# CONVOLVED FIBONACCI NUMBERS AND THEIR APPLICATIONS

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ABSTRACT. In this paper, we present a new approach to the convolved Fibonacci numbers arising from the generating function of them and give some new and explicit identities for the convolved Fibonacci numbers.

## 1. INTRODUCTION

As is well known, the Fibonacci numbers are given by the numbers in the following integer sequnce:

$$1,1,2,3,5,8,13,21,34,55,89,144,\ldots.$$

The sequence $F_n$ of Fibonacci numbers is defined by the recurrence relation as follows:

$$(1.1)\quad F_0=1,\quad F_1=1,\quad F_n=F_{n-1}+F_{n-2},\quad (n\geq 2),\quad (\text{see [1–8]}).$$

The sequence can be extended to negative index $n$ arising from the rearranged recurrence relation

$$(1.2)\qquad F_{n-2}=F_n-F_{n-1},\quad (\text{see [1–13]})$$

which yields the sequence of "negafibonacci" numbers satisfying

$$(1.3)\qquad F_{-n}=(-1)^{n+1}F_n,\quad (\text{see [11, 12]}).$$

It is well known that the generating function of Fibonacci numbers is given by

$$(1.4)\qquad \frac{1}{1-t-t^2}=\sum_{n=0}^{\infty}F_nt^n,\quad (\text{see [3–6]}).$$

The convolved Fibonacci numbers $p_n(x)$, $(n\geq 0)$, are defined by the generating function

$$(1.5)\qquad \left(\frac{1}{1-t-t^2}\right)^x=\sum_{n=0}^{\infty}p_n(x)\frac{t^n}{n!},\quad (x\in\mathbb{R}).$$

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From (1.4) and (1.5), we note that

$$\frac{p_n(1)}{n!} = F_n, \quad (n \geq 0). \tag{1.6}$$

In this paper, we present a new approach to the convolved Fibonacci numbers arising from the generating function of them and give some new and explicit identities for the convolved Fibonacci numbers.

## 2. Convolved Fibonacci numbers and their applications

From (2.45), we note that

$$\begin{aligned} \sum_{n=0}^{\infty} p_n(x)\frac{t^n}{n!} &= \left(\frac{1}{1-t-t^2}\right)^x = \left(\frac{1}{1-t-t^2}\right)\left(\frac{1}{1-t-t^2}\right)^{x-1} \\ &= \left(\sum_{l=0}^{\infty} p_l(1)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} p_m(x-1)\frac{t^m}{m!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}p_l(1)\,p_{n-l}(x-1)\right)\frac{t^n}{n!}. \end{aligned} \tag{2.1}$$

By comparing the coefficients on both sides of (2.1), we obtain the following proposition.

**Proposition 1.** *For $n \geq 0$, $x \in \mathbb{R}$, we have*

$$p_n(x) = \sum_{l=0}^{n}\binom{n}{l}p_l(1)\,p_{n-l}(x-1).$$

Let us take $x = r \in \mathbb{N}$. Then, by Proposition 1, we get

$$\begin{aligned} p_n(r) &= \sum_{l_1=0}^{n}\binom{n}{l_1}p_{l_1}(1)\,p_{n-l_1}(r-1) \\ &= \sum_{l_1=0}^{n}\sum_{l_2=0}^{n-l_1}\binom{n}{l_1}\binom{n-l_1}{l_2}p_{l_1}(1)\,p_{l_2}(1)\,p_{n-l_1-l_2}(r-2) \\ &= \sum_{l_1=0}^{n}\sum_{l_2=0}^{n-l_1}\sum_{l_3=0}^{n-l_1-l_2}\binom{n}{l_1}\binom{n-l_1}{l_2}\binom{n-l_1-l_2}{l_3}p_{l_1}(1)\,p_{l_2}(1) \\ &\quad \times p_{l_3}(1)\,p_{n-l_1-l_2-l_3}(r-3) \\ &\ \ \vdots \\ &= \sum_{l_1=0}^{n}\sum_{l_2=0}^{n-l_1}\cdots\sum_{l_{r-1}=0}^{n-l_1-\cdots-l_{r-2}}\binom{n}{l_1}\binom{n-l_1}{l_2}\cdots \end{aligned} \tag{2.2}$$

$$\times \binom{n-l_1-l_2-\cdots-l_{r-2}}{l_{r-1}}$$
$$\times \left(\prod_{k=1}^{r-1} p_{l_k}(1)\right) p_{n-l_1-l_2-\cdots-l_{r-1}}(1).$$

Therefore, by (2.2), we obtain the following corollary.

**Corollary 2.** *For $r \in \mathbb{N}$ and $n \geq 0$, we have*

$$p_n(r) = \sum_{l_1=0}^{n} \sum_{l_2=0}^{n-l_1} \cdots \sum_{l_{r-1}=0}^{n-l_1-\cdots-l_{r-2}} \binom{n}{l_1}\binom{n-l_1}{l_2}\cdots$$
$$\times \binom{n-l_1-l_2-\cdots-l_{r-2}}{l_{r-1}} \left(\prod_{k=1}^{r-1} p_{l_k}(1)\right) p_{n-l_1-l_2-\cdots-l_{r-1}}(1).$$

We observe that

$$\left(\frac{1}{1-t-t^2}\right)^x = \left(\frac{1}{1-t-t^2}\right)^r \left(\frac{1}{1-t-t^2}\right)^{x-r}$$
$$= \left(\sum_{l=0}^{\infty} p_l(r)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} p_m(x-r)\frac{t^m}{m!}\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} p_l(r)\, p_{n-l}(x-r)\right)\frac{t^n}{n!}. \tag{2.3}$$

Therefore, by (2.45) and (2.3), we obtain the following theorem.

**Theorem 3.** *For $n \geq 0$, $r \in \mathbb{N}$, we have*

$$p_n(x) = \sum_{l=0}^{n}\binom{n}{l} p_l(r)\, p_{n-l}(x-r) = \sum_{l=0}^{n}\binom{n}{l} p_{n-l}(r)\, p_l(x-r).$$

Let us take $x = r+1$ in Theorem 3. Then, we have

$$p_n(r+1) = \sum_{l=0}^{n}\binom{n}{l} p_{n-l}(r)\, p_l(1)$$
$$= \sum_{l=0}^{n}\binom{n}{l} p_{n-l}(r)\, l!\frac{p_l(1)}{l!}$$
$$= \sum_{l=0}^{n} (n)_l\, p_{n-l}(r)\, F_l, \tag{2.4}$$

where $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \geq 1)$, $(x)_0 = 1$.

**Corollary 4.** *For $r \in \mathbb{N}$, $n \geq 0$, we have*

$$p_n(r+1) = \sum_{l=0}^{n} (n)_l \, p_{n-l}(r) \, F_l.$$

Taking $r = 1$ in Corollary 4, we have

$$\begin{aligned} p_n(2) &= \sum_{l=0}^{n} (n)_l \, p_{n-l}(1) \, F_l \\ &= \sum_{l=0}^{n} (n)_l \, (n-l)! \frac{p_{n-l}(1)}{(n-l)!} F_l \\ &= n! \sum_{n=0}^{n} \binom{n}{l} \binom{n}{l}^{-1} F_{n-l} F_l \\ &= n! \sum_{l=0}^{n} F_l F_{n-l}. \end{aligned} \tag{2.5}$$

Thus, by (2.5), we get

$$\frac{p_n(2)}{n!} = \sum_{l=0}^{n} F_l F_{n-l}, \quad (n \geq 0). \tag{2.6}$$

From (2.6), we can also derive the following equation.

$$\begin{aligned} p_n(3) &= \sum_{l_1=0}^{n} (n)_{l_1} \, p_{n-l_1}(2) \, F_{l_1} \\ &= \sum_{l_1=0}^{n} (n)_{l_1} \, (n-l_1)! \frac{p_{n-l_1}(2)}{(n-l_1)!} F_{l_1} \\ &= n! \sum_{l_1=0}^{n} \sum_{l_2=0}^{n-l_1} F_{l_1} F_{l_2} F_{n-l_1-l_2}. \end{aligned} \tag{2.7}$$

Thus, by (2.7), we get

$$\frac{p_n(3)}{n!} = \sum_{l_1=0}^{n} \sum_{l_2=0}^{n-l_1} F_{l_1} F_{l_2} F_{n-l_1-l_2}. \tag{2.8}$$

For $r = 3$ in Corollary 4, we have

$$\begin{aligned} p_n(4) &= \sum_{l_1=0}^{n} (n)_{l_1} \, p_{n-l_1}(3) \, F_{l_1} \\ &= n! \sum_{l_1=0}^{n} \frac{p_{n-l_1}(3)}{(n-l_1)!} F_{l_1} \end{aligned} \tag{2.9}$$

$$= n! \sum_{l_1=0}^{n} \sum_{l_2=0}^{n-l_1} \sum_{l_3=0}^{n-l_1-l_2} F_{l_1} F_{l_2} F_{l_3} F_{n-l_1-l_2-l_3}.$$

From (2.9), we note that

$$\frac{p_n(4)}{n!} = \sum_{l_1=0}^{n} \sum_{l_2=0}^{n-l_1} \sum_{l_3=0}^{n-l_1-l_2} F_{l_1} F_{l_2} F_{l_3} F_{n-l_1-l_2-l_3}. \tag{2.10}$$

Continuing this process, we have

$$\begin{aligned} &\frac{p_n(r+1)}{n!} \\ &= \sum_{l_1=0}^{n} \sum_{l_2=0}^{n-l_1} \cdots \sum_{l_r=0}^{n-l_1-\cdots-l_{r-1}} F_{l_1} F_{l_2} \cdots F_{l_r} F_{n-l_1-l_2-\cdots-l_r}, \end{aligned} \tag{2.11}$$

where $r \in \mathbb{N}$.

**Theorem 5.** *For $r \in \mathbb{N}$ and $n \geq 0$, we have*

$$\frac{p_n(r+1)}{n!} = \sum_{l_1=0}^{n} \sum_{l_2=0}^{n-l_1} \cdots \sum_{l_r=0}^{n-l_1-\cdots-l_{r-1}} F_{l_1} F_{l_2} \cdots F_{l_r} F_{n-l_1-l_2-\cdots-l_r}.$$

Let

$$\begin{aligned} F(t,x) &= \left(1-t-t^2\right)^{-x} \\ &= e^{-x \log\left(1-t-t^2\right)}. \end{aligned} \tag{2.12}$$

Then, by (2.12), we get

$$\begin{aligned} F^{(1)}(t,x) &= \frac{dF}{dt}(t,x) \\ &= x(1+2t)\left(1-t-t^2\right)^{-x-1} \\ &= x(1+2t)F(t,x+1), \end{aligned} \tag{2.13}$$

$$\begin{aligned} F^{(2)}(t,x) &= \frac{dF^{(1)}}{dt}(t,x) \\ &= 2xF(t,x+1) + \langle x\rangle_2 (1+2t)^2 F(t,x+2), \end{aligned} \tag{2.14}$$

where $\langle x\rangle_n = x(x+1)\cdots(x+n-1)$, $(n \geq 1)$, $\langle x\rangle_0 = 1$.

From (2.14), we note that

$$\begin{aligned} F^{(3)}(t,x) &= \frac{dF^{(2)}}{dt}(t,x) \\ &= 6\langle x\rangle_2 (1+2t) F(t,x+2) \\ &\quad + \langle x\rangle_3 (1+2t)^3 F(t,x+3). \end{aligned} \tag{2.15}$$

$$(2.16)\quad F^{(4)}(t,x) = \frac{dF^{(3)}}{dt}(t,x)$$
$$= 12\langle x\rangle_2 F(t,x+2) + 12\langle x\rangle_3 (1+2t)^2 F(t,x+3)$$
$$+ \langle x\rangle_4 (1+2t)^4 F(t,x+4),$$

(2.17)
$$F^{(5)}(t,x) = \frac{dF^{(4)}}{dt}(t,x)$$
$$= 60\langle x\rangle_3 (1+2t) F(t,x+3) + 20\langle x\rangle_4 (1+2t)^3 F(t,x+4)$$
$$+ \langle x\rangle_5 (1+2t)^5 F(t,x+5)$$

and
(2.18)
$$F^{(6)}(t,x) = \frac{dF^{(5)}}{dt}(t,x)$$
$$= 120\langle x\rangle_3 F(t,x+3) + 180\langle x\rangle_4 (1+2t)^2 F(t,x+4)$$
$$+30\langle x\rangle_5 (1+2t)^4 F(t,x+5) + \langle x\rangle_6 (1+2t)^6 F(t,x+6).$$

Thus, we are led to put

$$(2.19)\quad F^{(N)}(t,x) = \left(\frac{d}{dt}\right)^N F(t,x)$$
$$= \sum_{i=0}^{[\frac{N}{2}]} a_i(N)\langle x\rangle_{N-i} (1+2t)^{N-2i} F(t,x+N-i)$$

where $N \in \mathbb{N}$.

Taking the derivatives of (2.19) with respect to $t$, we have

$$(2.20)\quad F^{(N+1)}(t,x)$$
$$= \sum_{i=0}^{[\frac{N}{2}]} a_i(N)\langle x\rangle_{N-i} (1+2t)^{N-2i} F^{(1)}(t,x+N-i)$$
$$+ \sum_{i=0}^{[\frac{N}{2}]} a_i(N)\langle x\rangle_{N-i}\, 2(N-2i)(1+2t)^{N-2i-1} F(t,x+N-i)$$
$$= \sum_{i=0}^{[\frac{N}{2}]} 2(N-2i)\, a_i(N)\langle x\rangle_{N-i} (1+2t)^{N-2i-1} F(t,x+N-i)$$
$$+ \sum_{i=0}^{[\frac{N}{2}]} a_i(N)\langle x\rangle_{N-i+1} (1+2t)^{N-2i+1} F(t,x+N-i+1)$$

$$
\begin{aligned}
&= \sum_{i=1}^{[\frac{N}{2}]+1} 2(N-2i+2)\, a_{i-1}(N)\, \langle x \rangle_{N-i+1} \\
&\times (1+2t)^{N-2i+1} F(t, x+N-i+1) \\
&+ \sum_{i=0}^{[\frac{N}{2}]} a_i(N)\, \langle x \rangle_{N-i+1} (1+2t)^{N-2i+1} F(t, x+N-i+1).
\end{aligned}
$$

On the other hand, by replacing $N$ by $N+1$ in (2.19), we get

$$
\begin{aligned}
F^{(N+1)}(t,x) & \\
= \sum_{i=0}^{[\frac{N+1}{2}]} & a_i(N+1)\, \langle x \rangle_{N-i+1} (1+2t)^{N-2i+1} \\
\times & F(t, x+N-i+1).
\end{aligned} \tag{2.21}
$$

*Case 1.* Let $N$ be an even number. Then we have

$$
\begin{aligned}
&\sum_{i=1}^{\frac{N}{2}+1} 2(N-2i+2)\, a_{i-1}(N)\, \langle x \rangle_{N-i+1} \\
&\quad \times (1+2t)^{N-2i+1} F(t, x+N-i+1) \\
&+ \sum_{i=0}^{\frac{N}{2}} a_i(N)\, \langle x \rangle_{N-i+1} (1+2t)^{N-2i+1} F(t, x+N-i+1) \\
&= \sum_{i=0}^{\frac{N}{2}} a_i(N+1)\, \langle x \rangle_{N-i+1} \\
&\quad \times (1+2t)^{N-2i+1} F(t, x+N-i+1).
\end{aligned} \tag{2.22}
$$

Comparing the coefficients on both sides of (2.22), we get

$$
a_0(N+1) = a_0(N), \tag{2.23}
$$

$$
a_i(N+1) = 2(N-2i+2)\, a_{i-1}(N) + a_i(N), \quad \left(1 \le i \le \frac{N}{2}\right). \tag{2.24}
$$

*Case 2.* Let $N$ be an odd number. Then, we have

$$
\begin{aligned}
&\sum_{i=1}^{\frac{N+1}{2}} 2(N-2i+2)\, a_{i-1}(N)\, \langle x \rangle_{N-i+1} \\
&\times (1+2t)^{N-2i+1} F(t, x+N-i+1) \\
&+ \sum_{i=0}^{\frac{N-1}{2}} a_i(N)\, \langle x \rangle_{N-i+1} (1+2t)^{N-2i+1} F(t, x+N-i+1)
\end{aligned} \tag{2.25}
$$

$$= \sum_{i=0}^{\frac{N+1}{2}} a_i (N+1) \langle x \rangle_{N-i+1} (1+2t)^{N-2i+1} F(t, x+N-i+1).$$

Comparing the coefficients on both sides of (2.25), we have

$$(2.26) \qquad a_0 (N+1) = a_0 (N), \quad a_{\frac{N+1}{2}} (N+1) = 2a_{\frac{N-1}{2}} (N),$$

and

(2.27)

$$a_i (N+1) = 2(N-2i+2) a_{i-1} (N) + a_i (N), \quad \left(1 \le i \le \frac{N-1}{2}\right).$$

In addition, we have the following "initial conditions":

$$(2.28) \qquad F^{(0)} (t,x) = F(t,x) = a_0 (0) F(t,x).$$

Thus, by (2.28), we get $a_0 (0) = 1$.

From (2.13) and (2.19), we note that

$$(2.29) \quad F^{(1)} (t,x) = a_0 (1) x (1+2t) F(t, x+1) = x(1+2t) F(t,x+1).$$

Thus, by (2.29), we see that $a_0 (1) = 1$.

By (2.14) and (2.19), we easily get

$$\begin{aligned}(2.30) \quad F^{(2)} (t,x) &= \sum_{i=0}^{1} a_i (2) \langle x \rangle_{2-i} (1+2t)^{2-2i} F(t, x+2-i) \\ &= a_0 (2) \langle x \rangle_2 (1+2t)^2 F(t,x+2) + a_1 (2) x F(t,x+1) \\ &= 2xF(t,x+1) + \langle x \rangle_2 (1+2t)^2 F(t,x+2).\end{aligned}$$

Thus, by comparing the coefficients on both sides of (2.30), we get

$$(2.31) \qquad a_0 (2) = 1, \quad \text{and} \quad a_1 (2) = 2.$$

In (2.19), it is not difficult to show that

$$(2.32) \qquad a_{\frac{N+1}{2}} (N) = 0, \quad \text{for all positive integers } N.$$

From (2.32), we note that

$$(2.33) \qquad a_1 (1) = a_2 (3) = a_3 (5) = a_4 (7) = \cdots = 0.$$

By (2.32), we get

$$(2.34) \quad F^{(N)} (t,x) = \sum_{i=0}^{\left[\frac{N+1}{2}\right]} a_i (N) \langle x \rangle_{N-i} (1+2t)^{N-2i} F(t, x+N-i),$$

where

(2.35)

$$a_0 (N+1) = a_0 (N), \quad a_{\frac{N+1}{2}} (N) = 0, \quad \text{for all positive integers } N,$$

and
(2.36)

$$a_i(N+1) = 2(N-2i+2)\,a_{i-1}(N) + a_i(N), \qquad \left(1 \le i \le \left[\frac{N+1}{2}\right]\right).$$

From (2.35), we note that

$$a_0(N+1) = a_0(N) = a_0(N-1) = \cdots = a_0(1) = 1. \tag{2.37}$$

For $i = 1, 2, 3$ in (2.36), we have

$$a_1(N+1) = 2\sum_{k=0}^{N-1}(N-k)\,a_0(N-k), \tag{2.38}$$

$$a_2(N+1) = 2\sum_{k=0}^{N-3}(N-2-k)\,a_1(N-k), \tag{2.39}$$

and

$$a_3(N+1) = 2\sum_{k=0}^{N-5}(N-4-k)\,a_2(N-k). \tag{2.40}$$

Thus, we can deduce that, for $1 \le i \le \left[\frac{N+1}{2}\right]$,

$$\begin{aligned} a_i(N+1) &= 2\sum_{k=0}^{N-2i+1}(N-k-2i+2)\,a_{i-1}(N-k) \\ &= 2\sum_{k=0}^{N+2-2i} k a_{i-1}(k+2i-2). \end{aligned} \tag{2.41}$$

Now, we give explicit expressions for $a_i(N+1)$.
From (2.37), (2.38), (2.39) and (2.40), we have

$$a_1(N+1) = 2\sum_{k_1=1}^{N} k_1 a_0(k_1) = 2\sum_{k_1=1}^{N} k_1 \tag{2.42}$$

$$a_2(N+1) = 2\sum_{k_2=1}^{N-2} k_2 a_1(k_2+2) = 2^2\sum_{k_2=1}^{N-2}\sum_{k_1=1}^{k_2+1} k_2 k_1, \tag{2.43}$$

$$a_3(N+1) = 2\sum_{k_3=1}^{N-4} k_3 a_2(k_3+4) = 2^3\sum_{k_3=1}^{N-4}\sum_{k_2=1}^{k_3+1}\sum_{k_1=1}^{k_2+1} k_3 k_2 k_1 \tag{2.44}$$

and

$$a_4(N+1) = 2^4\sum_{k_4=1}^{N-6}\sum_{k_3=1}^{k_4+1}\sum_{k_2=1}^{k_3+1}\sum_{k_1=1}^{k_2+1} k_4 k_3 k_2 k_1. \tag{2.45}$$

Continuing this process, we have

$$(2.46)\qquad a_i(N+1) = 2^i \sum_{k_i=1}^{N-2i+2} \sum_{k_{i-1}=1}^{k_i+1} \cdots \sum_{k_1=1}^{k_2+1} \left(\prod_{l=1}^{i} k_l\right), \quad \left(1 \le i \le \left[\frac{N+1}{2}\right]\right).$$

Therefore, by (2.46), we obtain the following theorem.

**Theorem 6.** *For $N = 0, 1, 2, \ldots$, the family of differential equations*

$$F^{(N)}(t,x) = \left(\frac{d}{dt}\right)^N F(t,x) = \left(\sum_{i=0}^{\left[\frac{N+1}{2}\right]} a_i(N)\langle x\rangle_{N-i}(1+2t)^{N-2i}\left(1-t-t^2\right)^{-N+i}\right)F(t,x)$$

*have a solution*

$$F(t,x) = \left(1-t-t^2\right)^{-x}$$

*where*

$$a_0(N) = 1, \quad a_{\frac{N+1}{2}}(N) = 0, \quad \textit{for all positive integers } N,$$

*and*

$$a_i(N) = 2^i \sum_{k_i=1}^{N-2i+1} \sum_{k_{i-1}=1}^{k_i+1} \cdots \sum_{k_1=1}^{k_2+1} \left(\prod_{l=1}^{i} k_l\right), \quad \left(1 \le i \le \left[\frac{N}{2}\right]\right).$$

From (1.4), we note that

$$(2.47)\qquad 1 = \sum_{k=0}^{\infty} F_k t^k \left(1-t-t^2\right) = \sum_{k=0}^{\infty} F_k t^k - \sum_{k=1}^{\infty} F_{k-1} t^k - \sum_{k=2}^{\infty} F_{k-2} t^k.$$

Comparing the coefficients on the both sides of (2.47), we have

$$(2.48)\qquad F_0 = 1, \quad F_1 - F_0 = 0 \iff F_1 = F_0 = 1,$$

and

$$(2.49)\qquad F_k - F_{k-1} - F_{k-2} = 0 \quad \text{if } k \ge 2.$$

By (1.4), we easily get

$$(2.50)\qquad F(t,x) = \left(1-t-t^2\right)^{-x} = \sum_{k=0}^{\infty} p_k(x)\frac{t^k}{k!},$$

and

$$F^{(N)}(t,x)=\left(\frac{d}{dt}\right)^{N}F(t,x)=\sum_{k=0}^{\infty}p_{k+N}(x)\frac{t^k}{k!}. \tag{2.51}$$

On the other hand, by Theorem 6, we get

(2.52)

$$\begin{aligned}
&F^{(N)}(t,x)\\
&=\sum_{i=0}^{\left[\frac{N+1}{2}\right]}a_i(N)\langle x\rangle_{N-i}(1+2t)^{N-2i}F(t,x+N-i)\\
&=\sum_{i=0}^{\left[\frac{N+1}{2}\right]}a_i(N)\langle x\rangle_{N-i}(1+2t)^{N-2i}\sum_{m=0}^{\infty}p_m(x+N-i)\frac{t^m}{m!}\\
&=\sum_{i=0}^{\left[\frac{N+1}{2}\right]}a_i(N)\langle x\rangle_{N-i}\sum_{l=0}^{\infty}(N-2i)_l2^l\frac{t^l}{l!}\sum_{m=0}^{\infty}p_m(x+N-i)\frac{t^m}{m!}\\
&=\sum_{i=0}^{\left[\frac{N+1}{2}\right]}a_i(N)\langle x\rangle_{N-i}\sum_{k=0}^{\infty}\left(\sum_{l=0}^{k}\binom{k}{l}(N-2i)_l2^lp_{k-l}(x+N-i)\right)\frac{t^k}{k!}\\
&=\sum_{k=0}^{\infty}\left(\sum_{i=0}^{\left[\frac{N+1}{2}\right]}\sum_{l=0}^{k}\binom{k}{l}(N-2i)_l2^la_i(N)\langle x\rangle_{N-i}p_{k-l}(x+N-i)\right)\frac{t^k}{k!}.
\end{aligned}$$

Therefore, by comparing the coefficients on both sides of (2.51) and (2.52), we obtain the following theorem.

**Theorem 7.** *For $k,N=0,1,2,\dots$, we have*

$$p_{k+N}(x)=\sum_{i=0}^{\left[\frac{N+1}{2}\right]}\sum_{l=0}^{k}\binom{k}{l}(N-2i)_l2^la_i(N)\langle x\rangle_{N-i}p_{k-l}(x+N-i),$$

*where*

$$a_0(N)=1,\quad a_{\frac{N+1}{2}}(N)=0,\quad \textit{for all positive integers } N,$$

$$a_i(N)=2^i\sum_{k_i=1}^{N-2i+1}\sum_{k_{i-1}=1}^{k_i+1}\cdots\sum_{k_1=1}^{k_2+1}\left(\prod_{l=1}^{i}k_l\right),\quad \left(1\le i\le\left[\frac{N}{2}\right]\right).$$

When $k=0$ in Theorem 7, we have the following corollary.

**Corollary 8.** *For $N = 0, 1, 2, \ldots$, we have*

$$p_N(x) = \sum_{i=0}^{\left[\frac{N+1}{2}\right]} a_i(N) \langle x \rangle_{N-i}.$$

Let us take $x = 1$ in Corollary 8. Then, we easily get

$$p_N(1) = \sum_{i=0}^{\left[\frac{N+1}{2}\right]} a_i(N)(N-i)! = N! + \sum_{i=1}^{\left[\frac{N+1}{2}\right]} a_i(N)(N-i)! \tag{2.53}$$

Thus, by (2.53), we get

$$\begin{aligned} \frac{p_N(1)}{N!} &= 1 + \frac{1}{N!} \sum_{i=1}^{\left[\frac{N+1}{2}\right]} a_i(N)(N-i)! \\ &= 1 + \frac{1}{N!} \sum_{i=1}^{\left[\frac{N+1}{2}\right]} \sum_{k_i=1}^{N+1-2i} \sum_{k_{i-1}=1}^{k_i+1} \cdots \sum_{k_1=1}^{k_2+1} 2^i \left( \prod_{l=1}^{i} k_l \right) (N-i)! \end{aligned} \tag{2.54}$$

Therefore, by (1.6) and (2.54), we obtain the following corollary.

**Corollary 9.** *For $N = 0, 1, 2, \ldots$, we have*

$$F_N - 1 = \frac{1}{N!} \left( \sum_{i=1}^{\left[\frac{N+1}{2}\right]} \sum_{k_i=1}^{N+1-2i} \sum_{k_{i-1}=1}^{k_i+1} \cdots \sum_{k_1=1}^{k_2+1} 2^i \left( \prod_{l=1}^{i} k_l \right) (N-i)! \right).$$

*Remark.* Recently, several authors have studied special polynomials and sequences arising from the generating functions (see [1–16]).

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