$\mathbb{Z}$-Cyclic Generalized Ordered Whist Tournaments

Elyssa Cipriano[*] Stephanie Costa[†] Rebecca Sparks[‡]

Abstract

Generalized whist tournament designs and ordered whist tournament designs are relatively new specializations of whist tournament designs having first appeared in 2003 and 1996 respectively. In this paper, we extend the concept of an ordered whist tournament to a generalized whist tournament and introduce an entirely new combinatorial design, which we call a generalized ordered whist tournament. We focus specifically on generalized whist tournaments for games of size 6 and teams of size 3 where the number of players is a prime of the form $6n + 1$, and prove that these tournaments exist for all primes p of the form $p = 6n + 1$, with the possibile exception of $p \in \{7, 13, 19, 37, 61, 67\}$.

1 Introduction

Whist is a card game that originated in Turkey, but became prominent in England. It is an international card game that has transformed into other popular card games such as Bid Whist, Spades, and Bridge [3].

Definition 1.1 *[2] A whist tournament, $Wh(4n+1)$, for $4n+1$ players is a schedule of games each involving two players playing against two others, having the following properties:*

a. The games are arranged in $4n + 1$ rounds, with n games per round.

b. Each player plays in one game in all but one of the rounds.

c. Each player partners every other player exactly once.

[*]Rhode Island College class of 2013, E-mail ecipriano_4212@email.ric.edu
[†]Rhode Island College, Providence, RI. E-mail scosta@ric.edu
[‡]Rhode Island College, Providence, RI. E-mail rsparks@ric.edu

d. Each player opposes every other player exactly twice.

A whist tournament, $Wh(4n)$, for $4n$ players is similarly defined except that the games are arranged in $4n-1$ rounds such that each player plays in exactly one game in every round.

Thus a $Wh(v)$, $v \equiv 0, 1 \pmod 4$ is a (near) resolvable $(v, 4, 3)$-BIBD where the (near) resolution classes can be thought of as the rounds of the tournament.
The four players in any game can be thought of as sitting around a circular table where partners sit across from each other. Partners of the first kind are defined to be partners sitting in the North-South positions while partners of the second kind sit in the East-West positions.

In the 1970s, it was established that whist tournaments for $4n$ and $4n+1$ players exist for all positive integers through the work of R.D. Baker, H. Hanani, and R.M. Wilson [2]. Beginning in the 1990s, mathematicians turned their focus to different specializations of whist tournaments. There are many specializations, but one of particular concern in this study is an ordered whist tournament which was first introduced in an unpublished paper by Y.Lu [9] and is defined below.

Definition 1.2 *[4] An ordered whist tournament, $OWh(v)$, for v players is a $Wh(v)$ having the following properties:*

a. Each player opposes every other player exactly once as a partner of the first kind.

b. Each player opposes every other player exactly once as a partner of the second kind.

Example 1.1 *An $OWh(5)$ is given by the following five games: (1,2,4,3),(2,3,0,4),(3,4,1,0),(4,0,2,1),(0,1,3,2).*

Through the work of Costa, Finizio, and Leonard it is known that ordered whist tournaments exist for all $v = 4n+1$ and do not exist for multiples of 4 [8].

Another type of specialization of a whist tournament is a generalized whist tournament.

Definition 1.3 *[1] Let t, e, k, v be integers such that $k = et$ and $v \equiv 0, 1$ $\pmod k$. A (t, k) Generalized Whist Tournament Design on v players, denoted $(t, k)GWhD(v)$ is a (near) resolvable $(v, k, k-1)$-BIBD having the following properties:*

a. Each block is considered to be a game involving e teams of t players each.

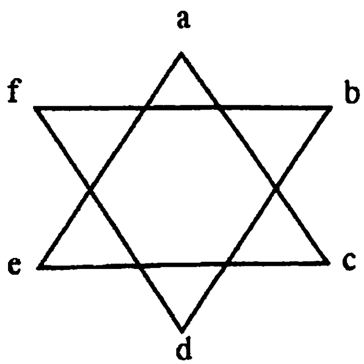
b. Every pair of players appear together in the same game exactly $t-1$ times as teammates.

c. Every pair of players appear together in the same game exactly $k-t$ times as opponents.

We denote such a tournament $(t,k)GWhD(v)$.

Definition 1.4 *[1] If $v \equiv 1 \pmod{k}$, a $(t,k)GWhD(v)$ is said to be $\mathbb{Z}$-cyclic if the players are elements of $\mathbb{Z}_v$ and the rounds are cyclically generated so that each round can be obtained from the previous round by adding 1 to each element $\pmod{v}$. In such a tournament, each player plays in one game in all but one round of the tournament. We traditionally label the round by the player that sits out. Thus the initial round of the tournament is the round that omits player 0.*

If we focus specifically on $(3,6)GWhD(6n+1)$, games of the tournament are denoted by the 6-tuple (a,b,c,d,e,f) where (a,c,e) and (b,d,f) are the two opposing teams. Players would sit at the table as follows (note partners are situated at vertices of the same triangle):



Using symmetric differences it follows that a collection of n games $(a_i,b_i,c_i,d_i,e_i,f_i)$, $i=1,...,n$ form the initial round game of a $\mathbb{Z}$-cyclic generalized whist tournament on $6n+1$ players, if all of the following three conditions are satisfied:

$$\bigcup_{i=1}^{n}\{a_i,b_i,c_i,d_i,e_i,f_i\} = \mathbb{Z}_{6n+1} - \{0\} \quad (1.1)$$

$$\bigcup_{i=1}^{n}\{\pm(a_i-c_i),\pm(a_i-e_i),\pm(c_i-e_i),\pm(b_i-d_i),\pm(b_i-f_i),\pm(d_i-f_i)\} \quad (1.2)$$

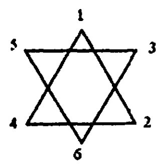
gives every element of $\mathbb{Z}_{6n+1} - \{0\}$ exactly twice.

$$\bigcup_{i=1}^{n} \{a_i - b_i, a_i - d_i, a_i - f_i, d_i - a_i, d_i - c_i, d_i - e_i, b_i - c_i, b_i - a_i, b_i - e_i,$$
$$e_i - b_i, e_i - d_i, e_i - f_i, c_i - b_i, c_i - d_i, c_i - f_i, f_i - a_i, f_i - c_i, f_i - e_i\} \quad (1.3)$$

gives every element of $\mathbb{Z}_{6n+1} - \{0\}$ exactly three times.

We refer to the differences in (1.2) and (1.3) as the partner and opponent differences, respectively.

Example 1.2 *The initial round of a* $\mathbb{Z}$*-cyclic* $(3,6)GWhD(7)$:

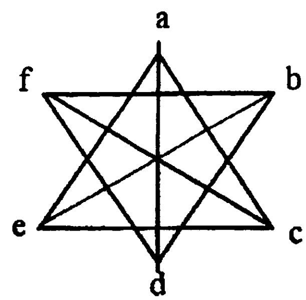


Round 0: $(1,3,2,6,4,5)$.

In order to introduce some notation, let $\mathbb{F}_p$ be the finite field of order p (thus p is a prime power) and $\mathbb{F}_p^*$ be its multiplicative group. If d is a divisor of $q-1$, we denote by C^d the subgroup of index d of $\mathbb{F}_p^*$, i.e., the group of all non-zero d-th powers of $\mathbb{F}_p$. The set of cosets of C^d in $\mathbb{F}_p^*$ (called cyclotomic classes of index d) is $\{r^i C^d | 0 \leq i \leq d-1\}$ where r is a fixed primitive element of $\mathbb{F}_p$, i.e., a generator of $\mathbb{F}_p^*$. The coset $r^i C^d$ will be denoted by C_i^d.

2 Main Results

The goal of this project was to determine if we could "order" a $(3,6)GWhD(6n)$ or $(3,6)GWhD(6n+1)$. In particular, we aimed to balance the three occasions where any pair of players meet as opponents. In order to introduce some notation, consider the game (a,b,c,d,e,f) again where (a,c,e) and (b,d,f) are the two opposing teams. The players would sit around the table as shown below:



We say a player is on Axis N if he is sitting in position a or d; a player is on Axis E if he is sitting in position b or e; a player is on Axis W if he is sitting in position f or c. Using this notation, we define our new specialization as follows:

Definition 2.1 *A* $(3,6)GWhD(v)$, $v \equiv 0,1 \pmod 6$ *is ordered if each player opposes every other player exactly once while sitting on Axis N, Axis W, and Axis E. We denote such a tournament by* $(3,6)OGWhD(v)$.

Theorem 2.1 *If a (3,6)GWhD(v) is ordered then* $v \equiv 1 \pmod 6$.

Proof: Suppose the $(3,6)GWhD(v)$ is based on the set X with $|X| = v$. Let $x \in X$ and consider the totality of games in which x sits on Axis N. Suppose there are k such games. In each game x opposes three distinct players each sitting on three different axes. Since the Wh(v) is ordered, then the $3k$ players that x opposes in these k games must contain the totality of players in the tournament distinct from x. We conclude that $v = 3k + 1$. This means v will always be a multiple of six plus one and therefore $v = 6n + 1$.

Theorem 2.2 *Let* $(a_i,b_i,c_i,d_i,e_i,f_i)$, $0 \le i \le n-1$ *denote non-identity elements in the cyclic group* $\mathbb{Z}_{6n+1}$. *Suppose that the collection of games* $(a_i,b_i,c_i,d_i,e_i,f_i)$, $0 \le i \le n-1$ *constitutes an initial round of a* $\mathbb{Z}$*-cyclic (3,6) GWhD(v). This* $\mathbb{Z}$*-cyclic (3,6) GWhD(v) is ordered if and only if*

$$\bigcup_{i=0}^{n-1} = \{(a_i-b_i),(a_i-d_i),(a_i-f_i),(d_i-a_i),(d_i-c_i),(d_i-e_i)\} = \mathbb{Z}_{6n+1}-\{0\} \tag{2.4}$$

and

$$\bigcup_{i=0}^{n-1} = \{(b_i - a_i), (b_i - c_i), (b_i - e_i), (e_i - b_i), (e_i - d_i), (e_i - f_i)\} = \mathbb{Z}_{6n+1} - \{0\} \quad (2.5)$$

and

$$\bigcup_{i=0}^{n-1} = \{(c_i - b_i), (c_i - d_i), (c_i - f_i), (f_i - a_i), (f_i - c_i), (f_i - e_i)\} = \mathbb{Z}_{6n+1} - \{0\}. \quad (2.6)$$

Proof: ($\Leftarrow$) Suppose that conditions (2.4), (2.5) and (2.6) from above are true. This means that the $6n$ differences are all unique.

Assume that the tournament is not ordered. Then there exists at least one pair, (x, y), having the property that in their three meetings as opponents, x, say, sits on one of the Axes, N, E, or W more than once. If x sits on Axis N more than once, without loss of generality, we can assume that x and y meet as opponents in the following two games:

$$(x, y, \star, \diamond, \circ, \triangleleft)(x, \dagger, \wr, y, ?, !)$$

Since these two games are translates of games in the initial round, it follows that

$x - y = a_i - b_i$ for some initial round table $(a_i, b_i, c_i, d_i, e_i, f_i)$
$x - y = a_j - d_j$ for some initial round table $(a_j, b_j, c_j, d_j, e_j, f_j)$

Therefore, $a_i - b_i = a_j - d_j$ which contradicts the facts that differences are distinct. The above proof is similar if we assume x is sitting on Axis E or on Axis W more than once. Thus the tournament is ordered.

($\Rightarrow$) Suppose that the $(3, 6)GWhD(6n + 1)$ is ordered.

Assume:

$$\bigcup_{i=0}^{n-1} = \{(a_i - b_i), (a_i - d_i), (a_i - f_i), (d_i - a_i), (d_i - c_i), (d_i - e_i)\} \neq \mathbb{Z}_{6n+1} - \{0\} \quad (2.7)$$

or

$$\bigcup_{i=0}^{n-1} = \{(b_i - a_i), (b_i - c_i), (b_i - e_i), (e_i - b_i), (e_i - d_i), (e_i - f_i)\} \neq \mathbb{Z}_{6n+1} - \{0\} \quad (2.8)$$

or

$$\bigcup_{i=0}^{n-1} = \{(c_i-b_i),(c_i-d_i),(c_i-f_i),(f_i-a_i),(f_i-c_i),(f_i-e_i)\} \neq \mathbb{Z}_{6n+1}-\{0\}. \tag{2.9}$$

Since none of the differences can equal the identity, this assumption implies that at least two differences have the same value. However, no two differences can be equal without violating the assumption that the $(3,6)GWhD(6n+1)$ is ordered. Therefore, if two differences are equal, they have to come from distinct initial round tables:

$$\text{Table } i = (a_i,b_i,c_i,d_i,e_i,f_i) \text{ and Table } j = (a_j,b_j,c_j,d_j,e_j,f_j)$$

Suppose that $a_i-b_i = a_j-d_j$. Define x by the requirement that $a_j+x = a_i$. Then in round x, Table j becomes

$$(a_j+x, b_j+x, c_j+x, d_j+x, e_j+x, f_j+x)$$

which is equivalent to

$$(a_i, b_j+x, c_j+x, b_i, e_j+x, f_j+x) \tag{2.10}$$

Comparing Table i with table (2.10), we see that a_i opposes b_i as a partner sitting on Axis N at both tables, contradicting the fact that the tournament is ordered. The above proof is similar for Axis E and Axis W seating positions. Similar contradictions occur for all the other possible matchings of the differences.

In the following theorem, we present a construction together with necessary conditions to guarantee the existence of a $\mathbb{Z}$-cyclic $(3,6)OGWhD(p)$, p a prime of the form $6n+1$. The authors would like to thank the referee for suggesting the use of the patterned construction as an alternative to the constructions initially submitted and for making us aware of the recent result of Buratti and Pasotti (Theorem 2.4 in [6]) which allow us to obtain the asymptotic existence result given in Theorem 2.5.

The following theorem is modeled after the Triplewhist and Splittable whist constructions presented by Buratti in [5] and Butler and Finizio in [7], respectively.

Theorem 2.3 *Let p be a prime, $p = 6n+1$. Let S denote a complete system of representatives for the cosets of $\{-1,1\}$ in C^3. Let (a,b,c) be a triple of elements of $\mathbb{Z}_p-\{0\}$ such that each of the sets $\{a-c, a+b, c+b\}$, $\{2a, a-b, a+c\}$, $\{2b, a-b, b-c\}$, $\{2c, a+c, b-c\}$ is a system of representatives for the cyclotomic classes of index 3, then*

$$\{as, bs, cs, -as, -bs, -cs \mid s \in S\}$$

is the initial round of a $\mathbb{Z}$-cyclic $(3,6)GOWhD(p)$.

Proof: In order to show that this result holds, we must show that the construction above satisfies conditions (1.1), (1.2), (2.4), (2.5) and (2.6). Notice if $\{\alpha, \beta, \gamma\}$ is a system of representatives for the cyclotomic classes of index 3, $\alpha, \beta, \gamma \in \mathbb{Z}_p - \{0\}$, then $\{\pm\alpha s, \pm\beta s, \pm\gamma s\} = \{\alpha, \beta, \gamma\}\{-1, 1\}S = \{\alpha, \beta, \gamma\}C^3 = \mathbb{Z}_p - \{0\}$.
By the argument above and our assumptions, it is clear that our partner differences:

$$\{\pm(a-c)s, \pm(a+b)s, \pm(c+b)s | s \in S\}$$

as well as our Axis N, E and W differences:

$$\{\pm 2as, \pm(a-b)s, \pm(a+c)s | s \in S\}$$
$$\{\pm 2bs, \pm(a-b)s, \pm(b-c)s | s \in S\}$$
$$\{\pm 2cs, \pm(a+c)s, \pm(b-c)s | s \in S\}$$

are each equal to $\mathbb{Z}_p - \{0\}$, satisfying conditions (1.2), (2.4), (2.5) and (2.6), respectively.

Since $\{2a, a-b, a+c\}, \{2b, a-b, b-c\}, \{2c, a+c, b-c\}$ are each systems of representatives for the cyclotomic classes of index 3, if we assume $a-b \in C_i^3$ for some $i \in \{0,1,2\}$, $a+c \in C_j^3$ for some $j \in \{0,1,2\} - \{i\}$, and $b-c \in C_k^3$ where $\{k\} = \{0,1,2\} - \{i,j\}$, then we must have $2a \in C_k^3, 2b \in C_j^3$ and $2c \in C_i^3$ satisfying condition (1.1).

Recently, Weil's theorem on multiplicative character sums has been used by many to obtain asymptotic existence results for various combinatorial designs. In [6], Buratti and Pasotti present a result which eliminates many of the tedious calculations normally required in order to apply Weil's theorem.

In keeping with the notation presented in [6], given integers $e \geq 2, t \geq 1$ and $n \geq 0$, we denote by $Q(e,t,n)$ the number defined by

$$Q(e,t,n) = \frac{1}{4}(U + \sqrt{U^2 + 4e^{t-1}(t+en)})^2 \text{ where } U = \Sigma_{h=1}^{t}\tbinom{t}{h}(e-1)^h(h-1).$$

In the case where $n = 0$, $Q(e,t,n)$ is simply expressed as $Q(e,t)$. It is also worth noting that Buratti and Pasotti show that $Q(e,t,n) < Q(e,t,n')$ for $n < n'$[6].

Theorem 2.4 *[6] Let $p \equiv 1 \pmod e$ be a prime power, let $B = \{b_1, ..., b_t\}$ be an arbitrary t-subset of $\mathbb{F}_p$ and let $(\beta_1, ..., \beta_t)$ be an arbitrary element of $\mathbb{Z}_e^t$. Set $X = \{x \in \mathbb{F}_p : x - b_i \in C_{\beta_i}^e \text{ for } i = 1, ..., t\}$. Then we have*

$$|X| \geq \frac{p - U\sqrt{p} - e^{t-1}t}{e^t} \text{ and hence } |X| > n \text{ as soon as } p > Q(e,t,n).$$

Thus in particular, X is nonempty for $p > Q(e,t)$

Using an approach similar to the one presented in application 2 of [6], we can conclude that if i is the integer satisfying $2 \in C_i^3$, a, b and c are elements of $\mathbb{Z}_{6n+1} - \{0\}$ such that, for example,

$$\begin{array}{lllll} a \in C_{3-i}^3; & & & & \\ b \in C_{2-i}^3; & b-a \in C_1^3; & b+a \in C_0^3; & & \\ c \in C_{1-i}^3; & c-b \in C_0^3; & c+a \in C_2^3; & c-a \in C_1^3; & c+b \in C_2^3; \end{array}$$

then the triple (a, b, c) satisfies the conditions of Theorem 2.3.

Theorem 2.5 *There exists a $\mathbb{Z}$-cyclic $(3,6)OGWhD(p)$ for every prime p of the form $p = 6n + 1$ whenever $p > 323,434$.*

Proof: Let C_i^3 be the cyclotomic class of index 3 containing 2 and let a be a fixed element of C_{3-i}^3. Applying Theorem 2.4 with $e = 3, t = 3, B = \{0, a, -a\}$ and $(\beta_1, \beta_2, \beta_3) = (2 - i, 1, 0)$, we see that the set $X_1 =$

$$\{x \in \mathbb{Z}_p : x \in C_{2-i}^3 \text{ and } x - a \in C_1^3 \text{ and } x + a \in C_0^3\}$$

is nonempty whenever $p > Q(3, 3)$. (Note, we could have chosen β_3 to be any element of the set $\{0, 1, 2\}$, but any of the options lead to the requirement that $p > Q(3, 3)$ in order to guarantee X_1 is nonempty.) Since $p > Q(3,5)$ implies $p > Q(3,3)$ we can claim that X_1 is nonempty whenever $p > Q(3,5)$. Let b be a fixed element of X_1.
Applying Theorem 2.4 again with $e = 3, t = 5, B = \{0, b, -a, a, -b\}$ and $(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5) = (1 - i, 0, 2, 1, 2)$, we see that the set $X_2 =$

$$\{x \in \mathbb{Z}_p : x \in C_{1-i}^3 \text{ and } x-b \in C_0^3 \text{ and } x+a \in C_2^3 \text{ and } x-a \in C_1^3 \text{ and } x+b \in C_2^3\}$$

is nonempty whenever $p > Q(3, 5)$. (Again, we could have chosen $\beta_4 = 2$ and $\beta_5 = 1$, but either option requires $p > Q(3,5)$ in order to guarantee X_2 is nonempty). Let c be a fixed element of X_2.

It is clear that the triple (a, b, c) satisfies the conditions of Theorem 2.3 and therefore there exists a $\mathbb{Z}$-cyclic $(3,6)GOWhD(p)$ whenever $p = 6n+1$ is a prime, $p > Q(3,5)$. Since $\lceil Q(3,5) \rceil = 323,434$, we can guarantee the existence of a $\mathbb{Z}$-cyclic $(3,6)GOWhD(p)$ whenever $p = 6n + 1$ prime, $p > 323,434$.

Theorem 2.6 *There exists a $\mathbb{Z}$-cyclic $(3,6)OGWhD(p)$ for every prime p of the form $p = 6n + 1$, whenever $p \notin \{7, 13, 19, 37, 61, 67\}$.*

Proof: The existence of a $\mathbb{Z}$-cyclic $(3,6)OGWh(p)$ for $p > 323,433$ is guaranteed by Theorem 2.5. For $p < 323,424$, the initial round of a $\mathbb{Z}$-cyclic $(3,6)OGWhD(p)$ has been found by computer and it is available from the authors. For $p < 109$ and $p \notin \{7, 13, 19, 37, 61, 67\}$ we give an explicit solution in the examples below.

Example 2.1 *The initial round of a* $\mathbb{Z}$*-cyclic* $(3,6)OGWhD(79)$*:*

$$\{(22s, 53s, 32s, 34s, 25s, 71s) | s \in C^6\}$$

where $C^6 = \{34^{6k} | 0 \leq k \leq 12\}$*. (Note this design is not of the patterned type given in the construction of Theorem 2.3 but does satisfy conditions* (1.1), (1.2), (2.4), (2.5), *and* (2.6).*)*

Example 2.2 *The initial round of a* $\mathbb{Z}$*-cyclic* $(3,6)OGWhD(31)$*:*

$$\{(1s, 6s, 5s, 30s, 25s, 26s) | s \in S\}$$

where $S = \{17^i | 0 \leq i \leq 4\} = \{1, 15, 8, 27, 2\}$.

Example 2.3 *The initial round of a* $\mathbb{Z}$*-cyclic* $(3,6)OGWhD(43)$*:*

$$\{(1s, 7s, 6s, 42s, 36s, 37s) | s \in S\}$$

where $S = \{28^i | 0 \leq i \leq 6\} = \{1, 22, 11, 27, 35, 49, 41\}$.

Example 2.4 *The initial round of a* $\mathbb{Z}$*-cyclic* $(3,6)OGWh(73)$*:*

$$\{(1s, 53s, 37s, 72s, 20s, 36s) | s \in S)\}$$

where $S = \{59^i | 0 \leq i \leq 11\} = \{1, 30, 24, 63, 65, 52, 27, 7, 64, 22, 3, 17\}$.

Example 2.5 *The initial round of a* $\mathbb{Z}$*-cyclic* $(3,6)OGWh(109)$*:*

$$\{(1s, 15s, 58s, 108s, 94s, 51s) | s \in S)\}$$

where $S = \{37^i | 0 \leq i \leq 17\}$.

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