On edge contractible graphs *

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Abstract

Let G be a connected graph with a perfect matching on $2n$ vertices ($n \geq 2$). A graph H is a contraction of G if it can be obtained from G by a sequence of edge contractions. Then G is said to be *edge contractible* if for any contraction G' of G with $|V(G')|$ is even, G' has a perfect matching. In this note, we obtain a sufficient and necessary condition for a graph to be an edge contractible graph.

Key words perfect matching; edge contractible graphs; n-extendable graphs

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1 Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *connectivity* of G is denoted by $\kappa(G)$. The number of odd components of G is denoted by $c_o(G)$. Given an edge e of G, the graph G/e is obtained from G by contracting e to a single vertex. A graph G' is said to be a contraction of G, written $G' < G$, if G' can be obtained from G by a sequence of edge contractions. For a subset $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S, and by G/S the graph obtained from G by contracting S to a single vertex.

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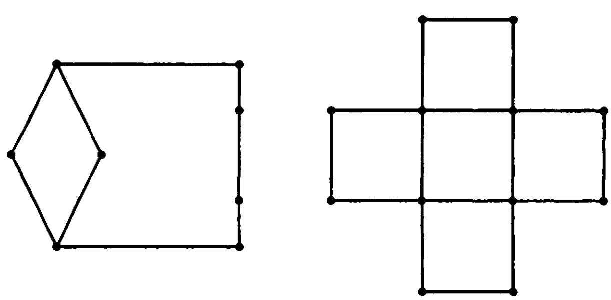
The n-extendable graphs (see [4, 5, 6, 7]) and the k-critical graphs (see [1, 2, 3, 8]) are two extensively studied classes of graphs in matching. A connected graph G is said to be *n-extendable* if G has n independent edges and any n independent edges are contained in a perfect matching of G. A connnected graph G is said to be *k-critical* if the deletion of any k vertices of $V(G)$ results in a graph with a perfect matching. Therefore a $2k$-critical graph is k-extendable. Roughly speaking, above mentioned graphs mainly focus on the property that for a graph G with a perfect matching, after deleting some specific vertices, the resulting graph can also contain a perfect matching. It is natural to ask what more structural properties of graphs with perfect matchings can be revealed by contracting some edges?

Motivated by the above question, in the present paper we introduce a new class of graphs which can preserve containing a perfect matching by taking a contracting operation.

. **Edge contractible graphs:** A connected graph G with a perfect matching on $2n$ vertices ($n \geq 2$) is said to be *edge contractible* if for any contraction G' of G with $|V(G')|$ is even, G' has a perfect matching. In other words, a connected graph G of even order is edge contractible if for any sequence of graphs G_0, G_1, ..., G_{2t} ($t \leq n-1$) such that $G_0 = G$, $G_{i+1} = G_i/e_i$, where e_i is an arbitrary edge of G_i, the graph G_{2t} has a perfect matching.

For example, the graph T_1 in Figure 1 is edge contractible. But the graph T_2 in Figure 1 is not edge contractible, since $K_{1,3}$ is a contraction of T_2.



T_1 T_2

Fig.1. Two graphs T_1 and T_2

Given three positive integers r_1, r_2, r_3 with $r_1 + r_2 + r_3$ even, form a graph F_{r_1,r_2,r_3} as follows: join each vertex of a complete graph K_{r_2} to each vertex of two disjoint complete graphs K_{r_1} and K_{r_3}. It is obvious that F_{r_1,r_2,r_3} is edge contractible with $\kappa(F_{r_1,r_2,r_3}) = r_2$. This family of graphs shows that the connectivity of an edge contractible graph can be any integer. While Plummer [5] proved that the connectivity of an n–extendable graph is not less than $n+1$, which implies that the structures of edge contractible graphs may be quite different from those of n–extendable graphs. In this paper, we will investigate some structural properties of these graphs.

The main work of this note is as follows which gives a simple characterization of edge contractible graphs.

Theorem 1. Let G be a connected graph of order $2n$ ($n \geq 2$) with a perfect matching. Then G is edge contractible if and only if G contains no $K_{1,3}$ as a contraction.

The proof of this theorem will use the following fundamental result in matching theory.

Tutte's 1-factor Theorem ([4]). A graph G has a perfect matching if and only if $c_o(G - S) \leq |S|$, for all $S \subseteq V(G)$.

2 Proof of Theorem 1

The necessity is obvious.

To prove the sufficiency, suppose that G contains no $K_{1,3}$ as a contraction, but G is not edge contractible. We shall derive a contradiction.

Since G is not edge contractible, G has a contraction G', that is

$$G' < G, \tag{1}$$

such that G' has even number of vertices and does not have a perfect matching.

By Tutte's 1-factor Theorem, there exists a vertex set $S' \subseteq V(G')$ such that $c_o(G' - S') \geq |S'| + 1$. Since the order of G' is even, by parity, we have

$$c_o(G' - S') \geq |S'| + 2. \tag{2}$$

Construct a bipartite graph G'' from G' as follows: for each non-singleton component of $G' - S'$, or of $G'[S']$, contract it into a single vertex by a series of edge contractions respectively. Clearly,

$$G'' < G'. \tag{3}$$

Suppose the vertex set S' of G' becomes the vertex set S'' of G'' under above contractions. By (2), we may assume G'' has bipartition $(S'', V(G'')\backslash S'')$ with

$$|V(G'')\backslash S''| \geq |S''| + 2. \tag{4}$$

Case 1. $|S''| = 1$.

Let $|V(G'')\backslash S''| = t$. By (4), $t \geq 3$, so $G'' = K_{1,t}$. Therefore

$$K_{1,3} < G''. \tag{5}$$

Since the relation $<$ on graphs is transitive, by (1), (3) and (5), we obtain $K_{1,3} < G$, a contradiction.

Case 2. $|S''| \geq 2$.

Since G'' is connected, the condition $|S''| \geq 2$ implies that there exists a vertex, say x, in $V(G'')\backslash S''$ with degree at least 2. Let y_1, y_2, ..., y_d $(d \geq 2)$ be the neighbors of x and let $G_1 = G''/\{x, y_1, y_2, ..., y_d\}$. Then

$$G_1 < G''. \tag{6}$$

Note that G_1 is again a bipartite graph. By (4) and the construction of G_1, we may assume that G_1 has bipartition (R_1, T_1) with $|T_1| \geq |R_1| + 2 \geq 3$. If $|R_1| \geq 2$, repeating above argument, we can get a sequence of bipartite graphs G_1, G_2, ..., G_n such that

$$G_n < G_{n-1} < ... < G_1, \tag{7}$$

G_i has the bipartition (R_i, T_i) with $|T_i| \geq |R_i| + 2 \geq 3$ for $i \geq 2$ and $|R_n| = 1$. Set $|T_n| = p$. Thus $p \geq 3$ and $G_n = K_{1,p}$, so we have

$$K_{1,3} < G_n. \tag{8}$$

Now by (1), (3), (6), (7) and (8), we have $K_{1,3} < G$, a contradiction. □

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