SEMISYMMETRIC CUBIC GRAPHS OF ORDER $10p^n$

A. BABAI[1] & B. KHOSRAVI[2]

ABSTRACT. A regular graph Γ is said to be semisymmetric if its full automorphism group acts transitively on its edge set but not on its vertex set. Some authors classified semisymmetric cubic graphs of orders $10p$ and $10p^2$. Also it is proved that there is no connected semisymmetric cubic graph of order $10p^3$. In this paper, we continue this work and prove that there is no connected semisymmetric cubic graph of order $10p^n$, where $n \geq 4$, $p \geq 7$ and $p \neq 11$.

1. Introduction

Let Γ be a finite, simple, connected and undirected graph. We denote by $V(\Gamma)$, $E(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ its vertex set, edge set and full automorphism group, respectively. For $u, v \in V(\Gamma)$, $u \sim v$ means that u is adjacent to v. Let $G \leq \mathrm{Aut}(\Gamma)$. If G acts transitively on $V(\Gamma)$ we say that Γ is *G-vertex-transitive*, and if G acts transitively on $E(\Gamma)$ we say that Γ is *G-edge-transitive*. In the special case, when $G = \mathrm{Aut}(\Gamma)$, we say that Γ is vertex-transitive and Γ is edge-transitive, respectively. A regular graph is *G-semisymmetric* if it is G-edge-transitive but not G-vertex-transitive. It can be shown that a G-edge but not G-vertex-transitive graph Γ is bipartite, where the bipartition sets are orbits of G. So G is transitive on each bipartition set. Moreover, if Γ is regular, then these two parts have equal cardinality.

Let Γ be a graph and $u \in V(\Gamma)$. Then $N_\Gamma(u)$ denotes the set of vertices adjacent to u in Γ which is called *the neighborhood* of u in Γ. Let N be a subgroup of $\mathrm{Aut}(\Gamma)$. *The quotient graph* Γ_N or Γ/N of Γ relative to N is defined as the graph such that the set Σ of N-orbits in $V(\Gamma)$ is the vertex set of Γ/N and $B, C \in \Sigma$ are adjacent if and only if there exist $u \in B$ and $v \in C$ such that $u \sim v$ in Γ.

A graph $\tilde{\Gamma}$ is called a *covering* of graph Γ with projection $\wp : \tilde{\Gamma} \to \Gamma$, if $\wp$ is a surjection from $V(\tilde{\Gamma})$ to $V(\Gamma)$ such that $\wp|_{N_{\tilde{\Gamma}}(\tilde{v})} : N_{\tilde{\Gamma}}(\tilde{v}) \to N_\Gamma(v)$

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is a bijection for any vertices $v \in V(\Gamma)$ and $\tilde{v} \in \wp^{-1}(v)$. The *fibre* of an edge or a vertex is its preimage under $\wp$. If $\tilde{\Gamma}$ is connected, then any vertex or edge fibres are of the same cardinality n. This number is called *the fold number* of the covering, and we say that $\wp$ is an n-fold covering. A covering $\tilde{\Gamma}$ of Γ with a projection $\wp$ is said to be *regular* (or K-covering) if there is a semiregular subgroup K of the automorphism group $\mathrm{Aut}(\tilde{\Gamma})$ such that graph Γ is isomorphic to the quotient graph $\tilde{\Gamma}/K$, say by h, and the quotient map $\tilde{\Gamma} \to \tilde{\Gamma}/K$ is the composition $\wp h$ of $\wp$ and h.

The study of semisymmetric graphs was started by Folkman [12]. He proved that there are no semisymmetric graphs of order $2p$ or $2p^2$ for p a prime. Also he posed a number of problems which spurred the interest in this topic (see [8, 11, 13, 16]). Semisymmetric cubic graphs of orders $2p^3$ and $6p^2$ are classified in [14, 15]. Also in [2, 5] it is proved that every cubic edge-transitive graph of order $8p$ or $8p^2$ is vertex-transitive. In [6], the semisymmetric cubic graphs of order $16p^2$ are determined. It is proved that there is no semisymmetric cubic graphs of order $4p^n$ or $34p^2$ (see [1, 4]). It is given an overview of known families of semisymmetric cubic graphs in [10] and the semisymmetric cubic graphs of order up to 768 are determined. We know that the orders of smallest semisymmetric cubic graphs are 54, 110, 112 and 126.

Semisymmetric cubic graphs of orders $10p$ and $10p^2$ are classified in [11, 17]. Also it is proved that there is no semisymmetric cubic graph of order $10p^3$ (see [3]). In this paper we continue this work and show that there is no semisymmetric cubic graph of order $10p^n$, where $n \geq 4$, $p \geq 7$ and $p \neq 11$.

Throughout the paper we use the classification of finite simple groups. Also we denote by $\pi(n)$ the set of all prime divisors of n. If G is a finite group, then $\pi(|G|)$ is denoted by $\pi(G)$.

2. Preliminary Results

Let Γ be a connected G-semisymmetric graph of degree d with bipartition sets $U(\Gamma)$ and $W(\Gamma)$, where $G \leq \mathrm{Aut}(\Gamma)$.

Lemma 2.1. ([14, Lemma 3.1]) *Let $\{B_1, B_2, \ldots, B_s\}$ be a complete imprimitive block system of action of G on $U(\Gamma)$. There exists a positive constant integer m_U such that*

$$|N_\Gamma(w) \cap B_i| = 0, \text{or } m_U, \quad \forall w \in W(\Gamma), \quad 1 \leq i \leq s,$$

and m_U is a divisor of d.

Lemma 2.2. ([14, Lemma 3.2]) *Suppose that $\{B_1, B_2, \ldots, B_s\}$ is the set of all orbits of a normal subgroup N of G on $U(\Gamma)$.*

(1) *If $s > 1$ and d is a prime, then $m_U = 1$;*

(2) *If the action of N on $W(\Gamma)$ is transitive, then $d|B_i| = nm_U$, where $|U(\Gamma)| = n$;*

(3) *If the action of N on $W(\Gamma)$ is intransitive, let $\Omega = \{C_1, C_2, \ldots, C_t\}$ be the set of all orbits of N on $W(\Gamma)$, then there exists a positive constant integer m_W such that*

$$|N_\Gamma(u) \cap C_i| = 0, \text{or } m_W, \quad \forall u \in U(\Gamma), \quad 1 \leq i \leq t,$$

and $|B_i|m_W = |C_j|m_U$. In particular, if $m_U = m_W = 1$, then N acts semiregularly on both $U(\Gamma)$ and $W(\Gamma)$, and Γ is an N-regular covering of a G/N-semisymmetric graph.

The next lemma is a special case of the above lemma.

Lemma 2.3. *Let Γ be a connected G-semisymmetric cubic graph with bipartition sets $U(\Gamma)$ and $W(\Gamma)$, where $G \leq \mathrm{Aut}(\Gamma)$. Moreover, suppose that N is a normal subgroup of G. If N is intransitive on bipartition sets, then N acts semiregularly on both $U(\Gamma)$ and $W(\Gamma)$, and Γ is an N-regular covering of a G/N-semisymmetric graph.*

Lemma 2.4. ([15, Proposition 2.4]) *The vertex stabilizers of a connected G-edge-transitive cubic graph X have order $2^r \cdot 3$, $r \geq 0$. Moreover, if u and v are two adjacent vertices, then $|G : \langle G_u, G_v \rangle| \leq 2$ and the edge stabilizer $G_u \cap G_v$ is a common Sylow 2-subgroup of G_u and G_v.*

Lemma 2.5. ([7]) *If $\tilde{\Gamma}$ is a bipartite covering of a non-bipartite graph Γ, then the fold number is even.*

3. Main Results

Theorem 3.1. *Let Γ be a connected semisymmetric cubic graph of order $10p^n$, where $n \geq 4$, $p \geq 7$ and $p \neq 11$. Let $A = \mathrm{Aut}(\Gamma)$, and $Q = O_p(A)$ be the maximal normal p-subgroup of A. Then $|Q| = p^n$.*

Proof. Let Γ be a connected semisymmetric graph of order $10p^n$ with bipartition sets $U(\Gamma)$ and $W(\Gamma)$. So we have $|U(\Gamma)| = |W(\Gamma)| = 5p^n$. Also we know that A acts transitively on $U(\Gamma)$. Therefore $|A| = 2^r \cdot 3 \cdot 5 \cdot p^n$ where $r \geq 0$, by Lemma 2.4. We will show that $|Q| = p^n$.

First, suppose that $|Q| = 1$. Let N be a minimal normal subgroup of A. Therefore N is a characteristically simple group and so $N \cong T^k$, where T is a simple subgroup of N. Let T be a nonabelian simple group. We know that the order of each nonabelian finite simple group T has at least three prime divisors, containing 2. Since $\pi(T) \subseteq \pi(A) = \{2, 3, 5, p\}$ and $\pi(Sz(q)) \nsubseteq \pi(A)$, where $q = 2^{2m+1}$, it follows that $T \ncong Sz(q)$. Therefore $3 \mid |T|$. By the order of A, we have $k = 1$, so $N = T$. Therefore by the classification of $\{2, 3, 5, p\}$-simple groups [9], $N \cong \mathrm{PSL}(2,m)$, where $m \in \{5, 7, 16, 31\}$. If N acts transitively on $U(\Gamma)$ or $W(\Gamma)$, then $p^n \mid |N|$

and we get a contradiction, since $n \geq 4$. Therefore N acts intransitively and so N acts semiregularly, by Lemma 2.3. Therefore there exists $k \in \mathbb{N}$, such that $|U(\Gamma)| = 5p^n = k|N|$, which is a contradiction, since $|\pi(N)| \geq 3$. Therefore T is abelian and so N is elementary abelian.

If N acts transitively on $U(\Gamma)$ or $W(\Gamma)$, then $5p^n \mid |N|$, which is a contradiction. Therefore N acts intransitively on both $U(\Gamma)$ and $W(\Gamma)$. Then using Lemma 2.3, N acts semiregularly on both $U(\Gamma)$ and $W(\Gamma)$. Therefore, $|N| = 5, p, p^2, \ldots, p^{n-1}$ or p^n. Since $|Q| = 1$, hence $|N| = 5$. Then the quotient graph Γ_N is A/N-semisymmetric, by Lemma 2.3. Therefore $|U(\Gamma_N)| = |W(\Gamma_N)| = p^n$. Let M/N be a minimal normal subgroup of A/N. Similarly to the above M/N is a nonabelian simple group or an elementary abelian group. If M/N is a nonabelian simple group, then $\pi(M/N) = \{2, 3, p\}$ and so $M/N \cong \mathrm{PSL}(2, 7)$ and similarly to the above discussion we get a contradiction. Therefore M/N is an elementary abelian group.

- If M/N acts transitively on $U(\Gamma_N)$ or $W(\Gamma_N)$, then $p^n \mid |M/N|$. Therefore by the order of A/N, we have $|M/N| = p^n$, consequently $|M| = 5p^n$.
- If M/N acts intransitively on $U(\Gamma_N)$ and $W(\Gamma_N)$, then M/N acts semiregularly on both $U(\Gamma_N)$ and $W(\Gamma_N)$, by Lemma 2.3. Consequently, there exists $1 \leq \alpha < n$ such that $|M/N| = p^\alpha$, since $|U(\Gamma_N)| = p^n$. Hence $|M| = 5p^\alpha$.

Let $P \in Syl_p(M)$, so $|P| = p^\beta$, where $\beta \in \{n, \alpha\}$. By Sylow theorem we have $P \trianglelefteq M$, and so P is characteristic in M. Therefore P is normal in A, which is a contradiction.

Now assume that $|Q| = p^s$, where $1 \leq s \leq n-2$. We know that Q acts intransitively on $U(\Gamma)$ and $W(\Gamma)$, since $5p^n \nmid |Q|$. Therefore the quotient graph Γ_Q is an A/Q-semisymmetric graph, by Lemma 2.3. Therefore $|U(\Gamma_Q)| = |W(\Gamma_Q)| = 5p^{n-s}$. Let N/Q be a minimal normal subgroup of A/Q so $N/Q \cong (T/Q)^k$, where T/Q is simple.
Let T/Q be a nonabelian simple group. Therefore $N/Q \cong \mathrm{PSL}(2, m)$, where $m \in \{5, 7, 16, 31\}$. If N/Q acts intransitively on both $U(\Gamma_Q)$ and $W(\Gamma_Q)$ so N/Q acts semiregularly on both of them. Therefore there exists $t \in \mathbb{N}$, such that $|U(\Gamma_Q)| = t|N/Q|$, which is a contradiction. Consequently, N/Q acts transitively on $U(\Gamma_Q)$ or $W(\Gamma_Q)$, and so $p^{n-s} \mid |N/Q|$, we get a contradiction, by the order of N/Q.
Therefore T/Q is abelian and so N/Q is an elementary abelian group. If N/Q acts transitively on $U(\Gamma_Q)$ or $W(\Gamma_Q)$, then $5p^{n-s} \mid |N/Q|$, which is a contradiction. So N/Q acts semiregularly on both $U(\Gamma_Q)$ and $W(\Gamma_Q)$, by Lemma 2.3. So $|N/Q| = 5, p, p^2, \ldots, p^{n-s-1}$ or p^{n-s}. Since $|Q| = p^s$, $|N/Q| = 5$, and so $|N| = 5p^s$. Also we can see that the quotient graph Γ_N

is A/N-semisymmetric, by Lemma 2.3, because N acts intransitively on $U(\Gamma)$ and $W(\Gamma)$, since $|U(\Gamma)| \nmid |N|$. Therefore $|U(\Gamma_N)| = |W(\Gamma_N)| = p^{n-s}$.

Suppose that M/N is a minimal normal subgroup of A/N. Therefore $M/N \cong (T/N)^k$, where T/N is simple. If T/N is a nonabelian simple group, then $M/N \cong T/N$. Therefore $M/N \cong \mathrm{PSL}(2,7)$. Let M/N act transitively on $U(\Gamma_N)$ or $W(\Gamma_N)$. So $|U(\Gamma_N)| \mid |M/N|$, which is a contradiction. Therefore M/N acts semiregularly on both $U(\Gamma_N)$ and $W(\Gamma_N)$, hence there exists $t \in \mathbb{N}$, such that $t|M/N| = |U(\Gamma_N)|$, which is a contradiction. Therefore T/N is abelian and so M/N is an elementary abelian group.

- If M/N acts transitively on $U(\Gamma_N)$ or $W(\Gamma_N)$, then similarly to the above $|M/N| = p^{n-s}$, consequently $|M| = 5p^n$.
- If M/N acts intransitively on $U(\Gamma_N)$ and $W(\Gamma_N)$, so M/N acts semiregularly on both $U(\Gamma_N)$ and $W(\Gamma_N)$, by Lemma 2.3. Consequently, there exists $\alpha \in \mathbb{N}$ such that $1 \leq \alpha < n - s$ and $|M/N| = p^\alpha$, hence $|M| = 5p^{\alpha+s}$.

Let $P \in Syl_p(M)$. So $|P| = p^\beta$, where $\beta \in \{n, \alpha + s\}$. We can see that P is normal in A, by Sylow theorem, which is a contradiction.

Finally, let $|Q| = p^{n-1}$. Since we can show that Q acts intransitively on both $U(\Gamma)$ and $W(\Gamma)$, so by Lemma 2.3, the quotient graph Γ_Q is a cubic semisymmetric or symmetric graph of order $10p$, which is a contradiction by [10, 18], since $p \neq 11$. Therefore $|Q| = p^n$. □

Theorem 3.2. *Let $n \geq 4$, $p \geq 7$ and $p \neq 11$. There is no connected semisymmetric cubic graph of order $10p^n$.*

Proof. Let Γ be a semisymmetric cubic graph with two partition sets $U(\Gamma)$ and $W(\Gamma)$. Moreover let $Q = O_p(A)$, where $A = \mathrm{Aut}(\Gamma)$. We claim that Q acts intransitively on both partitions of Γ. We know that $|U(\Gamma)| = |W(\Gamma)| = 5p^n$. If Q acts transitively on $U(\Gamma)$ or $W(\Gamma)$, then $5p^n \mid |Q|$, which is a contradiction, by Theorem 3.1. Therefore, Q acts intransitively on both of them and so Γ_Q is A/Q-semisymmetric of order 10, by Lemma 2.3 and Theorem 3.1. By [12], we know that there exists no semisymmetric graph of order 10 and so Γ_Q is a symmetric cubic graph of order 10. Therefore Γ_Q is the Peterson graph O_3. Since Γ is bipartite and O_3 is nonbipartite, the fold number p^n must be even, by Lemma 2.5, which is a contradiction. Therefore there is no connected semisymmetric cubic graph of order $10p^n$. □

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[1] DEPARTMENT OF MATHEMATICS, UNIVERSITY OF QOM, QOM, IRAN. P.O.BOX: 37185-3766

[2] DEPT. OF PURE MATH., FACULTY OF MATH. AND COMPUTER SCI., AMIRKABIR UNIVERSITY OF TECHNOLOGY (TEHRAN POLYTECHNIC), 424, HAFEZ AVE., TEHRAN 15914, IRAN

[2]SCHOOL OF MATHEMATICS, INSTITUTE FOR RESEARCH IN FUNDAMENTAL SCIENCES (IPM), P.O.BOX: 19395-5746, TEHRAN, IRAN

E-mail address: `a_babai@aut.ac.ir`

E-mail address: `khosravibbb@yahoo.com`