Results for rooted trees

Theorem 2.3 ([9]). *Let T be a rooted tree of order n. Then $\gamma(T) \leq \lceil n/2 \rceil$.*

We are now in a position to provide a characterization of all the rooted trees for which attain the upper bound given in Theorem 2.3. For this purpose, we first give some definitions and properties involving rooted trees.

Let T be a rooted tree with root r. We choose a vertex x_0 of T_0 $(= T)$ such that $d_{T_0}(r, x_0) = h(T_0) - 1$. Note that $T_1 = T_0 - N_{T_0}^+[x_0]$ is still a rooted tree with root r or empty. This means that we can take a sequence of such operations on T and hence have a sequence $T_0, T_1, \ldots, T_l$ of digraphs until the resulting digraph T_l is empty or the isolated vertex r. Let $\mathcal{T}_i = T_i - T_{i+1}$ for each $i \in \{0, 1, \ldots, l-1\}$ and let $\mathcal{T}_l = T_l$. Then we call the spanning subdigraph $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_l$ a *vertex disjoint star cover* (*vds-cover*) of T.

Definition 2.4. *Let $\mathscr{T}_1$ (resp. $\mathscr{T}_2$) be the family of all rooted trees T with a vds-cover $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_l$ $(l \geq 0)$ such that*

(i) For each $i \in \{0, 1, \ldots, l-1\}$, $\mathcal{T}_i$ is isomorphic to $\overrightarrow{S_2}$.

(ii) $\mathcal{T}_l = \emptyset$ (resp. $\mathcal{T}_l$ is an isolated vertex).

Definition 2.5. *Let $\mathscr{T}_3$ be the family of all rooted trees T with a vds-cover $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_l$ $(l \geq 1)$ such that*

(i) *One of* $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_{l-1}$ *is isomorphic to* $\overrightarrow{S_3}$ *and the others are isomorphic to* $\overrightarrow{S_2}$.

(ii) $\mathcal{T}_l$ *is an isolated vertex.*

Lemma 2.6. *Let* $T \in \mathscr{T}_1 \cup \mathscr{T}_2 \cup \mathscr{T}_3$ *be a rooted tree of order* n. *Then*

$$\gamma(T) = \begin{cases} n/2, & \text{if } T \in \mathscr{T}_1 \cup \mathscr{T}_3, \\ (n+1)/2, & \text{if } T \in \mathscr{T}_2. \end{cases}$$

Proof. We proceed by induction on n. If $n = 1$ or 2, then the statement is valid. Hence we may assume that $n \geq 3$. This implies that $h(T) \geq 2$.

Let $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_l$ be a vds-cover of T and let $T' = T - \mathcal{T}_0$. By the definition of $\mathscr{T}_i$ ($i \in \{1,2,3\}$), we have that if $T \in \mathscr{T}_i$ ($i \in \{1,2\}$), then $T' \in \mathscr{T}_i$ and $\mathcal{T}_0 \cong \overrightarrow{S_2}$; if $T \in \mathscr{T}_3$, then $T' \in \mathscr{T}_3$ and $\mathcal{T}_0 \cong \overrightarrow{S_2}$, or $T' \in \mathscr{T}_2$ and $\mathcal{T}_0 \cong \overrightarrow{S_3}$. Thus, if $T \in \mathscr{T}_1$, then $T' \in \mathscr{T}_1$ and $\mathcal{T}_0 \cong \overrightarrow{S_2}$, and hence by the induction hypothesis, $\gamma(T') = |V(T')|/2 = (n-2)/2$. Then it follows from Lemma 2.2 that $\gamma(T) = \gamma(T') + 1 = (n-2)/2 + 1 = n/2$. The discussion for the case when $T \in \mathscr{T}_i$ ($i \in \{2,3\}$) is analogous, which completes our proof. □

Theorem 2.7. *Let* T *be a rooted tree of order* n. *Then*

$$\gamma(T) = \begin{cases} n/2, & \text{if and only if } T \in \mathscr{T}_1 \cup \mathscr{T}_3, \\ (n+1)/2, & \text{if and only if } T \in \mathscr{T}_2. \end{cases}$$

Proof. By Lemma 2.6, the sufficiency is trivial. We proceed by induction on n to show the necessity. Let $\gamma(T) = n/2$ or $(n+1)/2$. If $n = 1$ or 2, then the statement is valid. Hence we may assume that $n \geq 3$. If $h(T) = 1$, then $\gamma(T) = 1 < n/2$.

Suppose next that $h(T) \geq 2$. Let $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_l$ be a vds-cover of T and let $T' = T - \mathcal{T}_0$. By Theorem 2.3, we have that $\gamma(T') \leq (|V(T')| + 1)/2$. Moreover, by Lemma 2.2, we have that $\gamma(T) = \gamma(T') + 1$.

Claim 2.8. *If* $\gamma(T) = (n+1)/2$, *then* $T \in \mathscr{T}_2$.

Proof of Claim 2.8: If $|V(\mathcal{T}_0)| \geq 3$, then $\gamma(T) = \gamma(T') + 1 \leq (|V(T')| + 1)/2 + (|V(\mathcal{T}_0)| - 1)/2 = n/2$, a contradiction. Hence $|V(\mathcal{T}_0)| = 2$, that is, $\mathcal{T}_0 \cong \overrightarrow{S_2}$. Since $\gamma(T') = \gamma(T) - 1 = (n+1)/2 - 1 = (|V(T')| + 1)/2$, $T' \in \mathscr{T}_2$ by the induction hypothesis. Then it follows that $T \in \mathscr{T}_2$. So, this claim is true.

Claim 2.9. *If* $\gamma(T) = n/2$, *then* $T \in \mathscr{T}_1 \cup \mathscr{T}_3$.

Proof of Claim 2.9: If $|V(\mathcal{T}_0)| \geq 4$, then

$$\gamma(T) = \gamma(T') + 1 < (|V(T')| + 1)/2 + (|V(\mathcal{T}_0)| - 1)/2 = n/2,$$

a contradiction. Hence $|V(\mathcal{T}_0)| = 2$ or 3. If $|V(\mathcal{T}_0)| = 2$, that is, if $\mathcal{T}_0 \cong \overrightarrow{S_2}$, then $\gamma(T') = \gamma(T) - 1 = n/2 - 1 = |V(T')|/2$ and hence by the induction hypothesis,

$T' \in \mathscr{T}_1 \cup \mathscr{T}_3$. Then it follows that $T \in \mathscr{T}_1 \cup \mathscr{T}_3$. If $|V(T_0)| = 3$, that is, if $T_0 \cong \overrightarrow{S_3}$, then $\gamma(T') = \gamma(T) - 1 = n/2 - 1 = (|V(T')| + 1)/2$ and hence as proven earlier, $T' \in \mathscr{T}_2$. Then it also follows that $T \in \mathscr{T}_3 \subset \mathscr{T}_1 \cup \mathscr{T}_3$. So, this claim is true. $\square$

2.2 Results for connected contrafunctional digraphs

In [4], Harary et al. showed that every connected contrafunctional digraph has a unique directed cycle and the removal of any one arc of the directed cycle results in a rooted tree. The simplest connected contrafunctional digraph of order n is the *directed cycle* $\overrightarrow{C_n}$ of length n, for which we have the following result.

$$\gamma(\overrightarrow{C_n}) = \begin{cases} n/2, & \text{if } n \text{ is even,} \\ (n+1)/2, & \text{if } n \text{ is odd.} \end{cases} \tag{1}$$

Theorem 2.10 ([9]). *Let D be a connected contrafunctional digraph of order n. Then $\gamma(D) \le \lceil n/2 \rceil$.*

We are now in a position to provide a characterization of all the connected contrafunctional digraphs for which attain the upper bound given in Theorem 2.10. For this purpose, we first give some definitions and properties involving connected contrafunctional digraphs.

Let D be a connected contrafunctional digraph. We define the *height* of D, denoted by $h(D)$, to be the maximum distance from its unique directed cycle C to all vertices of D, i.e., $h(D) = \max\{d_D(C, v) : v \in V(D)\}$. In particular, the height of a directed cycle is exactly equal to 0.

Definition 2.11. *Let $\mathscr{D}$ be the family of all connected contrafunctional digraphs D satisfying*

(i) $h(D) = 1$.

(ii) There exist two vertices $u, v \in V(C)$ with $(u, v) \in A(C)$ and $d_D^+(v) > 1$, where C is the unique directed cycle of D, such that $D - (u, v) \in \mathscr{T}_1$.

Lemma 2.12. *Let D be a connected contrafunctional digraph of order n with $h(D) = 1$. Then*

$$\gamma(D) \le n/2$$

with equality if and only if $D \in \mathscr{D}$.

Proof. Note that $h(D) = 1$. Therefore, we may assume that C is the unique directed cycle of D, $u, v \in V(C)$, $x \notin V(C)$ and $(u, v), (v, x) \in A(D)$. Let $T = D - (u, v)$. It is easy to see that T is a rooted tree with root v and $T \notin \mathscr{T}_2 \cup \mathscr{T}_3$.

It follows from Theorems 2.3 and 2.7 that $\gamma(D) \le \gamma(T) \le n/2$, establishing the desired upper bound. To prove the sufficiency, suppose that $D \in \mathscr{D}$. Without loss of generality, we may assume that $T \in \mathscr{T}_1$. Hence by Theorem 2.7, we have that $\gamma(T) = n/2$. Morerover, by Lemma 2.1, we have that there exists a $\gamma(D)$-set $S(D)$ such that $v \in S(D)$. Then it is easy to verify that $S(D)$ is also a dominating set of T and hence $\gamma(T) \leq |S(D)| = \gamma(D)$. On the other hand, $\gamma(D) \leq \gamma(T)$ since T is a spanning subdigraph of D. As a result, we have that $\gamma(D) = \gamma(T) = n/2$.

To prove the necessity, suppose that $\gamma(D) = n/2$. We next show that $D \in \mathscr{D}$. Suppose, to the contrary, that $D \notin \mathscr{D}$. Then it is easy to see that $T \notin \mathscr{T}_1$. Note that $T \notin \mathscr{T}_1 \cup \mathscr{T}_2 \cup \mathscr{T}_3$. Hence it follows from Theorems 2.3 and 2.7 that $\gamma(D) \leq \gamma(T) < n/2$, which is a contradiction. Hence $D \in \mathscr{D}$. □

Let D be a connected contrafunctional digraph with a unique directed cycle C. We choose a vertex x_0 of D_0 $(= D)$ such that $d_{D_0}(C, x_0) = h(D_0) - 1 \geq 1$. Note that $D_1 = D_0 - N^+_{D_0}[x_0]$ is still a connected contrafunctional digraph with the unique directed cycle C. Similar to the discussion for a rooted tree, we can take a sequence of such operations on D and hence have a sequence $D_0, D_1, \ldots, D_l$ of connected contrafunctional digraphs until the resulting digraph D_l is the directed cycle C or a connected contrafunctional digraph with height 1. Let $\mathcal{T}_i = D_i - D_{i+1}$ for each $i \in \{0, 1, \ldots, l-1\}$ and let $\mathcal{D}_l = D_l$. Then we call the spanning subdigraph $\mathcal{D} = \mathcal{T}_0 \cup \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_{l-1} \cup \mathcal{D}_l$ a *vertex disjoint star and connected contrafunctional digraph cover* (*vdsc-cover*) of D.

Definition 2.13. *Let $\mathscr{D}_1$ (resp. $\mathscr{D}_2$) be the family of all connected contrafunctional digraphs D with a vdsc-cover $\mathcal{D} = \mathcal{T}_0 \cup \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_{l-1} \cup \mathcal{D}_l$ $(l \geq 0)$ such that*

(i) For each $i \in \{0, 1, \ldots, l-1\}$, $\mathcal{T}_i$ is isomorphic to $\overrightarrow{S_2}$.

(ii) $\mathcal{D}_l$ is an even directed cycle or $\mathcal{D}_l \in \mathscr{D}$ (resp. $\mathcal{D}_l$ is an odd directed cycle).

Definition 2.14. *Let $\mathscr{D}_3$ be the family of all connected contrafunctional digraphs D with a vdsc-cover $\mathcal{D} = \mathcal{T}_0 \cup \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_{l-1} \cup \mathcal{D}_l$ $(l \geq 1)$ such that*

(i) One of $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_{l-1}$ is isomorphic to $\overrightarrow{S_3}$ and the others are isomorphic to $\overrightarrow{S_2}$.

(ii) $\mathcal{D}_l$ is an odd directed cycle.

Lemma 2.15. *Let $D \in \mathscr{D}_1 \cup \mathscr{D}_2 \cup \mathscr{D}_3$ be a connected contrafunctional digraph of order n. Then*

$$\gamma(D) = \begin{cases} n/2, & \text{if } D \in \mathscr{D}_1 \cup \mathscr{D}_3, \\ (n+1)/2, & \text{if } D \in \mathscr{D}_2. \end{cases}$$

Proof. Let $\overrightarrow{C_k}$ be the unique directed cycle of D. We fix k and proceed by induction on n. If $n = k$ is even (resp. odd), then $D \in \mathscr{D}_1 \subset \mathscr{D}_1 \cup \mathscr{D}_3$ (resp. $\mathscr{D}_2$). On the other hand, by (1), we have that $\gamma(D) = n/2$ (resp. $(n+1)/2$). Hence we may assume that $n \geq k+1$. If $h(D) = 1$, then by Lemma 2.12, $D \in \mathscr{D} \subset \mathscr{D}_1 \subset \mathscr{D}_1 \cup \mathscr{D}_3$ and $\gamma(D) = n/2$.

Suppose next that $h(D) \geq 2$. Let $\mathcal{D} = \mathcal{T}_0 \cup \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_{l-1} \cup \mathcal{D}_l$ be a vdsc-cover of D and let $D' = D - \mathcal{T}_0$. By the definition of $\mathscr{D}_i$ $(i \in \{1, 2, 3\})$, we have that if $D \in \mathscr{D}_i$ $(i \in \{1,2\})$, then $D' \in \mathscr{D}_i$ and $\mathcal{T}_0 \cong \overrightarrow{S_2}$; if $D \in \mathscr{D}_3$, then $D' \in \mathscr{D}_3$ and $\mathcal{T}_0 \cong \overrightarrow{S_2}$, or $D' \in \mathscr{D}_2$ and $\mathcal{T}_0 \cong \overrightarrow{S_3}$. Thus, if $D \in \mathscr{D}_1$, then $D' \in \mathscr{D}_1$ and $\mathcal{T}_0 \cong \overrightarrow{S_2}$, and hence by the induction hypothesis, $\gamma(D') = |V(D')|/2 = (n-2)/2$. Then by Lemma 2.2, $\gamma(D) = \gamma(D')+1 = (n-2)/2+1 = n/2$. The discussion for the case when $D \in \mathscr{D}_i$ $(i \in \{2,3\})$ is analogous, which completes our proof. □

Theorem 2.16. *Let D be a connected contrafunctional digraph of order n. Then*

$$\gamma(D) = \begin{cases} n/2, & \textit{if and only if } D \in \mathscr{D}_1 \cup \mathscr{D}_3, \\ (n+1)/2, & \textit{if and only if } D \in \mathscr{D}_2. \end{cases}$$

Proof. By Lemma 2.15, the sufficiency is trivial. To show the necessity, suppose that $\gamma(D) = n/2$ or $(n+1)/2$. Let $\overrightarrow{C_k}$ be the unique directed cycle of D. We fix k and proceed by induction on n. If $n = k$ is even (resp. odd), then by (1), we have that $\gamma(D) = n/2$ (resp. $(n+1)/2$). On the other hand, clearly $D \in \mathscr{D}_1 \subset \mathscr{D}_1 \cup \mathscr{D}_3$ (resp. $\mathscr{D}_2$). Hence we may assume that $n \geq k+1$. If $h(D) = 1$, then by Lemma 2.12, $\gamma(D) = n/2$ and $D \in \mathscr{D} \subset \mathscr{D}_1 \subset \mathscr{D}_1 \cup \mathscr{D}_3$.

Suppose next that $h(D) \geq 2$. Let $\mathcal{D} = \mathcal{T}_0 \cup \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_{l-1} \cup \mathcal{D}_l$ be a vdsc-cover of D and let $D' = D - \mathcal{T}_0$. By Theorem 2.10, we have that $\gamma(D') \leq (|V(D')|+1)/2$. Moreover, by Lemma 2.2, we have that $\gamma(D) = \gamma(D')+1$.

Claim 2.17. *If $\gamma(D) = (n+1)/2$, then $D \in \mathscr{D}_2$.*

Proof of Claim 2.17: If $|V(\mathcal{T}_0)| \geq 3$, then $\gamma(D) = \gamma(D')+1 \leq (|V(D')|+1)/2 + (|V(\mathcal{T}_1)|-1)/2 = n/2$, a contradiction. Hence $|V(\mathcal{T}_0)| = 2$, that is, $\mathcal{T}_0 \cong \overrightarrow{S_2}$. Since $\gamma(D') = \gamma(D) - 1 = (n+1)/2 - 1 = (|V(D')|+1)/2$, $D' \in \mathscr{D}_2$ by the induction hypothesis. Then it follows that $D \in \mathscr{D}_2$. So, this claim is true.

Claim 2.18. *If $\gamma(D) = n/2$, then $D \in \mathscr{D}_1 \cup \mathscr{D}_3$.*

Proof of Claim 2.18: If $|V(\mathcal{T}_0)| \geq 4$, then $\gamma(D) = \gamma(D')+1 < (|V(D')|+1)/2 + (|V(\mathcal{T}_0)|-1)/2 = n/2$, a contradiction. Hence $|V(\mathcal{T}_0)| = 2$ or 3. If $|V(\mathcal{T}_0)| = 2$, that is, if $\mathcal{T}_0 \cong \overrightarrow{S_2}$, then $\gamma(D') = \gamma(D) - 1 = n/2 - 1 = |V(D')|/2$ and hence by the induction hypothesis, $D' \in \mathscr{D}_1 \cup \mathscr{D}_3$. Then it follows that $D \in \mathscr{D}_1 \cup \mathscr{D}_3$. If $|V(\mathcal{T}_0)| = 3$, that is, if $\mathcal{T}_0 \cong \overrightarrow{S_3}$, then $\gamma(D') = \gamma(D)-1 = n/2-1 = (|V(D')|+1)/2$ and hence as proven earlier, $D' \in \mathscr{D}_2$. Then it also follows that $D \in \mathscr{D}_3 \subset \mathscr{D}_1 \cup \mathscr{D}_3$. So, this claim is true. □

2.3 Results for other digraphs

Let D be a digraph with $\delta^-(D) \geq 1$. We choose an arbitrary incoming arc of v for each vertex v of D. Then all such arcs induce a spanning subdigraph H of D consisting of some connected components, say $H_1, H_2, \ldots, H_l$. Moreover, H_i $(i \in \{1,2,\ldots,l\})$ is a connected contrafunctional subdigraph of D since each vertex of

H_i has in-degree 1. Then we call the spanning subdigraph $H = H_1 \cup H_2 \cup \cdots \cup H_l$ a *vertex disjoint connected contrafunctional digraph cover* (*vdc-cover*) of D.

Theorem 2.19. *Let D be a digraph of order n with $\delta^-(D) \geq 1$. Then*

$$\gamma(D) \leq (k+1)n/(2k+1),$$

where $2k+1$ is the length of a shortest odd directed cycle in D. In particular, if D contains no odd directed cycles, then $\gamma(D) \leq n/2$.

Proof. Let $H = H_1 \cup H_2 \cup \cdots \cup H_l$ be a vdc-cover of D. For each $i \in \{1, 2, \ldots, l\}$, if H_i contains an even directed cycle, then clearly $H_i \notin \mathscr{D}_2$ and hence by Theorems 2.10 and 2.16, $\gamma(H_i) \leq |V(H_i)|/2 < (k+1)|V(H_i)|/(2k+1)$; if H_i contains an odd directed cycle, say $\vec{C}_{2s+1}$, then again by Theorem 2.10,

$$\gamma(H_i) \leq (|V(H_i)|+1)/2 \leq (s+1)|V(H_i)|/(2s+1) \leq (k+1)|V(H_i)|/(2k+1)$$

since $|V(H_i)| \geq 2s+1$ and $s \geq k$. Then it follows that

$$\gamma(D) \leq \gamma(H) = \sum_{i=1}^{l} \gamma(H_i) \leq \frac{k+1}{2k+1} \sum_{i=1}^{l} |V(H_i)| = \frac{k+1}{2k+1} n.$$

In particular, if D contains no odd directed cycles, then for each $i \in \{1, 2, \ldots, l\}$, H_i contains an even directed cycle (that is, $H_i \notin \mathscr{D}_2$) and hence by Theorems 2.10 and 2.16, we have that $\gamma(H_i) \leq |V(H_i)|/2$. Then it follows that

$$\gamma(D) \leq \gamma(H) = \sum_{i=1}^{l} \gamma(H_i) \leq \frac{1}{2} \sum_{i=1}^{l} |V(H_i)| = \frac{n}{2},$$

which completes our proof. □

Lemma 2.20. *Let D be a contrafunctional digraph of order n. If $2k+1$ is the length of a shortest odd directed cycle, then*

$$\gamma(D) = (k+1)n/(2k+1)$$

if and only if D is a disjoint union of copies of $\vec{C}_{2k+1}$.

Proof. By (1), the sufficiency is trivial. To prove the necessity, suppose that $\gamma(D) = (k+1)n/(2k+1)$. Let $H_1, H_2, \ldots, H_l$ be the connected components of D. This implies that for each $i \in \{1, 2, \ldots, l\}$, H_i is a connected contrafunctional subdigraph of D. Hence by Theorem 2.19, it is easy to see that for each $i \in \{1, 2, \ldots, l\}$, $\gamma(H_i) \leq (k+1)|V(H_i)|/(2k+1)$. If there exists at least a connected component, say H_1, of D such that $\gamma(H_1) < (k+1)|V(H_1)|/(2k+1)$, then

$$\gamma(D) = \sum_{i=1}^{l} \gamma(H_i) < \frac{k+1}{2k+1} \sum_{i=1}^{l} |V(H_i)| = \frac{k+1}{2k+1} n,$$

which is a contradiction. Then it follows that for each $i \in \{1,2,\ldots,l\}$, $\gamma(H_i) = (k+1)|V(H_i)|/(2k+1) > |V(H_i)|/2$ and hence by Theorems 2.10 and 2.16, we have that $H_i \in \mathscr{D}_2$ and $\gamma(H_i) = (k+1)|V(H_i)|/(2k+1) = (|V(H_i)|+1)/2$, implying that $|V(H_i)| = 2k+1$. Note that the length of a shortest odd directed cycle of D is $2k+1$. Therefore, for each $i \in \{1,2,\ldots,l\}$, $H_i \cong \vec{C}_{2k+1}$, which completes our proof. □

As an immediate consequence of Theorem 2.19 and Lemma 2.20, we have the following result due to Lee [9].

Corollary 2.21 ([9]). *Let D be a contrafunctional digraph of order n. Then*

$$\gamma(D) = 2n/3$$

if and only if D is a disjoint union of copies of $\vec{C}_3$.

Theorem 2.22. *Let D be a connected digraph of order n with $\delta^-(D) \geq 1$. If $2k+1$ is the length of a shortest odd directed cycle, then*

$$\gamma(D) = (k+1)n/(2k+1)$$

if and only if $D \cong \vec{C}_{2k+1}$.

Proof. By (1), the sufficiency is trivial. To prove the necessity, suppose that $\gamma(D) = (k+1)n/(2k+1)$. Let $H = H_1 \cup H_2 \cup \cdots \cup H_l$ be a vdc-cover of D. If H contains no odd directed cycles, then by Theorem 2.19, $\gamma(D) \leq \gamma(H) \leq n/2 < (k+1)n/(2k+1)$, a contradiction. Hence we may assume that H contains odd directed cycles. Let $2s+1$ be the length of a shortest odd directed cycle of H. This implies that $s \geq k$. Then again by Theorem 2.19, $\gamma(H) \leq (s+1)n/(2s+1)$ and hence

$$\frac{k+1}{2k+1}n = \gamma(D) \leq \gamma(H) \leq \frac{s+1}{2s+1}n \leq \frac{k+1}{2k+1}n.$$

It follows that $\gamma(H) = \gamma(D) = (k+1)n/(2k+1)$ and $s = k$. Note that H is a contrafunctional digraph of order n. Hence by Lemma 2.20, H is a disjoint union of copies of $\vec{C}_{2k+1}$, implying that for each $i \in \{1,2,\ldots,l\}$, $H_i \cong \vec{C}_{2k+1}$.

We now claim that $l = 1$. Suppose, to the contrary, that $l \geq 2$. Note that D is connected but H is not. Therefore, there exist at least two vertices $u \in V(H_i)$ and $v \in V(H_j)$, where $i \neq j$, such that $(u,v) \in A(D)$. Then it is easy to verify that $\gamma(H_i \cup H_j \cup (u,v)) = \gamma(H_i) + \gamma(H_j) - 1$ and hence

$$\gamma(D) \leq \gamma(H \cup (u,v)) = \gamma(H) - 1 = \frac{k+1}{2k+1}n - 1 < \frac{k+1}{2k+1}n,$$

a contradiction. Therefore, $l = 1$, implying that $H = H_1 \cong \vec{C}_{2k+1}$ and $n = 2k+1$.

We next claim that $D = H \cong \vec{C}_{2k+1}$. Suppose, to the contrary, that $D \neq H = v_1v_2\ldots v_{2k+1}v_1$. Then there exist two vertices, say v_1 and v_i $(2 \leq i \leq 2k)$, of D such that $(v_i, v_1) \in A(D) - A(H)$. It is easy to observe that $D' = H \cup$

$(v_i, v_1) - (v_{2k+1}, v_1)$ is also a spanning connected contrafunctional subdigraph of D. Note that the length of a shortest odd directed cycle of D is $2k+1$. Therefore, D' contains an even directed cycle and hence $D' \notin \mathscr{D}_2$. Then it follows from Theorems 2.10 and 2.16 that

$$\gamma(D) \leq \gamma(D') \leq (2k+1)/2 < k+1 = (k+1)n/(2k+1),$$

a contradiction. Thus, $D = H \cong \overrightarrow{C}_{2k+1}$, which completes our proof. □

As a consequence of Theorems 2.19 and 2.22, we have the following result.

Corollary 2.23. *Let D be a digraph of order n with $\delta^-(D) \geq 1$. If $2k+1$ is the length of a shortest odd directed cycle, then*

$$\gamma(D) = (k+1)n/(2k+1)$$

if and only if D is a disjoint union of copies of $\overrightarrow{C}_{2k+1}$.

References

[1] H. Cai, J. Liu, J. Meng, Domination number in iterated line digraph of a complete bipartite digraph, Ars Combin. 107 (2012) 515-520.

[2] Y. Fu, *Dominating set and converse dominating set of a directed graph*, Amer. Math. Monthly 75 (1968) 861-863.

[3] G. Hao, J. Qian, On the sum of out-domination number and in-domination number of digraphs, Ars Combin. 119 (2015) 331–337.

[4] F. Harary, R.Z. Norman, D. Cartwright, Structural Models, Wiley, New York, 1965.

[5] T.W. Haynes, S.T. Hedetniemi, P.J. Slater (Eds.), Domination in Graphs: Advanced Topics, Marcel Dekker, Inc., New York, 1998.

[6] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, Inc., New York, 1998.

[7] V. Koltun, C.H. Papadimitriou, Approximately dominating representatives, Theoret. Comput. Sci. 371 (2007) 148-154.

[8] C. Lee, On the domination number of a digraph. Ph.D. Dissertation, Michigan State University (1994).

[9] C. Lee, *Domination in digraphs*, J. Korean Math. Soc. 35 (1998) 843-853.

[10] L. Niepel, M. Knor, Domination in the cross product of digraphs, Ars Combin. 97 (2010) 271-279.

[11] E.F. Shan, T.C.E. Cheng, L.Y. Kang, Absorbant of generalized de Bruijn digraphs, Inform. Process. Lett. 105 (2007) 6-11.

[12] J. Wu, *Extended dominating-set-based routing in ad hoc wireless networks with unidirectional links*, IEEE Trans. Parallel Distrib. Syst. 13 (2002) 866-881.