# All good (bad) words consisting of four blocks *

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**Abstract**

The generalized Fibonacci cube $Q_d(f)$ is the graph obtained from the hypercube $Q_d$ by removing all vertices that contain a given binary word $f$. A binary word $f$ is called good if $Q_d(f)$ is an isometric subgraph of $Q_d$ for all $d \geq 1$, and bad otherwise. A non-extendable sequence of contiguous equal digits in a word $f$ is called a block of $f$. The question to determine the good (bad) words consisting of at most three blocks was solved by Ilić, Klavžar and Rho. This question is further studied in the present paper. All the good (bad) words consisting of four blocks are determined completely, and all bad 2-isometric words among consisting of at most four blocks words are found to be 1100 and 0011.

**Key words:** generalized Fibonacci cube, isometric subgraph, good word, bad word

## 1 Introduction

Hsu [4] introduced *Fibonacci cubes* as a model for interconnection networks, which has similar properties as hypercubes. The vertex set of the Fibonacci cube $\Gamma_d$ is the set of all words of length $d$ that contain no two consecutive 1s and two vertices are adjacent in $\Gamma_d$ if they differ in precisely one bit. For more about Fibonacci cubes, see [7] for a survey.

A word $f$ is called a *factor* of a word $\mu$ if $f$ appears as a sequence of $|f|$ consecutive bits of $\mu$, where $|f|$ denotes the length of $f$. Fibonacci cube $\Gamma_d$ can be seen as the graph obtained from $Q_d$ by removing all words that contain 11 as a factor. Inspired by this, Ilić, Klavžar and Rho [5]

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introduced the *generalized Fibonacci cube*, $Q_d(f)$, as the graph obtained from $Q_d$ by removing all words that contain a given word $f$ as a factor. In this notation the Fibonacci cube $\Gamma_d$ is the graph $Q_d(11)$. The subclass $Q_d(1^s)$ of generalized Fibonacci cube has been studied in [9, 10] (also under the name generalized Fibonacci cube). Generalized Fibonacci cubes have been studied from several points of view, see, for example, the recent papers of Azarija et al. [1, 2, 3]. In paper [1] the Wiener index of $Q_d(1^s)$ was studied, in paper [2] isomorphism classes of generalized Fibonacci cubes were studied and in paper [3] the connectivity of generalized Fibonacci cubes was shown.

A natural problem on generalized Fibonacci cubes is when they embed isometrically into hypercubes. This question is first studied by Ilić, Klavžar and Rho in [5], and it leads to the concepts of the so called good and bad word [8]. A word $f$ is called *good* if $Q_d(f)$ is an isometric subgraph of $Q_d$ for all $d \geq 1$, and *bad* otherwise. For example,the word $f = 1^s01^s0$ is good [5]. Infinite families of bad words were found [5, 6, 11]. Klavžar and Shpectorov [8] asserted that about eight percent of all words are good, and they showed that if $Q_d(f)$ is not an isometric subgraph of $Q_d$ for some dimension $d$, then $Q_{d'}(f)$ is not an isometric subgraph of $Q_{d'}$ for all $d' > d$. Thus, for a bad word $f$, there exists the smallest integer $d$ such that $Q_d(f)$ is not an isometric subgraph of $Q_d$. This integer $d$ is called the *index* of $f$, which is denoted by $B(f)$. For a good word $f$, it can be set that $B(f) = \infty$. For a bad word $f$, Ilić et al. [6] showed that $B(f) < |f|^2$, and further they conjectured that $B(f) < 2|f|$. This conjecture was proved by Wei and Zhang [12].

Ilić et al. [6] studied good (bad) words from another angle. A word $\mu$ is called *$f$-free* if it does not contain $f$ as a factor. Let $s$ be a positive integer. Then $f$ is called *$s$-isometric* if for any $f$-free words $\mu$ and $\nu$ of the same length that differ in $s$ bits, the following holds : $\mu$ can be transformed into and $\nu$ by complementing one by one all the $s$ bits on which $\mu$ differs from $\nu$, such that all of the new words obtained in this process are $f$-free. Such a transformation is called an *$f$-free transformation* of $\mu$ to $\nu$. It is not difficult to see that a word $f$ is good if and only if $f$ is $s$-isometric for all $s \geq 1$, and $f$ is bad if and only if $f$ is not $s$-isometric for some one $s$.

In view of the result $B(f) < 2|f|$ one might be tempted that as soon as a word is bad, it is not 2-isometric. Ilić et al. [6] showed that this is not the case by demonstrating the words among the family $\{0^{2^{r+1}}10^{2^r-1}10^{2^r-1}|r \geq 0\}$ are bad and 2-isometric, and they conjectured the words of this family are all the words that are bad 2-isometric among those with exactly two 1s. However, Wei and Zhang [11] showed that this conjecture is not true by showing it is the family $\{0^{2r+2}10^r10^r|r \geq 0\}$ but not the above one. Obviously, by this result the word 0011 is bad and 2-isometric.

A non-extendable sequence of contiguous equal digits in a word $\alpha$ is called a *block* of $\alpha$. Let $F' = \{1^r0^s1^t \mid r \geq 1, s \geq 0 \text{ and } t \geq 0\}$. Then $F'$ is the set of words consisting of at most three blocks such that the beginning bit is 1. Ilić et al. [5] studied the family $F'$, and the result is summarized in Table 1.

Table 1: Classification of the isometricity of the word $f \in F'$.

| $(i')$ | $r$ | $s$ | $t$ | good or bad |
|---|---|---|---|---|
| $(1')$ | $r \geq 1$ | $s = 0$ | $t = 0$ | good |
| $(2')$ | $r \geq 1$ | $s = 1$ | $t = 0$ | good |
| $(3')$ | $r = 2$ | $s = 2$ | $t = 0$ | bad (2-isometric) |
| $(4')$ | $r \geq 2$ | $s \geq 3$ | $t = 0$ | bad (not 2-isometric) |
| $(5')$ | $r \geq 1$ | $s \geq 1$ | $t \geq 1$ | bad (not 2-isometric) |

Klavžar [7] suggested determining more words $f$ and integers $n$, for which $Q_n(f)$ is isometric or non-isometric in $Q_n$, ideally, classifying embeddable words. But he sensed that it seems a difficult question. We pay a special attention to the words consisting of four blocks in the present paper. The result is shown in the following theorem.

**Theorem 1.1.** *Let $F = \{1^r0^s1^t0^k \mid r,\ s,\ t \text{ and } k \geq 1\}$. Then all the good (bad) words among $F$ are shown in Table 2, and all the bad words are not 2-isometric.*

Table 2: Classification of the isometricity of the word $f \in F$.

| (i) | $r$ | $s$ | $t$ | $k$ | good or bad |
|---|---|---|---|---|---|
| (1) | $r \geq 2$ | $s \geq 1$ | $t \geq 1$ | $k \geq 2$ | bad (not 2-isometric) |
| (2) | $r = 1$ | $s \geq 1$ | $t \geq 1$ | $k \geq s+2$ | bad (not 2-isometric) |
| (3) | $r = 1$ | $s \geq 1$ | $t \geq 1$ | $k = s+1$ | good |
| (4) | $r = 1$ | $s = k$ | $t = 1$ | $k \geq 1$ | good |
| (5) | $r = 1$ | $s = k$ | $t \geq 2$ | $k \geq 1$ | bad (not 2-isometric) |
| (6) | $r = 1$ | $s = k+1$ | $t = 1$ | $k \geq 1$ | bad (not 2-isometric) |
| (7) | $r = 1$ | $s = k+1$ | $t \geq 2$ | $k \geq 1$ | good |
| (8) | $r = 1$ | $s \geq k+2$ | $t = 1$ | $k \geq 1$ | bad (not 2-isometric) |
| (9) | $r = 1$ | $s \geq k+2$ | $t = 2$ | $k \geq 1$ | good |
| (10) | $r = 1$ | $s \geq k+2$ | $t \geq 3$ | $k \geq 1$ | bad (not 2-isometric) |

In the rest of the section we introduce additional terminology and notation needed. Let $\alpha = a_1a_2\cdots a_d$ be any word. With $\alpha^R = a_d\cdots a_2a_1$ we denote the reverse of $\alpha$ and $\overline{\alpha} = \overline{a}_1\overline{a}_2\cdots\overline{a}_d$ the complement of $\alpha$, where $\overline{a}_i = 1 - a_i$, $i = 1,\ldots,d$. Ilić et al. [5] presented the following result.

**Proposition 1.2** ([5]). *Let $f$ be any word and $d \geq 1$. Then $Q_d(f) \cong Q_d(\overline{f}) \cong Q_d(f^R)$.*

Note that the isometricity of all the words consisting of at most four blocks can be covered by the words among sets $F'$ and $F$ in view of Proposition 1.2. For instance, by Table 2 (7) the words $10^5 1^3 0^4$, $01^5 0^3 1^4$, $0^4 1^3 0^5 1$ and $1^4 0^3 1^5 0$ all are good.

From the results of Tables 1 and 2, the following result is obvious.

**Corollary 1.3.** *Let $f$ be any bad word consisting of at most 4 blocks. Then $f$ is 2-isometric if and only if either $f = 1100$ or $f = 0011$.*

In the next section, Theorem 1.1 is proved.

## 2 Proof of the main results

Recall that $F = \{1^r 0^s 1^t 0^k | r, s, t \text{ and } k \geq 1\}$. By Proposition 1.2, the following Fig. 1 can produce a decomposition of $F$, and so $F$ can be divided into 10 subfamilies:

$$r,s,t,k \geq 1 \begin{cases} r \geq 2, k \geq 2 \cdots\cdots (1) \\ r = 1, k \geq 1 \begin{cases} k \geq s+2 \cdots\cdots (2) \\ k \leq s+1 \begin{cases} k = s+1 \cdots\cdots (3) \\ s = k \begin{cases} t = 1 \cdots\cdots (4) \\ t \geq 2 \cdots\cdots (5) \end{cases} \\ s = k+1 \begin{cases} t = 1 \cdots\cdots (6) \\ t \geq 2 \cdots\cdots (7) \end{cases} \\ s \geq k+2 \begin{cases} t = 1 \cdots\cdots (8) \\ t = 2 \cdots\cdots (9) \\ t \geq 3 \cdots\cdots (10) \end{cases} \end{cases} \end{cases} \end{cases}$$

Fig. 1. A decomposition of $F$.

$F_1 = \{1^r 0^s 1^t 0^k | \ r \geq 2,\ s \geq 1,\ t \geq 1 \text{ and } k \geq 2\}$,
$F_2 = \{10^s 1^t 0^k | \ s \geq 1,\ t \geq 1 \text{ and } k \geq s+2\}$,
$F_3 = \{10^s 1^t 0^{s+1} | \ s \geq 1 \text{ and } t \geq 1\}$,
$F_4 = \{10^s 10^s | \ s \geq 1\}$,
$F_5 = \{10^s 1^t 0^s | \ s \geq 1 \text{ and } t \geq 2\}$,
$F_6 = \{10^{k+1} 10^k | \ k \geq 1\}$,
$F_7 = \{10^{k+1} 1^t 0^k | \ t \geq 2 \text{ and } k \geq 1\}$,
$F_8 = \{10^s 10^k | \ s \geq k+2 \text{ and } k \geq 1\}$,
$F_9 = \{10^s 1^2 0^k | \ s \geq k+2 \text{ and } k \geq 1\}$, and
$F_{10} = \{10^s 1^t 0^k | \ s \geq k+2,\ t \geq 3 \text{ and } k \geq 1\}$.

The bad words among $F$ can be determined by Lemma 2.1:

**Lemma 2.1.** *Let i=1,2,5,6,8 and 10. Then the words from $F_i$ are bad and not 2-isometric.*

**Proof.** For any word $f \in F_i$ $(i = 1, 2, 5, 6, 8, 10)$, we would like to show that $f$ is not 2-isometric by giving $f$-free words $\alpha$ and $\beta$ which differ exactly in two bits, but none of the words $\mu$ and $\nu$ obtained from $\alpha$ by complementing one bit in which $\alpha$ differs from $\beta$ is $f$-free.

***For the word*** $f \in F_1$, let $\alpha = 1^r0^s1^t0^{k-2}101^{r-2}0^s1^t0^k$ and $\beta = 1^r0^s1^t0^{k-2}011^{r-2}0^s1^t0^k$. Obviously, both $\alpha$ and $\beta$ are $f$-free and that they differ in two bits. The two words obtained from $\alpha$ by complementing the bits in which $\alpha$ differs from $\beta$ are $\mu = 1^r0^s1^t0^{k-2}001^{r-2}0^s1^t0^k$ and $\nu = 1^r0^s1^t0^{k-2}111^{r-2}0^s1^t0^k$. Yet none of $\mu$ and $\nu$ is $f$-free, thus $f$ is bad and not 2-isometric.

***For the word*** $f \in F_2$, let $\alpha = 10^s1^t0^{k-s-2}00^s11^{t-1}0^k$ and $\beta = 10^s1^t0^{k-s-2}10^s01^{t-1}0^k$. Then both $\alpha$ and $\beta$ are $f$-free and they differ in two bits. The two words obtained from $\alpha$ by complementing the bit-s in which $\alpha$ differs from $\beta$ are $\mu = 10^s1^t0^{k-s-2}10^s11^{t-1}0^k$ and $\nu = 10^s1^t0^{k-s-2}00^s01^{t-1}0^k$. Since none of $\mu$ and $\nu$ is $f$-free, $f$ is not 2-isometric, and so $f$ is bad.

***For the word*** $f \in F_5$, let $\alpha = 10^s1^{t-1}10^{s-1}11^{t-1}0^s$ and $\beta = 10^s1^{t-1}0\,0^{s-1}01^{t-1}0^s$. Obviously, both $\alpha$ and $\beta$ are $f$-free and they differ in two bits. The two words obtained from $\alpha$ by complementing the bits in which $\alpha$ differs from $\beta$ are $\mu = 10^s1^{t-1}00^{s-1}11^{t-1}0^s$ and $\nu = 10^s1^{t-1}10^{s-1}01^{t-1}0^s$. Note that none of $\mu$ and $\nu$ is $f$-free. Hence $f$ is not 2-isometric.

***For the word*** $f \in F_6$ ***or*** $F_8$, let $\alpha = 10^{s-1}000^{s-1}10^k$ and $\beta = 10^{s-1}110^{s-1}10^k$. It can be checked that both $\alpha$ and $\beta$ are $f$-free and that they differ in two bits. The two words obtained from $\alpha$ by complementing the bits in which $\alpha$ differs from $\beta$ are $\mu = 10^{s-1}100^{s-1}10^k$ and $\nu = 10^{s-1}010^{s-1}10^k$. Yet none of $\mu$ and $\nu$ is $f$-free. Hence $f$ is not 2-isometric and so $f$ is bad.

***For the word*** $f \in F_{10}$, let $\alpha = 10^s1^{t-2}010^{s-2}1^t0^k$ and $\beta = 10^s1^{t-2}1\,00^{s-2}1^t0^k$. Obviously, both $\alpha$ and $\beta$ are $f$-free and that they differ in two bits. The two words obtained from $\alpha$ by complementing the bits in which $\alpha$ differs from $\beta$ are $\mu = 10^s1^{t-2}110^{s-2}1^t0^k$ and $\nu = 10^s1^{t-2}000^{s-2}1^t0^k$. Since none of $\mu$ and $\nu$ is $f$-free, $f$ is not 2-isometric. So $f$ is bad. □

Next we turn to the good words among $F$. Here we need the concept of *x-error overlap*, which can be found in [6, 8]. For a word $f$ of length $n$, let $b_l(f)$ is the beginning of $f$ of length $l$ and $e_l(f)$ is the end part of $f$ of the same length $l$, where $l \le n$. Suppose that $b_l(f)$ and $e_l(f)$ agree in all but $x$ positions. Then we say that $f$ has an $x$-error overlap of length $l$. Klavžar and Shpectorov showed the following result:

**Lemma 2.2** ([8]). *If $f$ is bad then $f$ has a 2-error overlap.*

The good words among $F$ can be determined by Lemma 2.3:

**Lemma 2.3.** *Let $i$=3,4,7 and 9. Then the words from $F_i$ are good.*

**Proof.** We need to show that any one word $f \in F_i$ ($i = 3, 4, 7, 9$) has no 2-error overlap by Lemma 2.2. This can be done by comparing $b_l(f)$ and $e_l(f)$ for all $l \in \{1, \ldots, |f|\}$. Suppose that $b_l(f)$ disagree from $e_l(f)$ in exactly $x$ positions. We would like to show that $x \neq 2$ for all $l \in \{1, \ldots, |f|\}$ to complete the proof. Convenience, take two copies $f^{(1)}$ and $f^{(2)}$, as illustrated in Fig. 2, where $l' := |f| - l$. Obviously, $x$ is the number of pairs

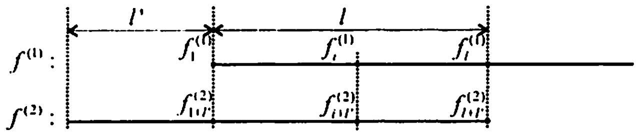


Fig. 2. Illustration of a error overlap of length $l$.

of $f_i^{(1)}$ and $f_{i+l'}^{(2)}$ such that $f_i^{(1)} \neq f_{i+l'}^{(2)}$, where $i \in L = \{1, \ldots, l\}$.

***For the word $f \in F_3$***, $|f| = 2s+t+2$, $f_1 = f_{s+2} = \cdots = f_{t+s+1} = 1$, and $f_2 = \cdots = f_{s+1} = f_{t+s+2} = \cdots = f_{|f|} = 0$. (a)

If $1 \leq l \leq s+1$, then $s+t+1 \leq l' \leq 2s+t+1$. By (a), $f_1^{(1)} = 1 \neq 0 = f_{1+l'}^{(2)}$ and $f_i^{(1)} = f_{i+l'}^{(2)} = 0$ for $i \in L \setminus \{1\}$. Hence $x = 1$.

If $1 = s+2$, then $l' = s+t$. By (a), $f_{s+2}^{(1)} = 1 \neq 0 = f_{s+2+l'}^{(2)}$, $f_1^{(1)} = f_{1+l'}^{(2)} = 1$ and $f_i^{(1)} = f_{i+l'}^{(2)} = 0$ for $i \in L \setminus \{1, s+2\}$. Hence $x = 1$.

For $l \geq s+3$, we distinguish two cases.

***Case* 1.** $t = 1$.

Clearly, if $s+3 \leq l \leq 2s+2$, then $1 \leq l' \leq s$, and so $f_1^{(1)} = 1 \neq 0 = f_{1+l'}^{(2)}$, $f_{s+1}^{(1)} = 0 \neq 1 = f_{s+1+l'}^{(2)}$, $f_{s+2}^{(1)} = 1 \neq 0 = f_{s+2+l'}^{(2)}$ and $f_i^{(1)} = f_{i+l'}^{(2)} = 0$ for $i \in L \setminus \{1, s+1, s+2\}$ by (a). Hence $x = 3$.

***Case* 2.** $t \geq 2$.

We distinguish three subcases: $l = s+i$ and $3 \leq i \leq t+1$, $s+t+2 \leq l \leq 2s+2$ and $l = 2s+2+j$ and $1 \leq j \leq t-1$.

First we consider $l = s+i$, where $3 \leq i \leq t+1$. Note that $s+1 \leq l' \leq s+t-1$. We claim that $x = 2s+1$ if $1 \leq s \leq i-2$, and that $x = 2i-3$ if $s \geq i-1$. In fact, if $1 \leq s \leq i-2$, then $l' = s+t-(i-2) \geq t$, $l'+i = s+t+2$ and so by (a) $f_2^{(1)} = \cdots = f_{s+1}^{(1)} = 0$, $f_{2+l'}^{(2)} = \cdots = f_{s+1+l'}^{(2)} = 1$, $f_i^{(1)} = \cdots = f_{s+i}^{(1)} = 1$, $f_{i+l'}^{(2)} = \cdots = f_{s+i+l'}^{(2)} = 0$, and $f_m^{(1)} = f_{m+l'}^{(2)} = 1$ for $m \in L \setminus \{2, \ldots, s+1, i, \ldots, s+i\}$. Hence $x = 2s+1$. If $s \geq i-1$, then $l' = s+(t+1-i)+1 \geq s+1$, $l' = t+(s+1-i)+1 \geq s+1$, $l'+i = s+t+2$ and so by (a) $f_2^{(1)} = \cdots = f_{i-1}^{(1)} = 0$, $f_{2+l'}^{(2)} = \cdots = f_{i-1+l'}^{(2)} = 1$, $f_{s+2}^{(1)} = \cdots = f_{s+i}^{(1)} = 1$, $f_{s+2+l'}^{(2)} = \cdots = f_{s+i+l'}^{(2)} = 0$, $f_{(1)}^{1} = f_{1+l'}^{(2)} = 1$ and $f_m^{(1)} = f_{m+l'}^{(2)} = 0$ for $m \in L \setminus \{1, 2, \ldots, i-1, s+2, \ldots, s+i\}$. Hence $x = 2i-3$.

Next we consider $s+t+2 \leq l \leq 2s+2$. Obviously, this holds only for $s \geq t$. In this subcase $t \leq l' \leq s$, and so by (a) $f_1^1 = 1 \neq 0 = f_{1+l'}^2$, $f_{s+2}^{(1)} = \cdots = f_{s+t+1}^{(1)} = 1$, $f_{s+2+l'}^{(2)} = \cdots = f_{s+t+1+l'}^{(2)} = 0$, $f_{s+2-l'}^{(1)} = \cdots = f_{s+t+1-l'}^{(1)} = 0$, $f_{s+2}^{(2)} = \cdots = f_{s+t+1}^{(2)} = 1$, and $f_m^{(1)} = f_{m+l'}^{(2)} = 0$ for $m \in L \setminus \{1, s+2-l', \ldots, s+t+1-l', s+2, \ldots, s+t+1\}$. Hence $x = 2t+1$.

Last we consider $l = 2s+2+j$ and $1 \leq j \leq t-1$. Obviously, $l' = t-j$ and $1 \leq l' \leq t-1$. We only need to consider the case $s \geq t-j$ since $s \leq t-j-1$ has been considered in the case $l = s+i$ for $i \in \{3, \ldots, t+1\}$. We claim that $x = 2t-2i+1$. In fact, by (a) $f_1^{(1)} = 1 \neq 0 = f_{1+l'}^{(2)}$, $f_{s-t+j+2}^{(1)} = \cdots = f_{s+1}^{(1)} = 0$, $f_{s-t+j+2+l'}^{(2)} = \cdots = f_{s+1+l'}^{(2)} = 1$, $f_{s+j+2}^{(1)} = \cdots = f_{s+t+1}^{(1)} = 1$, $f_{s+j+2+l'}^{(2)} = \cdots = f_{s+t+1+l'}^{(2)} = 0$, $f_i^{(1)} = f_{i+l'}^{(2)} = 1$ for $i \in \{s+2, \ldots, s+j+1\}$, and $f_i^{(1)} = f_{i+l'}^{(2)} = 0$ for $i \in L \setminus \{1, s-t+j+2, \ldots, s+t+1\}$. Hence $x = 2t-2j+1$.

By the above discussions, we know that any word $f \in F_3$ has no 2-error overlap, and so $f$ is good.

***For $F_4$***, note that $|f| = 2s+2$, $f_1 = f_{s+2} = 1$, and $f_2 = \cdots = f_{s+1} = f_{s+3} = \cdots = f_{|f|} = 0$. (b)

We distinguish three cases: $1 \leq l \leq s$, $l = s+1$ and $s+2 \leq l \leq 2s+1$.

If $1 \leq l \leq s$, then $s+2 \leq l' \leq 2s+1$, and so $f_1^{(1)} = 1 \neq 0 = f_{1+l'}^{(2)}$, and $f_i^{(1)} = f_{i+l'}^{(2)} = 0$ for $i \in L \setminus \{1\}$ by (b). Hence $x = 1$.

If $l = s+1$, then $l' = s+1$, and so $f_1^{(1)} = f_{1+l'}^{(2)} = 1$, and $f_i^{(1)} = f_{i+l'}^{(2)} = 0$ for $i \in L \setminus \{1\}$ by (b). Hence $x = 0$.

If $s+2 \leq l \leq 2s+1$, then $1 \leq l' \leq s$, and so $f_1^{(1)} = 1 \neq 0 = f_{1+l'}^{(2)}$, $f_{s+2}^{(1)} = 1 \neq 0 = f_{s+2+l'}^{(2)}$, $f_{l-s}^{(1)} = 0 \neq 1 = f_{l-s+l'}^{(2)}$, and $f_i^{(1)} = f_{i+l'}^{(2)} = 0$ for $i \in L \setminus \{1, l-s, s+2\}$ by (b). Hence $x = 3$.

Thus we know that any word $f \in F_4$ has no 2-error overlap, and so $f$ is good.

***For $F_7$***, note that $|f| = 2k+t+2$, $f_1 = f_{k+3} = \cdots = f_{k+t+2} = 1$, and $f_2 = \cdots = f_{k+2} = f_{k+t+3} = \cdots = f_{|f|} = 0$. (c)

First we consider $1 \leq l \leq k$. Clearly, $k+t+2 \leq l' \leq 2k+t+1$, and so $f_1^{(1)} = 1 \neq 0 = f_{1+l'}^{(2)}$, and $f_i^{(1)} = f_{i+l'}^{(2)} = 0$ for $i \in L \setminus \{1\}$ by (c). Hence $x = 1$.

Next we consider $l = k+1$. Clearly, $l' = k+t+1$, and so $f_1^{(1)} = f_{1+l'}^{(2)} = 1$ and $f_i^{(1)} = f_{i+l'}^{(2)} = 0$ for $i \in L \setminus \{1\}$ by (c). Hence $x = 0$.

Now we consider $l = k+2$. Obviously $l' = k+t$ and so $f_2^1 = 0 \neq 1 = f_{2+l'}^2$, $f_1^{(1)} = f_{1+l'}^{(2)} = 1$ and $f_i^{(1)} = f_{i+l'}^{(2)} = 0$ for $i \in L \setminus \{1, 2\}$ by (c). Hence $x = 1$.

For the case $l \geq k+3$, we distinguish two cases.

***Case* 1.** $t = 2$.

We distinguish three subcases to continue the discussion.

The first is that $l = k+3$. Obviously $l' = k+1$, and so by (c), $f_1^{(1)} = 1 \neq 0 = f_{1+l'}^{(2)}$, $f_{k+3-l'}^{(1)} = 0 \neq 1 = f_{k+3}^{(2)}$, $f_{k+4-l'}^{(1)} = 0 \neq 1 = f_{k+4}^{(2)}$, $f_{k+3}^{(1)} = 1 \neq 0 = f_{k+3+l'}^{(2)}$, and $f_i^{(1)} = f_{i+l'}^{(2)} = 0$ for $i \in L \setminus \{1, k+3-l', k+4-l', k+3\}$. Hence $x = 4$.

The second is that $k+4 \leq l \leq 2k+2$. Obviously, it holds only for $k \geq 2$. As $2 \leq l' \leq k$, $f_1^{(1)} = 1 \neq 0 = f_{1+l'}^{(2)}$, $f_{k+3-l'}^{(1)} = 0 \neq 1 = f_{k+3}^{(2)}$, $f_{k+4-l'}^{(1)} = 0 \neq 1 = f_{k+4}^{(2)}$, $f_{k+3}^{(1)} = 1 \neq 0 = f_{k+3+l'}^{(2)}$, $f_{k+4}^{(1)} = 1 \neq 0 = f_{k+4+l'}^{(2)}$, and $f_i^{(1)} = f_{i+l'}^{(2)} = 0$ for $i \in L \setminus \{1, k+3-l', k+4-l', k+3, k+4\}$ by (c). Hence $x = 5$.

The last is that $l = 2k+3$. Obviously, $l' = 1$ and so $f_1^{(1)} = 1 \neq 0 = f_{1+l'}^{(2)}$, $f_{k+3-l'}^{(1)} = 0 \neq 1 = f_{k+3}^{(2)}$, $f_{k+4}^{(1)} = 1 \neq 0 = f_{k+4+l'}^{(2)}$, $f_{k+3}^{(1)} = f_{k+3+l'}^{(2)} = 1$ and $f_i^{(1)} = f_{i+l'}^{(2)} = 0$ for $i \in L \setminus \{1, k+3-l', k+3, k+4\}$ by (c). Hence $x = 3$.

***Case* 2.** $t \geq 3$.

We distinguish four subcases to continue the discussion.

The first is that $l = k+i$ and $3 \leq i \leq t$. Obviously, $l' = k+t+2-i$ and $k+2 \leq l' \leq k+t-1$. We claim that $x = 2k+1$ if $1 \leq k \leq i-2$, and $x = 2i-3$ if $k \geq i-1$. In fact, if $1 \leq k \leq i-2$, then $f_2^{(1)} = \cdots = f_{k+2}^{(1)} = 0$, $f_{2+l'}^{(2)} = \cdots = f_{k+2+l'}^{(2)} = 1$, $f_{k+t+3-l'}^{(1)} = \cdots = f_{2k+t+2-l'}^{(1)} = 1$, $f_{k+t+3}^{(2)} = \cdots = f_{2k+t+2}^{(2)} = 0$, and $f_m^{(1)} = f_{m+l'}^{(2)} = 1$ for $m \in L \setminus \{2, \ldots, k+2, k+t+3-l', \ldots, 2k+t+2-l'\}$ by (c). Hence $x = 2k+1$. If $k \geq i-1$, then $f_2^{(1)} = \cdots = f_i^{(1)} = 0$, $f_{2+l'}^{(2)} = \cdots = f_{i+l'}^{(2)} = 1$, $f_{k+3}^{(1)} = \cdots = f_{k+i}^{(1)} = 1$, $f_{k+3+l'}^{(2)} = \cdots = f_{k+i+l'}^{(2)} = 0$, $f_1^{(1)} = f_{1+l'}^{(2)} = 1$ and $f_m^{(1)} = f_{m+l'}^{(2)} = 0$ for $m \in L \setminus \{1, 2, \ldots, i, k+3, \ldots, k+i\}$ by (c). Hence $x = 2i-3$.

The second is that $l = k+t+1$, and so $l' = k+1$. We claim that $x = 2k+2$ if $1 \leq k \leq t-2$, and that $x = 2t$ if $k \geq t-1$. In fact, if $1 \leq k \leq t-2$, then $f_1^{(1)} = 1$, $f_{1+l'}^{(2)} = 0$, $f_2^{(1)} = \cdots = f_{k+2}^{(1)} = 0$, $f_{2+l'}^{(2)} = \cdots = f_{k+2+l'}^{(2)} = 1$, $f_{k+t+3-l'}^{(1)} = \cdots = f_{2k+t+2-l'}^{(1)} = 1$, $f_{k+t+3}^{(2)} = \cdots = f_{2k+t+2}^{2} = 0$, and $f_i^{(1)} = f_{i+l'}^{(2)} = 1$ for $i \in L \setminus \{1, 2, \ldots, k+2, k+t+3-l', \ldots, 2k+t+2-l'\}$ by (c). Hence $x = 2k+2$. If $k \geq t-1$, then $f_1^{(1)} = 1 \neq 0 = f_{1+l'}^{(2)}$, $f_2^{(1)} = \cdots = f_{t+1}^{(1)} = 0$, $f_{2+l'}^{(2)} = \cdots = f_{t+1+l'}^{(2)} = 1$, $f_{k+3}^{(1)} = \cdots = f_{k+t+1}^{(1)} = 1$, $f_{k+3+l'}^{(2)} = \cdots = f_{k+t+1+l'}^{(2)} = 0$, and $f_i^{(1)} = f_{i+l'}^{(2)} = 1$ for $i \in L \setminus \{1, 2, \ldots, t+1, k+3, \ldots, k+t+1\}$ by (c). Hence $x = 2t$.

The third case is that $k+t+2 \leq l \leq 2k+2$. Obviously, it holds only for $k \geq t$. Note that $t \leq l' \leq k$. We get that $f_1^{(1)} = 1 \neq 0 = f_{1+l'}^{(2)}$, $f_{k+3-l'}^{(1)} = \cdots = f_{k+t+2-l'}^{(1)} = 0$, $f_{k+3}^{(2)} = \cdots = f_{k+t+2}^{(2)} = 1$, $f_{k+3}^{(1)} = \cdots = f_{k+t+2}^{(1)} = 1$, $f_{k+3+l'}^{(2)} = \cdots = f_{k+t+1+l'}^{(2)} = 0$, and $f_i^{(1)} = f_{i+l'}^{(2)} = 0$ for $i \in L \setminus \{1, k+3-l', \ldots, k+t+2-l', k+3, \ldots, k+t+2\}$. Hence $x = 2t+1$.

The last case is that $l = 2k+2+j$ and $1 \le j \le t-1$. We only need to consider the case $k \ge t-j$ since $k \le t-j-1$ has been discussed in the first case $l = k+i$ and $3 \le i \le t$. Obviously, $l' = t-j$ and so $f_1^{(1)} = 1$, $f_{1+l'}^{(2)} = 0$, $f_{k-t+j+3}^{(1)} = \cdots = f_{k+2}^{(1)} = 0$, $f_{k-t+j+3+l'}^{2} = \cdots = f_{k+2+l'}^{(2)} = 1$, $f_{k+j+3}^{(1)} = \cdots = f_{k+t+2}^{(1)} = 1$, $f_{k+j+3+l'}^{2} = \cdots = f_{k+t+2+l'}^{(2)} = 0$, $f_m^{(1)} = f_{m+l'}^{(2)} = 1$ for $m \in \{k+3, \ldots, k+i+2\}$, and $f_m^{(1)} = f_{m+l'}^{(2)} = 0$ for $m \in L \setminus \{1, k-t+i+3, \ldots, k+t+2\}$ by (c). Hence $x = 2t-2i+1$.

By the above discussions, we know that any word $f \in F_7$ has no 2-error overlap, and so $f$ is good.

***For*** $\boldsymbol{F_9}$, note that $|f| = s+k+3$, $f_1 = f_{s+2} = f_{s+3} = 1$ and $f_2 = \cdots = f_{s+1} = f_{s+4} = \cdots = f_{s+k+3} = 0$. (d)

We can divided the interval $1 \le l \le s+k+2$ into seven subintervals: $1 \le l \le k$, $l = k+1$, $l = k+2$, $k+3 \le l \le s+1$, $l = s+2$, $s+3 \le l \le s+k+1$ and $l = s+k+2$.

If $1 \le l \le k$, then $s+3 \le l' \le s+k+2$, and so $f_1^{(1)} = 1 \ne 0 = f_{1+l'}^{(2)}$ and $f_i^{(1)} = f_{i+l'}^{(2)} = 0$ for $i \in L \setminus \{1\}$ by (d). Hence $x = 1$.

If $l = k+1$, then $l' = s+2$, and so $f_1^{(1)} = f_{1+l'}^{(2)} = 1$ and $f_i^{(1)} = f_{i+l'}^{(2)} = 0$ for $i \in L \setminus \{1\}$ by (d). Hence $x = 0$.

If $l = k+2$, then $l' = s+1$, and so $f_2^{(1)} = 1 \ne 0 = f_{2+l'}^{(2)}$, $f_1^{(1)} = f_{1+l'}^{(2)} = 1$ and $f_i^{(1)} = f_{i+l'}^{(2)} = 0$ for $i \in L \setminus \{1,2\}$ by (d). Hence $x = 1$.

If $k+3 \le l \le s+1$, then $k+2 \le l' \le s$, and so $f_1^{(1)} = 1 \ne 0 = f_{1+l'}^{(2)}$, $f_{s+2-l'}^{(1)} = 0 \ne 1 = f_{s+2}^{(2)}$, $f_{s+3-l'}^{(1)} = 0 \ne 1 = f_{s+3}^{(2)}$ and $f_i^{(1)} = f_{i+l'}^{(2)} = 0$ for $i \in L \setminus \{1, s+2-l', s+3-l'\}$ by (d). Hence $x = 3$.

If $l = s+2$, then $l' = k+1$, and so $f_1^{(1)} = 1 \ne 0 = f_{1+l'}^{(2)}$, $f_{s+2-l'}^{(1)} = 0 \ne 1 = f_{s+2}^{(2)}$, $f_{s+3-l'}^{(1)} = 0 \ne 1 = f_{s+3}^{(2)}$, $f_{s+2}^{(1)} = 1 \ne 0 = f_{s+2+l'}^{(2)}$, and $f_i^{(1)} = f_{i+l'}^{(2)} = 0$ for $i \in L \setminus \{1, s+2-l', s+3-l', s+2\}$ by (d). Hence $x = 4$.

If $s+3 \le l \le s+k+1$, then $2 \le l' \le k$, and so $f_1^{(1)} = 1 \ne 0 = f_{1+l'}^{(2)}$, $f_{s+2-l'}^{(1)} = 0 \ne 1 = f_{s+2}^{(2)}$, $f_{s+3-l'}^{(1)} = 0 \ne 1 = f_{s+3}^{(2)}$, $f_{s+2}^{(1)} = 1 \ne 0 = f_{s+2+l'}^{(2)}$, $f_{s+3}^{(1)} = 1 \ne 0 = f_{s+3+l'}^{(2)}$, and $f_i^{(1)} = f_{i+l'}^{(2)} = 0$ for $i \in L \setminus \{1, s+2-l', s+3-l', s+2, s+3\}$. Hence $x = 5$.

If $l = s+k+2$, then $l' = 1$, and so $f_1^{(1)} = 1 \ne 0 = f_{1+l'}^{(2)}$, $f_s^{(1)} = 0 \ne 1 = f_{s+l'}^{(2)}$, $f_{s+3}^{(1)} = 1 \ne 0 = f_{s+3+l'}^{(2)}$, $f_{s+2}^{(1)} = f_{s+2+l'}^{(2)} = 1$, and $f_i^{(1)} = f_{i+l'}^{(2)} = 0$ for $i \in L \setminus \{1, s, s+2, s+3\}$. Hence $x = 5$.

By the above discussions, for any word $f \in F_3$ there exists no 2-error overlap, and so $f$ is good. □

**Proof of Theorem 1.1.** For $i = 1, 2, 5, 6, 8$ and $10$, all the words of $F_i$ are bad by Lemma 2.1, and for $i = 3, 4, 7$ and $9$, all the words of $F_i$ are good by Lemma 2.3. As $F = \bigcup_{i=1}^{n} F_i$, it proves Theorem 1.1. $\square$

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