Genus embeddings of two types of graphs

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Abstract In this paper, by joint tree model, we obtain the genera of two types of graphs, which are suspensions of cartesian products of two types of bipartite graphs from a vertex.

Keywords: minimum genus, orientable surface, joint tree, cartesian product

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1. Introduction

A *surface* is a compact 2-dimensional manifold without boundary. Since it can be obtained by pairwise identifying the edges of an even polygon along a given direction, throughout this paper, an *orientable surface*[5] is represented by a cyclic order S of letters satisfying the following conditions:

Con.1 If $a \in S$, then $a^- \in S$.
Con.2 For each letter a on S, both a and a^- occur once on S.

Let $o(S)$ be the genus of surface S and $\mathcal{S}$ be the set of all orientable surfaces. In order to determine $o(S)$, an *equivalence*[6], denoted by $\sim$, on $\mathcal{S}$ is introduced by the following operations:

Op.1 $A \sim Aaa^-$ where $A \in \mathcal{S}$ and $a \notin A$.
Op.2 $AabBb^-a^- \sim AcBc^- \sim Ac^-Bc$ where $AB \in \mathcal{S}$ and $a, b, c \notin AB$.
Op.3 $(Aa)(a^-B) \sim AB$ where $AB \in \mathcal{S}$ and $AB \neq \emptyset$.
Op.4 $AaBbCa^-Db^-E \sim ADCBEaba^-b^-$ for $ABCDE \in \mathcal{S}$ and $a, b \notin ABCDE$.
Then by applying these operations above, each orientable surface is equivalent to only one of the following canonical forms:

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$$O_i = \begin{cases} a_0 a_0^-, & \text{if the genus of a surface is 0;} \\ \Pi_{k=1}^{i} a_k b_k a_k^- b_k^-, & \text{if the genus of a surface is } i. \end{cases}$$

Lemma1.1[6] *Let S_1 and S_2 be two surfaces, $a, b, a^-, b^- \notin S_2$. If $S_1 \sim S_2 aba^-b^-$, then $o(S_1) = o(S_2) + 1$.*

An *embedding* of a graph G into a surface S is a homeomorphism τ: $G \to S$ such that each component of $S - \tau(G)$ is homeomorphic to an open disc. Two embeddings τ_1: $G \to S$ and τ_2: $G \to S$ are *equivalent* if there is a homeomorphism h: $S \to S$ such that $h\tau_2 = \tau_1$.

The *minimum genus* $\gamma(G)$ of a graph G is the minimum genus of the orientable surface into which G has an embedding. Determining the minimum genus of a graph is NP-complete as shown by Thomassen[13]. Some papers have given the genera of some graphs of good symmetry, such as [2,10-11], etc. Further more, White[15] obtained the minimum genus of cartesian products of bipartite graph $K_{2m,2m}$ with itself. Pisanski[9] studied repeated cartesian products of regular bipartite graphs. The methods used mainly involve quotient embeddings that are formed as voltage or current in the dual case. Till now, there is no approach to solve genus problem of general graphs. In 2003, Liu set up the joint tree model[6] of a graph embedding, such that a corresponding relation was established between the joint trees and the embeddings. By joint trees, some works have been done [3,4,12-13,16-17]. In this paper, on the basis of the joint tree model, by dividing the associated surfaces into segments layer by layer and doing a sequence of exchangers on it, we obtain the genera of two types of graphs, which are suspensions of cartesian products of two types of bipartite graphs from a vertex. And these two types of graphs concerned in this paper are apex graphs, which are the cartesian products of the planar graphs with a vertex. Mohar[8] obtained the linear lower bound of the genus for apex graphs by minimum face covers and proved that the minimum face cover problem is NP-hard for planar triangulations. Then computation of the genus of apex graphs is NP-hard. The approach used here provides a channel for solving the genus problem of some graphs.

Given a graph $G = (V, E)$ and a spanning tree T, the graph $(V, E \backslash E(T))$ is called the cotree of G. Split the cotree edge $(u[i], v[i])$ into two semi-edges i_u, i_v, which are, respectively, incident with $u[i]$ and $v[i]$ for $1 \leq i \leq \beta$(Betti number), to obtain a new tree $\hat{T} = (V + V_1, E(T) + E_1)$, where $E_1 = \{(u[i], v_i), (v[i], \bar{v_i}) | 1 \leq i \leq \beta\}$ and $V_1 = \{v_i, \bar{v_i} | 1 \leq i \leq \beta\}$. Let $\delta = (\delta_1, \delta_2, \cdots, \delta_\beta)$ be a binary vector. Write $\hat{T^\delta}$ instead of $\hat{T}$, when edges $(u[i], v_i)$ and $(v[i], \bar{v_i})$ are labeled by a same letter as a_i with indices:+(always omitted) or $-$ for $1 \leq i \leq \beta$, where $\delta_i = 0$ means that the two indices are the same, otherwise, distinct. A *rotation* σ_v at a vertex v is a cyclic permutation of edges incident with v. Let $\sigma_G = \Pi_{v\in V(G)}\sigma_v$ be a rotation system of G. The tree $\hat{T}^\delta$ with a rotation of G is called a *joint tree*[3] of G, denoted by $\hat{T}^\delta_\sigma$.

On $\hat{T}^\delta_\sigma$, according to a given orientation, list the letter a_i or a_i^- for $i \in \{1,2,\cdots,\beta\}$ to obtain a cyclic order of 2β letters, which represents a surface, and called an *associated surface*[7] of G. That two associated surfaces are *the same* is meant that they have the same cyclic order with the same δ. Otherwise, distinct. Then an embedding of a graph on a surface can be represented by an associated surface of it.

From[7], for a fixed spanning tree T of the graph G, there is a 1-to-1 correspondence between the associated surfaces and the embeddings of G.

The cartesian product of the graphs G and G' is denoted $G\times G'$ and defined to be the graph with vertex set $V_G \times V_{G'}$ and edge set $E_G\times V_{G'} \cup V_G \times E_{G'}$. If the edge $(e, v') \in E_G \times V_{G'}$, and if the endpoints of the edge e are v_1 and v_2, then the endpoints of the edge (e, v') are the vertices (v_1, v') and (v_2, v'). If the edge $(v, e') \in V_G \times E_{G'}$, and if the endpoints of the edge e' are v_1' and v_2', then the endpoints of the edge (v, e') are the vertices (v, v_1') and (v, v_2'). For example, for $(P_3 \times P_3) + v_0$, a spanning tree is presented with thick lines as shown in Fig.1.1 and a joint tree in Fig.1.2. Denote cotree edge $u_i^j u_{i+1}^j$ by a_i^j, $u_j^i u_j^{i+1}$ by b_j^i for $i = 1,2$ and $j = 1,2,3$. Let the joint tree have an anticlockwise rotation at each vertex. Then an associated surface of $(P_3 \times P_3) + v_0$ is shown as

$$S = b_1^1 a_1^1 a_1^{1-} b_2^1 a_2^1 a_2^{1-} b_3^1 b_3^{1-} a_2^2 b_3^2 b_3^{2-} a_2^3 a_2^{3-} b_2^2 a_1^3 a_1^{3-} b_1^2 b_1^{2-} b_1^{1-} a_1^2 a_1^{2-} b_2^{2-} b_2^{1-} a_2^{2-}.$$

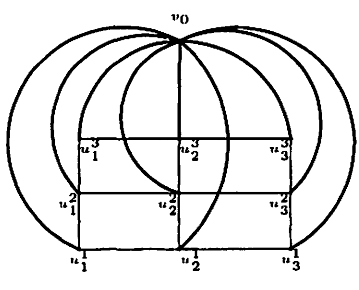

Fig.1.1 $(P_3 \times P_3) + v_0$

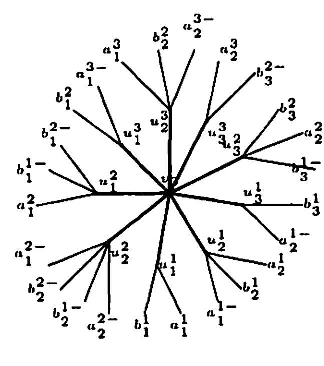

Fig.1.2 A joint tree of $(P_3 \times P_3) + v_0$

In order to obtain the associated surface of minimum genus, a layer division of an associated surface is defined for establishing an operation to transform this surface into another associated surface. For the descriptive convenience, firstly denote by $\langle S_v \rangle$ the associated surface S and call S_v a *layer segment*, where v is a vertex of the spanning tree T satisfying that $d_T(v) > 1$. Let $N(v) = \{v_1, v_2, \cdots, v_k\}$. Then by dividing S_v to obtain $S_v = \langle S_{v_1} S_{v_2} \cdots S_{v_k} \rangle$, where S_{v_j} is called a *layer segment* adjacent to vertex v for $1 \leq j \leq k$. Next $S_{v_j} = \langle S_{v_{j_1}} S_{v_{j_2}} \cdots S_{v_{j_t}} \rangle$, where $v_{j_l} \neq v$ is adjacent to the vertex v_j and $S_{v_{j_l}}$ is called a *layer segment* adjacent to vertex v_j for $1 \leq l \leq t$. Each layer segment except the layer segment S_{v_x} can be divided into other layer segments in the same way, where v_x is a vertex of T with $d_T(v_x) = 1$. According to a rotation system of G, the positions of any two layer segments adjacent to a same vertex in a layer division can be interchanged, and every interchange makes an associated surface S be transformed into another associated surface S'. So the operation in a layer division is called an *exchanger*[7], denoted by $S \to S'$. Hence the associated surface of minimum genus can be got by doing a sequence of exchangers on some associated surface.

As illustrated in the example above, let $S = \langle S_{v_0} \rangle$. Then

$$S = \langle S_{v_0} \rangle = \langle S_{u_1^1} S_{u_2^1} S_{u_3^1} S_{u_3^2} S_{u_3^3} S_{u_2^3} S_{u_1^3} S_{u_1^2} S_{u_2^2} \rangle$$

$$= \langle \langle b_1^1 a_1^1 \rangle \langle a_1^{1-} b_2^1 a_2^1 \rangle \langle a_2^{1-} b_3^1 \rangle \langle b_3^{1-} a_2^2 b_3^2 \rangle \langle b_3^{2-} a_2^3 \rangle \langle a_2^{3-} b_2^2 a_1^3 \rangle$$
$$\langle a_1^{3-} b_1^2 \rangle \langle b_1^{2-} b_1^{1-} a_1^2 \rangle \langle a_1^{2-} b_2^{2-} b_2^{1-} a_2^{2-} \rangle \rangle.$$

After doing an exchanger on S,

$$S \to S' = \langle \langle b_1^1 a_1^1 \rangle \langle a_1^{1-} b_2^1 a_2^1 \rangle \langle a_2^{1-} b_3^1 \rangle \langle b_3^{1-} a_2^2 b_3^2 \rangle \langle b_3^{2-} a_2^3 \rangle \langle a_2^{3-} b_2^2 a_1^3 \rangle$$
$$\langle a_1^{3-} b_1^2 \rangle \langle b_1^{2-} b_1^{1-} a_1^2 \rangle \langle a_1^{2-} b_2^{2-} a_2^{2-} b_2^{1-} \rangle \rangle,$$

namely an associated surface of minimum genus.

2. The minimum genus of $(P_n \times P_r) + v_0$

Let $P_n \times P_r$ be the cartesian product of the path on n vertices with the path on r vertices and $v_0 \notin P_{n,r}$. The suspension, denoted by $(P_n \times P_r) + v_0$, of $P_n \times P_r$ from v_0 is obtained by adjoining every vertex of $P_n \times P_r$ to v_0. Suppose that $P_n^{(i)} = u_1^i u_2^i \cdots u_n^i$ be a path on n vertices, of graph $P_n \times P_r$ for $1 \leq i \leq r$. Denote $u_j^i u_{j+1}^i$ by a_j^i, $u_k^s u_k^{s+1}$ by b_k^s, where all letters are distinct, $1 \leq i \leq r$, $1 \leq k \leq n$, $1 \leq s \leq r-1$ and $1 \leq j \leq n-1$.

For descriptive convenience, in the following, we omit the indices in the course of expression. But note that two indices of each letter appearing on each surface are distinct. And we denote by $P_{n,r}(H_{n,r})$ the graph $(P_n \times P_r) + v_0((C_n \times P_r) + v_0)$.

Let $A_i^{(n-2)} = A_0^{(i)} A_1^{(i)} \cdots A_{n-2}^{(i)}$, where

$A_j^{(i)} = \langle a_j^{i+1} b_{j+1}^{i+1} b_{j+1}^{i} a_{j+1}^{i+1} \rangle$, $b_k^0 = 0$, $0 \le j \le n-2$, $1 \le k \le n$ and $0 \le i \le r-1$.

Let $M = \langle a_{n-1}^1 b_n^1 \rangle \langle b_n^1 a_{n-1}^2 b_n^2 \rangle \langle b_n^2 a_{n-1}^3 b_n^3 \rangle \cdots \langle b_n^{r-2} a_{n-1}^{r-1} b_n^{r-1} \rangle \langle b_n^{r-1} a_{n-1}^r \rangle$,

$\underline{A}_{r-1}^{(n-2)} - (b_1^r, b_2^r, \cdots, b_{n-2}^r, b_{n-1}^r) = (A_0^{(r-1)} - b_1^r)(A_1^{(r-1)} - b_2^r) \cdots (A_{n-2}^{(r-1)} - b_{n-1}^r)$,

$\underline{B}_i^{(n-2)} = B_0^{(i)} B_1^{(i)} \cdots B_{n-2}^{(i)}$, where

$$B_j^{(i)} = \begin{cases} b_1^{2\lceil \frac{i+1}{2} \rceil - 1}, & j = 0; \\ \langle b_{j+1}^{i+1} b_{j+1}^{i} \rangle, & 1 \le j \le n-3; \\ \langle a_{n-1}^{i+1} b_{n-1}^{i} \rangle, & j = n-2, i \text{ is even}; \\ \langle b_{n-1}^{i+1} a_{n-1}^{i+1} \rangle, & j = n-2, i \ge 3 \text{ is odd}; \\ \langle b_{n-1}^{2} b_{n-1}^{1} a_{n-1}^{2} \rangle, & j = n-2, i = 1. \end{cases}$$

Theorem 2.1 $\gamma((P_n \times P_r) + v_0) = \lceil \frac{n-2}{2} \rceil \lceil \frac{r-2}{2} \rceil, \quad n, r \ge 2.$

Proof For $P_{n,r}$, choose all edges incident with v_0 to obtain a spanning tree. Then an associated surface of $P_{n,r}$(r is odd) can be shown as

$$S_1 = \underline{A}_0^{(n-2)} M (\overline{\underline{A}}_{r-1}^{(n-2)} - (b_{n-1}^r, \cdots, b_1^r)) \underline{A}_{r-2}^{(n-2)} \overline{\underline{A}}_{r-3}^{(n-2)} \cdots \underline{A}_3^{(n-2)} \overline{\underline{A}}_2^{(n-2)} \underline{A}_1^{(n-2)}$$

$$\sim \underline{B}_0^{(n-2)} (a_{n-1}^2 a_{n-1}^3 \cdots a_{n-1}^{r-2} a_{n-1}^{r-1}) (\overline{\underline{B}}_{r-1}^{(n-2)} - (b_{n-1}^r, b_{n-2}^r, \cdots, b_2^r, b_1^r b_1^{r-1})) \underline{B}_{r-2}^{(n-2)} \overline{\underline{B}}_{r-3}^{(n-2)} \cdots \underline{B}_3^{(n-2)} \overline{\underline{B}}_2^{(n-2)} \underline{B}_1^{(n-2)}$$

$$\to \underline{B}_0^{(n-4)} b_{n-2}^1 b_{n-1}^1 (a_{n-1}^2 a_{n-1}^3 \cdots a_{n-1}^{r-2} a_{n-1}^{r-1}) b_{n-1}^{r-1} b_{n-2}^{r-1} (\overline{\underline{B}}_{r-1}^{(n-4)} - (b_{n-3}^r, b_{n-2}^r, \cdots, b_2^r, b_1^r b_1^{r-1})) \underline{B}_{r-2}^{(n-4)} b_{n-2}^{r-2} b_{n-2}^{r-1} b_{n-1}^{r-1} a_{n-1}^{r-1} a_{n-1}^{r-2} b_{n-1}^{r-3} b_{n-2}^{r-3} b_{n-2}^{r-2} \overline{\underline{B}}_{r-3}^{(n-4)} \underline{B}_{r-4}^{(n-4)} b_{n-2}^{r-4} b_{n-2}^{r-3} b_{n-1}^{r-3} a_{n-1}^{r-3} \cdots \underline{B}_3^{(n-4)} b_{n-2}^3 b_{n-2}^4 b_{n-1}^4 a_{n-1}^4 a_{n-1}^3 b_{n-1}^2 b_{n-2}^2 b_{n-2}^3 \overline{\underline{B}}_2^{(n-4)} \underline{B}_1^{(n-4)} b_{n-2}^2 b_{n-2}^1 b_{n-1}^2 a_{n-1}^2 b_{n-1}^1$$

$$\sim \underline{B}_0^{(n-4)} b_{n-2}^1 b_{n-1}^1 f_{\frac{r-3}{2}} f_{\frac{r-5}{2}} \cdots f_2 f_1 b_{n-1}^{r-1} (\overline{\underline{B}}_{r-1}^{(n-4)} - (b_{n-3}^r, b_{n-2}^r, \cdots, b_2^r, b_1^r b_1^{r-1})) \underline{B}_{r-2}^{(n-4)} b_{n-2}^{r-2} b_{n-1}^{r-1} f_1 b_{n-1}^{r-3} b_{n-2}^{r-2} \overline{\underline{B}}_{r-3}^{(n-4)} \underline{B}_{r-4}^{(n-4)} b_{n-2}^{r-4} b_{n-1}^{r-3} f_2 \cdots \underline{B}_3^{(n-4)} b_{n-2}^3 b_{n-1}^4 f_{\frac{r-3}{2}} b_{n-1}^2 b_{n-2}^3 \overline{\underline{B}}_2^{(n-4)} \underline{B}_1^{(n-4)} b_{n-2}^1 b_{n-1}^2 b_{n-1}^1$$

$$\sim \underline{B}_0^{(n-4)} b_{n-2}^1 b_{n-1}^1 f_{\frac{r-5}{2}} \cdots f_2 f_1 b_{n-1}^{r-1} b_{n-1}^{r-1} (\overline{\underline{B}}_{r-1}^{(n-4)} - (b_{n-3}^r, b_{n-2}^r, \cdots, b_2^r, b_1^r b_1^{r-1})) \underline{B}_{r-2}^{(n-4)} b_{n-2}^{r-2} b_{n-1}^{r-1} f_1 b_{n-1}^{r-3} b_{n-2}^{r-2} \overline{\underline{B}}_{r-3}^{(n-4)} \underline{B}_{r-4}^{(n-4)} b_{n-2}^{r-4} b_{n-1}^{r-3} f_2 \cdots$$

$$\underline{B}_3^{(n-4)}\overline{\underline{B}}_2^{(n-4)}\underline{B}_1^{(n-4)} b_{n-2}^1 b_{n-1}^4 b_{n-1}^1 O_1$$

$$\sim \underline{B}_0^{(n-4)}(\overline{\underline{B}}_{r-1}^{(n-4)} - (b_{n-3}^r, b_{n-2}^r, \cdots, b_2^r, b_1^r b_1^{r-1}))\underline{B}_{r-2}^{(n-4)}\overline{\underline{B}}_{r-3}^{(n-4)}\underline{B}_{r-4}^{(n-4)}\cdots$$

$$\underline{B}_3^{(n-4)}\overline{\underline{B}}_2^{(n-4)}\underline{B}_1^{(n-4)} O_{\frac{r-1}{2}}$$

By applying Lemma1.1, when r is odd,

$$\gamma(P_{n,r}) \leq \gamma(P_{n-2,r}) + \frac{r-1}{2}.$$

Similarly, when r is even,

$$\gamma(P_{n,r}) \leq \gamma(P_{n-2,r}) + \frac{r-2}{2}.$$

Therefore for $n, r \geq 2$,

$$\gamma(P_{n,r}) \leq \gamma(P_{n-2,r}) + \lceil \frac{r-2}{2} \rceil \leq \lceil \frac{n-2}{2} \rceil \lceil \frac{r-2}{2} \rceil.$$

On the other hand, consider the path $Q_{n-2}^{(i)} = u_2^i u_3^i u_4^i \cdots u_{n-2}^i u_{n-1}^i$ for $2 \leq i \leq r-2$. The vertices $u_j^i u_{j+1}^i u_{j+2}^i (2 \leq j \leq n-3)$ can not be on a same face. Otherwise, without loss of generality, suppose that u_2^2, u_3^2, u_4^2 are on some face. Connecting $u_3^2 u_3^3 (u_3^2 u_3^1)$ by an edge must lead to a non-planar graph, see Fig.2.1. It is contrary to that $P_{n,r}$ is planar.

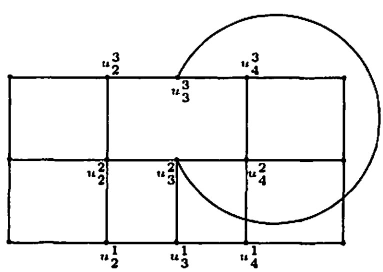

Fig.2.1 $P_4 \times P_3$

Therefore

$$\gamma(P_{2k+1,r}) \geq \gamma(P_{2k+2,r}).$$

Since $P_{2k+1,r} \subseteq P_{2k+2,r}$, $\gamma(P_{2k+2,r}) \geq \gamma(P_{2k+1,r})$.
Therefore

$$\gamma(P_{2k+2,r}) = \gamma(P_{2k+1,r}).$$

We prove $\gamma(P_{2k,r}) \geq \lceil \frac{2k-2}{2} \rceil \lceil \frac{r-2}{2} \rceil$ by induction on k. The result is true when $k = 1$, since $\gamma(P_{2,r}) = 0$. Suppose that it holds for any $m < k$. And apparently,

$$P_{3,r} \cup P_{2k-3,r} \subseteq P_{2k,r},$$

Then

$$\begin{aligned}\gamma(P_{2k,r}) &\geq \gamma(P_{3,r}) + \gamma(P_{2k-3,r}) \\ &= \gamma(P_{4,r}) + \gamma(P_{2k-2,r}) \\ &\geq \lceil \frac{r-2}{2} \rceil + \lceil \frac{2k-4}{2} \rceil \lceil \frac{r-2}{2} \rceil \\ &= \lceil \frac{2k-2}{2} \rceil \lceil \frac{r-2}{2} \rceil.\end{aligned}$$

And

$$\begin{aligned}\gamma(P_{2k-1,r}) = \gamma(P_{2k,r}) &\geq \lceil \frac{2k-2}{2} \rceil \lceil \frac{r-2}{2} \rceil \\ &= \lceil \frac{2k-3}{2} \rceil \lceil \frac{r-2}{2} \rceil.\end{aligned}$$

So for each $n \geq 2$,

$$\gamma(P_{n,r}) \geq \lceil \frac{n-2}{2} \rceil \lceil \frac{r-2}{2} \rceil.$$

Therefore for $n, r \geq 2$,

$$\gamma(P_{n,r}) = \lceil \frac{n-2}{2} \rceil \lceil \frac{r-2}{2} \rceil. \square$$

3. The minimum genus of $(C_n \times P_r) + v_0$

Let $C_n \times P_r$ be the cartesian product of a n-cycle and a path on r vertices and $v_0 \notin C_n \times P_r$. Firstly, consider $(C_n \times P_{2k+1}) + v_0$. Suppose that $C_n^{(i)} = u_1^i u_2^i \cdots u_{n-1}^i u_n^i u_1^i$ be the $n-$ cycle for $1 \leq i \leq 2k+1$. Denote $u_j^i u_{j+1}^i$ by a_{j+1}^i, $u_n^i u_1^i$ by a_n^i $(1 \leq i \leq 2k+1, 1 \leq j \leq n-1)$, $u_t^s u_t^{s+1}$ by a_t^s $(1 \leq s \leq 2k, 1 \leq t \leq n)$, where all letters are distinct.

Let $C_i^{(2k+1)} = C_0^{(i)} C_1^{(i)} \cdots C_{2k}^{(i)}$, $D_i^{(2k+1)} = D_0^{(i)} D_1^{(i)} \cdots D_{2k}^{(i)}$, where

$$C_j^{(i)} = \begin{cases} \langle b_i^j a_i^{j+1} a_{i+1}^{j+1} b_i^{j+1} \rangle, & 1 \leq i \leq n-1,\ 0 \leq j \leq 2k; \\ \langle b_n^j a_1^{j+1} a_n^{j+1} b_n^{j+1} \rangle, & i = n,\ 0 \leq j \leq 2k. \end{cases}$$

$b_i^{2k+1} = b_i^0 = 0 (1 \le i \le n)$.

$$D_j^{(i)} = \begin{cases} a^{2k+1}_{2\lceil \frac{i+1}{2} \rceil - 1}, & 1 \le i \le n-1,\ j = 2k; \\ a_1^{2k+1}, & i = n,\ j = 2k; \\ a^1_{2\lceil \frac{i}{2} \rceil}, & 1 \le i \le n,\ j = 0; \\ \langle a_i^{j+1} a_{i+1}^{j+1} \rangle, & 1 \le i \le n-1,\ 1 \le j \le 2k-1; \\ \langle a_1^{j+1} a_n^{j+1} \rangle, & i = n,\ 1 \le j \le 2k-1. \end{cases}$$

Theorem 3.1 $\gamma((C_n \times P_{2k+1}) + v_0) = k\lceil \frac{n-2}{2} \rceil, \quad k \ge 1,\ n \ge 2.$

Proof For $H_{n,2k+1}$, choose all edges incident with v_0 to obtain a spanning tree. Then an associated surface of $H_{n,2k+1}$ can be shown as

$$S_2 = C_1^{(2k+1)} C_2^{(2k+1)} C_3^{(2k+1)} \cdots C_{n-2}^{(2k+1)} C_{n-1}^{(2k+1)} C_n^{(2k+1)}$$

$$\sim D_1^{(2k+1)} D_2^{(2k+1)} D_3^{(2k+1)} \cdots D_{n-2}^{(2k+1)} D_{n-1}^{(2k+1)} D_n^{(2k+1)}$$

$$D_{n-2}^{(2k+1)} D_{n-1}^{(2k+1)} D_n^{(2k+1)} \to a_{n-1}^{2k+1} a_{n-1}^{2k} a_{n-2}^{2k} a_{n-2}^{2k-1} a_{n-1}^{2k-1} a_{n-1}^{2k-2} a_{n-2}^{2k-2} \cdots a_{n-2}^{3}$$
$$a_{n-1}^{3} a_{n-1}^{2} a_{n-2}^{2} a_{n-2}^{1} a_{n}^{1} a_{n}^{2} a_{n-1}^{2} a_{n-1}^{3} a_{n}^{3} \cdots a_{n}^{2k-2} a_{n-1}^{2k-2} a_{n-1}^{2k-1} a_{n}^{2k-1} a_{n}^{2k} a_{n-1}^{2k}$$
$$a_{n-1}^{2k+1} a_{1}^{2k+1} a_{1}^{2k} a_{n}^{2k} a_{n}^{2k-1} a_{1}^{2k-1} a_{1}^{2k-2} a_{n}^{2k-2} \cdots a_{n}^{3} a_{1}^{3} a_{1}^{2} a_{n}^{2} a_{n}^{1}$$

$$\sim a_{n-1}^{2k+1} a_{n-2}^{2k} a_{n-2}^{2k-1} a_{n-1}^{2k-1} a_{n-2}^{2k-2} a_{n-2}^{2k-3} a_{n-1}^{2k-3} a_{n-2}^{2k-4} a_{n-2}^{2k-5} a_{n-1}^{2k-5} \cdots a_{n-1}^{6} a_{n-2}^{6}$$
$$a_{n-2}^{5} a_{n-1}^{4} a_{n-2}^{4} a_{n-2}^{3} a_{n-1}^{3} a_{n-2}^{2} a_{n-2}^{1} a_{n}^{1} a_{n-1}^{2} a_{n}^{3} a_{n-1}^{4} a_{n}^{5} a_{n-1}^{6} a_{n}^{7} \cdots a_{n}^{2k-4}$$
$$a_{n-1}^{2k-3} a_{n}^{2k-2} a_{n-1}^{2k-1} a_{n}^{2k} a_{n-1}^{2k+1} a_{1}^{2k+1} a_{1}^{2k} a_{n}^{2k} a_{1}^{2k-1} a_{1}^{2k-2} a_{n}^{2k-2} a_{1}^{2k-3}$$
$$a_{1}^{2k-4} a_{n}^{2k-4} a_{1}^{2k-5} \cdots a_{1}^{6} a_{n}^{5} a_{1}^{5} a_{1}^{4} a_{n}^{3} a_{1}^{3} a_{1}^{2} a_{n}^{1}$$

$$\sim a_{1}^{2k+1} a_{1}^{2k} a_{n-2}^{2k} a_{n-2}^{2k-1} a_{n-1}^{2k-1} a_{n-2}^{2k-2} a_{n-2}^{2k-3} a_{n-1}^{2k-3} a_{n-2}^{2k-4} a_{n-2}^{2k-5} a_{n-1}^{2k-5} \cdots a_{n-1}^{6} a_{n-2}^{6}$$
$$a_{n-2}^{5} a_{n-1}^{4} a_{n-2}^{4} a_{n-2}^{3} a_{n-1}^{3} a_{n-2}^{2} a_{n-2}^{1} a_{n}^{1} a_{n-1}^{2} a_{n}^{3} a_{n-1}^{4} a_{n}^{5} a_{n-1}^{6} a_{n}^{7} \cdots a_{n}^{2k-4} a_{n-1}^{2k-3}$$
$$a_{n}^{2k-2} a_{n-1}^{2k-1} a_{1}^{2k-1} a_{1}^{2k-2} a_{n}^{2k-2} a_{1}^{2k-3} a_{1}^{2k-4} a_{n}^{2k-4} a_{1}^{2k-5} \cdots a_{n}^{5} a_{1}^{5} a_{1}^{4} a_{n}^{3} a_{1}^{3} a_{1}^{2} a_{n}^{1} O_1$$

$$\sim \cdots\cdots$$

$$\sim a_{1}^{2k+1} a_{1}^{2k} a_{n-2}^{2k} a_{n-2}^{2k-1} a_{1}^{2k-1} a_{1}^{2k-2} a_{n-2}^{2k-2} a_{n-2}^{2k-3} \cdots a_{1}^{5} a_{1}^{4} a_{n-2}^{4} a_{n-2}^{3} a_{1}^{3} a_{1}^{2} a_{n-2}^{1} O_k$$

Thus

$$S_2 \sim D_1^{(2k+1)} D_2^{(2k+1)} D_3^{(2k+1)} \cdots D_{n-3}^{(2k+1)} D_{n-2}^{(2k+1)} O_k.$$

From Lemma 1.1,

$$\gamma(H_{n,2k+1}) \le \gamma(H_{n-2,2k+1}) + k \le k\lceil \frac{n-2}{2} \rceil.$$

Since $H_{n,2k+1} \supseteq P_{n,2k+1}$,

$$\begin{aligned}\gamma(H_{n,2k+1}) &\geq \gamma(P_{n,2k+1}) \\ &= \lceil \frac{n-2}{2} \rceil \lceil \frac{2k+1-2}{2} \rceil \\ &= \lceil \frac{n-2}{2} \rceil \lceil \frac{2k-1}{2} \rceil \\ &= k \lceil \frac{n-2}{2} \rceil.\end{aligned}$$

Therefore

$$\gamma(H_{n,2k+1}) \geq \gamma(P_{n,2k+1}) = k \lceil \frac{n-2}{2} \rceil. \quad \square$$

Theorem 3.2

$\gamma((C_n \times P_2) + v_0) = 1, \gamma((C_2 \times P_r) + v_0) = 0$ for $n \geq 3, r \geq 1$.

Similar to the proof of Theorem 2.1,

$$\gamma(H_{2l+1,r}) = \gamma(H_{2l+2,r}), \ r \geq 3.$$

Let

$$E_j^{(n-1)} = E_1^{(j)} E_2^{(j)} \cdots E_{n-1}^{(j)}, \quad F_j^{(n-1)} = F_1^{(j)} F_2^{(j)} \cdots F_{n-1}^{(j)},$$

where

$$E_s^{(j)} = \begin{cases} \langle a_s^j b_s^{j-1} b_s^j a_{s+1}^j \rangle, & 2 \leq s \leq n-2; \\ \langle a_{n-1}^j b_{n-1}^{j-1} a_n^j b_{n-1}^j \rangle, & s = n-1, \ j \text{ is odd}; \\ \langle a_{n-1}^j b_{n-1}^j a_n^j b_{n-1}^{j-1} \rangle, & s = n-1, \ j \text{ is even}; \\ \langle b_1^{j-1} a_1^j b_1^j a_2^j \rangle, & s = 1, \ j \text{ is odd}; \\ \langle b_1^j a_1^j b_1^{j-1} a_2^j \rangle, & s = 1, \ j \text{ is even}. \end{cases}$$

$$F_s^{(j)} = \begin{cases} \langle b_s^{j-1} b_s^j \rangle, & 2 \leq s \leq n-2; \\ b_1^1, & s = 1, \ j = 1; \\ \langle a_1^j b_1^j \rangle, & s = 1, \ j \geq 3 \text{ is odd}; \\ \langle a_1^j b_1^{j-1} \rangle, & s = 1, \ j \text{ is even}; \\ \langle b_{n-1}^{j-1} a_n^j \rangle, & s = n-1, \ j \text{ is odd}; \\ \langle b_{n-1}^j a_n^j \rangle, & s = n-1, \ j \text{ is even}. \end{cases}$$

$$1 \leq j \leq 2k, b_s^0 = b_n^{2k} = 0.$$

Let

$$G_{n-1}^{(t)} = b_2^{x_t} b_3^{y_t} b_4^{x_t} b_5^{y_t} \cdots b_{n-2}^{x_t} b_{n-1}^{y_t}, H_{n-2}^{(i)} = b_2^i b_4^i b_6^i \cdots b_{n-4}^i b_{n-2}^i.$$

where $x_t = 2\lceil \frac{t-1}{2} \rceil + 1$, $y_t = 2\lceil \frac{t}{2} \rceil$ for $1 \le t \le 2k-2$.

Theorem 3.3
$(k-1)\lceil \frac{n-2}{2} \rceil \le \gamma((C_n \times P_{2k}) + v_0) \le \begin{cases} k\lceil \frac{n-2}{2} \rceil, & 3 \le n \le 2k; \\ (k-1)\lceil \frac{n}{2} \rceil, & n \ge 2k+1. \end{cases}$,
$k \ge 2$.

Proof An associated surface of $H_{n,2k}(n \ge 2k+1)$ can be shown as

$$S_3 = E_1^{(n-1)} E_2^{(n-1)} E_3^{(n-1)} \cdots \underline{E}_{2k-1}^{(n-1)} \overline{\underline{E}}_{2k}^{(n-1)} \overline{\underline{C}}_n^{(2k)}$$
$$\sim \underline{E}_1^{(n-1)} \overline{\underline{E}}_2^{(n-1)} \underline{E}_3^{(n-1)} \cdots \underline{E}_{2k-1}^{(n-1)} \overline{\underline{E}}_{2k}^{(n-1)} \overline{\underline{D}}_n^{(2k)}$$
$$\to H_{n-2}^{(1)} a_n^1 G_{n-1}^{(1)} a_1^3 G_{n-1}^{(2)} a_n^4 G_{n-1}^{(3)} a_1^5 G_{n-1}^{(4)} \cdots G_{n-1}^{(2k-4)} a_n^{2k-2} G_{n-1}^{(2k-3)}$$
$$a_1^{2k-1} G_{n-1}^{(2k-2)} a_n^k H_{n-2}^{(2k-1)} a_n^{2k} a_1^{2k-1} a_n^{2k-2} a_1^{2k-3} a_n^{2k-4} \cdots a_1^5 a_n^4 a_1^3 a_n^1$$
$$\sim H_{n-2}^{(1)} a_n^1 G_{n-1}^{(1)} a_1^3 G_{n-3}^{(2)} a_1^5 G_{n-1}^{(4)} \cdots G_{n-3}^{(2k-4)} G_{n-3}^{(2k-3)} a_1^{2k-1} G_{n-3}^{(2k-2)}$$
$$H_{n-4}^{(2k-1)} b_{n-1}^{2k-2} a_1^{2k-1} b_{n-1}^{2k-2} b_{n-1}^{2k-4} a_1^{2k-3} a_n^{2k-4} \cdots a_n^6 a_1^5 b_{n-1}^4 b_{n-1}^2 a_1^3 a_n^1 O_{k-1}$$
$$\sim \cdots\cdots \sim O_{(k-1)\lceil \frac{n}{2} \rceil + 1}$$

So for $n \ge 2k+1$,

$$\gamma(H_{n,2k}) \le (k-1)\lceil \frac{n}{2} \rceil + 1.$$

Since $H_{n,2k} \subseteq H_{n,2k+1}$ for $3 \le n \le 2k$,

$$\gamma(H_{n,2k}) \le \gamma(H_{n,2k+1}) \le k\lceil \frac{n-2}{2} \rceil.$$

Since $H_{n,2k-1} \subseteq H_{n,2k}$ for $n \ge 3$,

$$\gamma(H_{n,2k}) \ge \gamma(H_{n,2k-1}) \ge (k-1)\lceil \frac{n-2}{2} \rceil.$$

This completes the proof. □

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