Embedding of Hypercubes into Generalized Books

R. Sundara Rajan[1,*], Indra Rajasingh[1], Micheal Arockiaraj[2], T.M. Rajalaxmi[3] and B. Mahavir[4]

[1]School of Advanced Sciences, VIT University, Chennai, India, 600 127
[2]Department of Mathematics, Loyola College, Chennai, India, 600 034
[3]Department of Mathematics, SSN College of Engineering, Chennai, India, 603 110
[4]Department of Mathematics, A.M. Jain College, Chennai, India, 600 114

vprsundar@gmail.com

Abstract

Graph embedding has been known as a powerful tool for implementation of parallel algorithms or simulation of different interconnection networks. An embedding f of a guest graph G into a host graph H is a bijection on the vertices such that each edge of G is mapped into a path of H. In this paper, we introduce a graph called the generalized book and the main results obtained are: (1) For $r \geq 3$, the minimum wirelength of embedding r-dimensional hypercube Q^r into the generalized book $GB[2^{r_1}, 2^{r_2}, 2^{r_3}]$, where $r_1 + r_2 + r_3 = r$. (2) A linear time algorithm to compute the exact wirelength of embedding hypercube into generalized book. (3) An algorithm for embedding hypercube into generalized book with dilation 3 proving that the lower bound obtained by Manuel et al. [28] is sharp.

Keywords: Embedding, dilation, wirelength, hypercube, generalized book.

*This work is supported by National Board of Higher Mathematics (NBHM), No. 2/40(36)/2012-R&D-II/11622, Department of Atomic Energy, Government of India.

1 Introduction

A parallel algorithm or a massively parallel computer can each be modeled by a graph, in which the vertices of the graph represent the processes or processing elements, and the edges represent the communications among processes or processors. Thus, the problem of efficiently executing a parallel algorithm A on a parallel computer M can be often reduced to the problem of mapping the graph G, representing A, on the graph H, representing M, so that the mapping satisfies some predefined constraints. This is called a graph embedding [3].

An embedding of a guest graph G into a host graph H is a one-to-one mapping of the vertex set of G into that of H. The quality of an embedding can be measured by certain cost criteria. One of these criteria which is considered very often is the *dilation*. The dilation of an embedding is defined as the maximum distance between a pair of vertices of H that are images of adjacent vertices of G. It is a measure for the communication time needed when simulating one network on another [13].

Another important cost criteria is the *wirelength*. The wirelength of a graph embedding arises from VLSI designs, data structures and data representations, networks for parallel computer systems, biological models that deal with cloning and visual stimuli, parallel architecture, structural engineering and so on [25, 43].

One of the most efficient interconnection networks is the hypercube due to its structural regularity, potential for parallel computation of various algorithms, and the high degree of fault tolerance [39]. The hypercube has many excellent features and thus becomes the first choice of topological structure of parallel processing and computing systems. The machine based on hypercubes such as the Cosmic Cube from Caltech, the iPSC/2 from Intel and Connection Machines have been implemented commercially [12]. Hypercubes are very popular models for paralled computation because of their symmetry and relatively small number of interprocessor connections. The hypercube embedding problem is the problem of mapping a communication graph into a hypercube multiprocessor. Hypercubes are known to simulate other structures such as grids and binary trees [11, 29].

Graph embeddings have been well studied for binary trees into paths [25], hypercubes into grids [29], binary trees into hypercubes [11, 13], complete binary trees into hypercubes [2], meshes into crossed cubes [15], meshes into locally twisted cubes [20], meshes into faulty crossed cubes [45], generalized ladders into hypercubes [7], rectangular grids into hypercubes [9], rectangular grids into hypercubes [14], grids into grids [38], binary trees into grids [30], hypercubes into cycles [8, 19], star graph into path [44], snarks into torus [42], generalized wheels into arbitrary trees [37], m-sequencial k-ary trees into hypercubes [32], meshes into möbius cubes

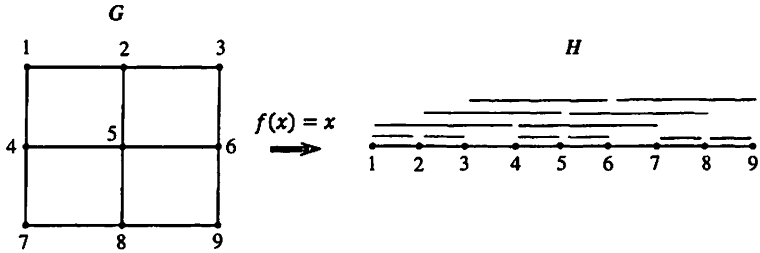

Figure 1: Wiring diagram of a grid G into path H with $WL_f(G,H) = 24$

[41], ternary tree into hypercube [18], enhanced and augmented hypercube into complete binary tree [26], circulant into arbitrary trees, cycles, certain multicyclic graphs and ladders [35], hypercubes into cylinders, snakes and caterpillars [27], tori and grids into twisted cubes [24], incomplete hypercube in books [16], 1-fault hamiltonian graphs into wheels and fans [1], hypercubes into necklace, windmill and snake graphs [33], embedding of special classes of circulant networks, hypercubes and generalized Petersen graphs [34], embedding variants of hypercubes with dilation 2 [28], circulant into necklace and windmill graphs [36].

Even though there are numerous results and discussions on the wirelength problem, most of them deal with only approximate results and the estimation of lower bounds [3, 8]. In this paper, we produce exact wirelength of embedding hypercubes into generalized books. The highlight of the paper is that the lower bound obtained in [28] for the dilation of embedding hypercubes into generalized books is sharp.

The rest of the paper is organized as follows. Section 2 gives definitions and other preliminaries. In Section 3, we compute the minimum wirelength of embedding r-dimensional hypercube Q^r into the generalized book $GB[2^{r_1}, 2^{r_2}, 2^{r_3}]$, where $r_1 + r_2 + r_3 = r$. In Section 4, we provide an $O(n)$-linear time algorithm to compute the minimum wirelength of embedding r-dimensional hypercube Q^r into the generalized book $GB[2^{r_1}, 2^{r_2}, 2^{r_3}]$, $n = 2^r$. In Section 5, we determine the exact dilation of embedding r-dimensional hypercube Q^r into generalized book $GB[4, 2, 2^{r-3}]$ using IPS Lemma. Also, the same technique can be used to estimate the exact dilation of embedding r-dimensional hypercube into a special class $GB[l]$ of books. Finally, concluding remarks and future study are given in Section 6.

2 Preliminaries

In this section we give the basic definitions and preliminaries related to embedding problems.

Definition 2.1. [3] *Let G and H be finite graphs. An embedding f of G into H is defined as follows:*

1. *f is a one-to-one map from $V(G) \rightarrow V(H)$*
2. *P_f is a one-to-one map from $E(G)$ to $\{P_f(u,v) : P_f(u,v)$ is a path in H between $f(u)$ and $f(v)$ for $(u,v) \in E(G)\}$.*

Definition 2.2. [3] *If $e = (u,v) \in E(G)$, then the length of $P_f(u,v)$ in H is called the dilation of the edge e. The maximal dilation over all edges of G is called the dilation of the embedding f. The dilation of embedding G into H denoted by $d(G,H)$ is the minimum dilation taken over all embeddings f of G into H. The expansion of an embedding f is the ratio of the number of vertices of H to the number of vertices of G.*

In this paper, we consider embeddings with expansion one.

The *edge congestion* of an embedding f of G into H is the maximum number of edges of the graph G that are embedded on any single edge of H. Let $EC_f(e)$ denote the number of edges (u,v) of G such that e is in the path $P_f(u,v)$ between $f(u)$ and $f(v)$ in H. In other words,

$$EC_f(e) = |\{(u,v) \in E(G) : e \in P_f(u,v)\}|$$

where $P_f(u,v)$ denotes the path between $f(u)$ and $f(v)$ in H with respect to f.

If we think of G as representing the wiring diagram of an electronic circuit, with the vertices representing components and the edges representing wires connecting them, then the edge congestion $EC(G,H)$ is the minimum, over all embeddings $f : V(G) \rightarrow V(H)$, of the maximum number of wires that cross any edge of H [4]. See Figure 1.

Definition 2.3. [29] *The wirelength of an embedding f of G into H is given by*

$$WL_f(G,H) = \sum_{(u,v)\in E(G)} |P_f(u,v)| = \sum_{e\in E(H)} EC_f(e)$$

where $|P_f(u,v)|$ denotes the length of the path $P_f(u,v)$ in H.

The wirelength of G into H is defined as

$$WL(G,H) = \min WL_f(G,H)$$

where the minimum is taken over all embeddings f of G into H.

The *wirelength problem* [3, 4, 8, 29, 30, 37] of a graph G into H is to find an embedding of G into H that induces the minimum wirelength $WL(G,H)$. The following two versions of the edge isoperimetric problem of a graph $G(V,E)$ have been considered in the literature [5], and are NP-complete [17].

Problem 1 : Find a subset of vertices of a given graph, such that the edge cut separating this subset from its complement has minimal size among all subsets of the same cardinality. Mathematically, for a given m, if $\theta_G(m) = \min_{A\subseteq V,\ |A|=m} |\theta_G(A)|$ where $\theta_G(A) = \{(u,v) \in E : u \in A, v \notin A\}$, then the problem is to find $A \subseteq V$ such that $|A| = m$ and $\theta_G(m) = |\theta_G(A)|$.

Problem 2 : Find a subset of vertices of a given graph, such that the number of edges in the subgraph induced by this subset is maximal among all induced subgraphs with the same number of vertices. Mathematically, for a given m, if $I_G(m) = \max_{A\subseteq V,\ |A|=m} |I_G(A)|$ where $I_G(A) = \{(u,v) \in E : u,v \in A\}$, then the problem is to find $A \subseteq V$ such that $|A| = m$ and $I_G(m) = |I_G(A)|$.

For a given m, where $m = 1,2,\ldots,n$, we consider the problem of finding a subset A of vertices of G such that $|A| = m$ and $|\theta_G(A)| = \theta_G(m)$. Such subsets are called optimal [5, 21].

Further, if a subset of vertices is optimal with respect to Problem 1, then its complement is also an optimal set. But, it is not true for Problem 2 in general. However for regular graphs a subset of vertices S is optimal with respect to Problem 1 if and only if S is optimal for Problem 2 [5]. In the literature, Problem 2 is defined as the maximum subgraph problem [17].

Lemma 2.4. (Congestion Lemma) [29] *Let G be an r-regular graph and f be an embedding of G into H. Let S be an edge cut of H such that the removal of edges of S leaves H into 2 components H_1 and H_2 and let $G_1 = f^{-1}(H_1)$ and $G_2 = f^{-1}(H_2)$. Also S satisfies the following conditions:*

(i) *For every edge $(a,b) \in G_i$, $i = 1,2$, $P_f(a,b)$ has no edges in S.*

(ii) *For every edge (a,b) in G with $a \in G_1$ and $b \in G_2$, $P_f(a,b)$ has exactly one edge in S.*

(ii) *G_1 is an optimal set.*

Then $EC_f(S)$ is minimum and $EC_f(S) = \sum_{e\in S} EC_f(e) = r\,|V(G_1)| - 2\,|E(G_1)|$.

Lemma 2.5. (Partition Lemma) [29] Let $f : G \to H$ be an embedding. Let $\{S_1, S_2, \ldots, S_p\}$ be a partition of $E(H)$ such that each S_i is an edge cut of H. Then

$$WL_f(G, H) = \sum_{i=1}^{p} EC_f(S_i). \quad \square$$

Definition 2.6. [43] *For $r \geq 1$, let Q^r denote the r-dimensional hypercube. The vertex set of Q^r is formed by the collection of all r-dimensional binary strings. Two vertices $x, y \in V(Q^r)$ are adjacent if and only if the corresponding binary strings differ exactly in one bit. The vertices of Q^r can also be identified with integers $0, 1, \ldots, n-1$.*

Equivalently if $n = 2^r$ then so that if a pair of vertices i and j are adjacent then $i - j = \pm 2^p$ for some $p \geq 0$.

Definition 2.7. [23] *An incomplete hypercube on i vertices of Q^r is the subcube induced by $\{0, 1, \ldots, i-1\}$ and is denoted by L_i, $1 \leq i \leq 2^r$.*

Theorem 2.8. [6, 10, 21] *Let Q^r be an r-dimensional hypercube. For $1 \leq i \leq 2^r$, L_i is an optimal set on i vertices.*

Lemma 2.9. [29] *Let Q^r be an r-dimensional hypercube. Let $m = 2^{t_1} + 2^{t_2} + \cdots + 2^{t_l}$ such that $r \geq t_1 > t_2 > \cdots > t_l \geq 0$. Then* $|E(Q^r[L_m])| = [t_1 \cdot 2^{t_1 - 1} + t_2 \cdot 2^{t_2 - 1} + \cdots + t_l \cdot 2^{t_l - 1}] + [2^{t_2} + 2 \cdot 2^{t_3} + \cdots + (l-1)2^{t_l}]$.

Lemma 2.10. [31] *For $1 \leq j < n$ and $i = 1, 2, \ldots, 2^{r_j}$*

$$\begin{aligned} Lex_j^i = \{ \; & m + x_{j+1} \cdot 2^{r_1 + r_2 + \cdots + r_j} + x_{j+2} \cdot 2^{r_1 + r_2 + \cdots + r_{j+1}} + \cdots \\ & + x_n \cdot 2^{r_1 + r_2 + \cdots + r_{n-1}} : 0 \leq m \leq i \cdot 2^{r - (r_j + \cdots + r_n)} - 1, \\ & 0 \leq x_k \leq 2^{r_k} - 1, \; k = j+1, j+2, \ldots, n \; \} \end{aligned}$$

is an optimal set on $i \times 2^{r - r_j}$ vertices in Q^r where $r_1 + r_2 + \cdots + r_n = r$, $r_1 \leq r_2 \leq \cdots \leq r_n$.

3 Wirelength of Embedding Hypercubes into Generalized Books

In this section, we compute the minimum wirelength of embedding r-dimensional hypercubes into generalized books $GB[2^{r_1}, 2^{r_2}, 2^{r_3}]$. For proving the main result, we need the following Lemma.

Lemma 3.1. *For $i = 1, 2, \ldots, r-1$ and $i < j \leq r$, $NcutS_i^{2^j} = \{2^j - 1, 2^j - 2, \ldots, x\}$, $x \geq 2^i$ is an optimal set in Q^r.*

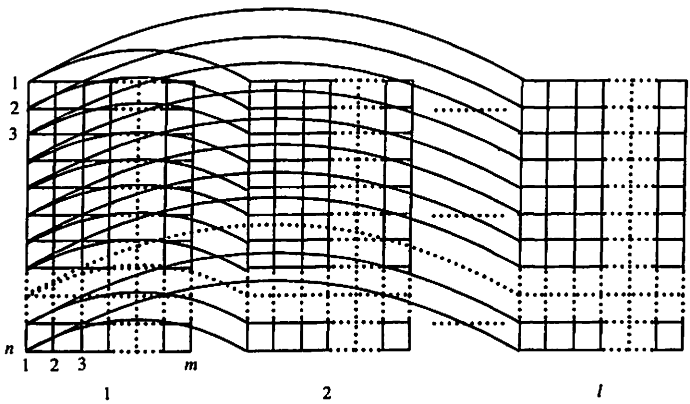

Figure 2: Generalized Book $GB[2^{r_1}, 2^{r_2}, 2^{r_3}]$

Proof. Define $\varphi : NcutS_i^{2^j} \to L_{2^j+2^i}$ by $\varphi(2^j - k) = k - 1$. If the binary representation of $2^j - k$ is $\underbrace{00\cdots00}_{r-j \text{ times}}\alpha_{r-j+1}\alpha_{r-j+2}\cdots\alpha_r$ then the binary representation of $k-1$ is $\underbrace{00\cdots00}_{r-j \text{ times}}\alpha_{r-j+1}\alpha_{r-j+2}\cdots\alpha_r$. Thus the binary representation of two numbers x and y differ in exactly one bit $\Leftrightarrow$ the binary representation of $\varphi(x)$ and $\varphi(y)$ differ in exactly one bit. Therefore (x, y) is an edge in $NcutS_i^{2^j}$ $\Leftrightarrow$ $(\varphi(x), \varphi(y))$ is an edge in $L_{2^j+2^i}$. Hence $NcutS_i^{2^j}$ and $L_{2^j+2^i}$ are isomorphic. By Theorem 2.8, $NcutS_i^{2^j}$ is an optimal set in Q^r.

An $n \times m$ *mesh* denoted by $M[n \times m]$ is nothing but the cartesian product $P_n \times P_m$. A mesh is also referred to as a grid.

Grid embedding plays an important role in computer architecture. VLSI Layout Problem, Crossing Number Problem, Graph Drawing, and Edge Embedding Problem are all a part of grid embedding. There are very few papers in the literature which provide the exact wirelength of grid embedding [29]. Generalized book is an extension of the grid network and is defined as follows.

Definition 3.2. *Let $M[n \times m]$ be a $n \times m$ mesh with n rows and m columns. A graph which is obtained from l copies of M, say $M_1, M_2, \ldots, M_l$, by joining each vertex of the 1st column of M_1 to the corresponding vertex of the 1st column of M_i by an edge for all $i = 2, 3, \ldots, l$ is called a generalized book and is denoted by $GB[n, m, l]$.*

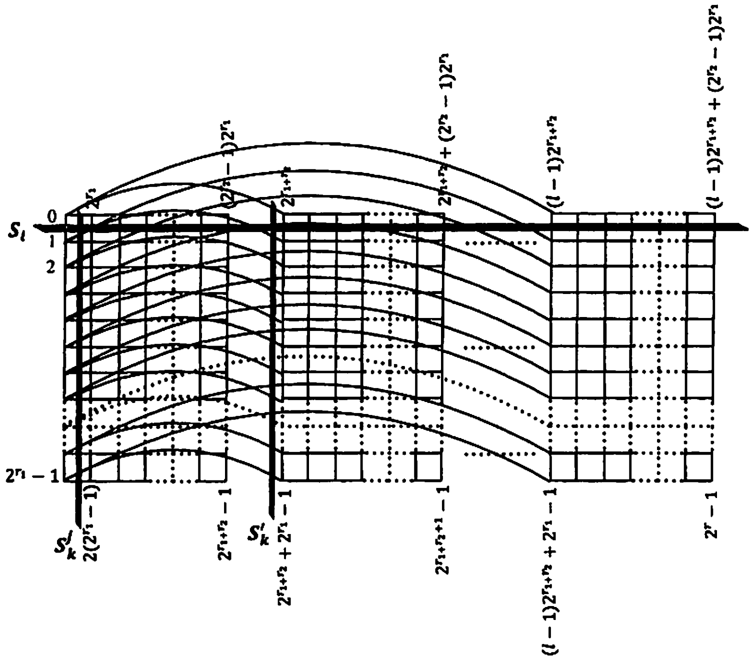

Figure 3: Edge cut of $GB[2^{r_1}, 2^{r_2}, 2^{r_3}]$

Remark 3.3. *$GB[n, m, l]$ has nml vertices and $ml(2n-1)-n$ edges. The diameter $d(GB[n, m, l]) = n + 2m - 1$. See Figure 2.*

Wirelength Algorithm

Input : The r-dimensional hypercube Q^r and the generalized book $GB[2^{r_1}, 2^{r_2}, 2^{r_3}]$, where $r_1 + r_2 + r_3 = r$.

Algorithm : The lexicographic embedding [3] of Q^r with the labeling 0 to $2^r - 1$ into $GB[2^{r_1}, 2^{r_2}, 2^{r_3}]$ is an assignment of labels to the nodes of $GB[2^{r_1}, 2^{r_2}, 2^{r_3}]$ as follows:

For $1 \leq k \leq 2^{r_3}$, $1 \leq j \leq 2^{r_2}$, the vertices in the j^{th} column of M_k of $GB[2^{r_1}, 2^{r_2}, 2^{r_3}]$ are labeled $(k-1)2^{r_1+r_2} + (j-1)2^{r_1}, (k-1)2^{r_1+r_2} + (j-1)2^{r_1} + 1, \ldots, (k-1)2^{r_1+r_2} + (j-1)2^{r_1} + 2^{r_1} - 1$ from top to bottom. See Figure 3.

Output : An embedding f of Q^r into $GB[2^{r_1}, 2^{r_2}, 2^{r_3}]$ given by $f(x) = x$ with minimum wirelength.

Proof of Correctness : We assume that the labels represent the vertices to which they are assigned. For $1 \leq i \leq 2^{r_1} - 1$, let $S_i = \{((i-1) + (j-1)2^{r_1} + (k-1)2^{r_1+r_2}, (i-1) + (j-1)2^{r_1} + (k-1)2^{r_1+r_2} + 1) : 1 \leq k \leq 2^{r_3}, 1 \leq j \leq 2^{r_2}\}$. For $1 \leq k \leq 2^{r_3} - 1$, let $S'_k = \{((i-1), k(2^{r_1+r_2}) + (i-1)) : 1 \leq i \leq 2^{r_1}\}$. For $1 \leq k \leq 2^{r_3}$ and $1 \leq j \leq 2^{r_2} - 1$, let $S^j_k = \{(k-1)2^{r_1+r_2} + (i-1) + (j-1)2^{r_1}, (k-1)2^{r_1+r_2} + (i-1) + j2^{r_1} : 1 \leq i \leq 2^{r_1}\}$. See Figure 3. Then $\{S_i : 1 \leq i \leq 2^{r_1} - 1\} \cup \{S'_k : 1 \leq k \leq 2^{r_3} - 1\} \cup \{S^j_k : 1 \leq k \leq 2^{r_3}, 1 \leq j \leq 2^{r_2} - 1\}$ is a partition of $[E(GB[2^{r_1}, 2^{r_2}, 2^{r_3}])]$.

For each i, $1 \leq i \leq 2^{r_1} - 1$, $E(GB[2^{r_1}, 2^{r_2}, 2^{r_3}]) \backslash S_i$ has two components H_{i1} and H_{i2}, where $V(H_{i1}) = \{(i-1) + (j-1)2^{r_1} + (k-1)2^{r_1+r_2} : 1 \leq k \leq 2^{r_3}, 1 \leq j \leq 2^{r_2}\}$. Let $G_{i1} = f^{-1}(H_{i1})$ and $G_{i2} = f^{-1}(H_{i2})$. By Lemma 2.10, G_{i1} is an optimal set and each S_i satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore $EC_f(S_i)$ is minimum.

For each k, $1 \leq k \leq 2^{r_3} - 1$, $E(GB[2^{r_1}, 2^{r_2}, 2^{r_3}]) \backslash S'_k$ has two components H'_{k1} and H'_{k2}, where $V(H'_{k1}) = \{k2^{r_1+r_2}, k2^{r_1+r_2} + 1, \ldots, k2^{r_1+r_2} + 2^{r_1+r_2} - 1\}$. Let $G'_{k1} = f^{-1}(H'_{k1})$ and $G'_{k2} = f^{-1}(H'_{k2})$. By Lemma 3.1, G'_{k1} is an optimal set and each S'_k satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore $EC_f(S'_k)$ is minimum.

For each k, j, $1 \leq k \leq 2^{r_3}$ and $1 \leq j \leq 2^{r_2} - 1$, $E(GB[2^{r_1}, 2^{r_2}, 2^{r_3}]) \backslash S^j_k$ has two components H^j_{k1} and H^j_{k2}, where $V(H^j_{k1}) = \{(k-1)2^{r_1+r_2}, (k-1)2^{r_1+r_2} + 1, \ldots, (k-1)2^{r_1+r_2} + 2^{r_1+r_2} - 1\} \backslash \{(k-1)2^{r_1+r_2} + (i-1) + (j-1)2^{r_1} : 1 \leq i \leq 2^{r_1}\}$. Let $G^j_{k1} = f^{-1}(H^j_{k1})$ and $G^j_{k2} = f^{-1}(H^j_{k2})$. By Lemma 3.1, G^j_{k1} is an optimal set, each S^j_k satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore $EC_f(S^j_k)$ is minimum. The Partition Lemma implies that the wirelength is minimum.

Theorem 3.4. *The minimum wirelength of Q^r into $GB[2^{r_1}, 2^{r_2}, 2^{r_3}]$, $r_1 + r_2 + r_3 = r$ is given by*

$$WL(Q^r, GB[2^{r_1}, 2^{r_2}, 2^{r_3}]) = \sum_{i=1}^{2^{r_1}-1} (r \cdot i(2^{r_2+r_3}) - 2\varepsilon_{i(2^{r_2+r_3})} + r_3(2^{r_3} - 1)2^{r_1+r_2} + \sum_{k=1}^{2^{r_3}} \sum_{j=1}^{2^{r_2}-1} [r(2^{r_1+r_2} - k \cdot 2^{r_1}) - 2\, \varepsilon_{2^{r_1+r_2} - k \cdot 2^{r_1}}],$$

where ε_i denotes the number of edges in L_i.

Proof. Label the vertices of Q^r and $GB[2^{r_1}, 2^{r_2}, 2^{r_3}]$ using Wirelength Algorithm. By Congestion Lemma,

(i) $EC_f(S_i) = r \cdot i(2^{r_2+r_3}) - 2\, \varepsilon_{i \cdot 2^{r_2+r_3}}$, $1 \leq i \leq 2^{r_1} - 1$

(ii) $EC_f(S'_k) = r_3 \cdot 2^{r_1+r_2}$, $1 \leq k \leq 2^{r_3} - 1$ and

(iii) $EC_f(S^j_k) = r(2^{r_1+r_2} - k \cdot 2^{r_1}) - 2\varepsilon_{2^{r_1+r_2}-k\cdot 2^{r_1}}$, $1 \leq k \leq 2^{r_3}$ and $1 \leq j \leq 2^{r_2} - 1$.

Then by Partition Lemma,

$$
\begin{aligned}
WL(Q^r, GB[2^{r_1}, 2^{r_2}, 2^{r_3}]) &= \sum_{i=1}^{2^{r_1}-1} (r \cdot i(2^{r_2+r_3}) - 2\varepsilon_{i.2^{r_2+r_3}}) \\
&\quad + \sum_{k=1}^{2^{r_3}-1} 2^{r_1+r_2}(r - (r_1 + r_2)) \\
&\quad + \sum_{k=1}^{2^{r_3}} \sum_{j=1}^{2^{r_2}-1} [r(2^{r_1+r_2} - k \cdot 2^{r_1}) - 2\, \varepsilon_{2^{r_1+r_2}-k\cdot 2^{r_1}}] \\
&= \sum_{i=1}^{2^{r_1}-1} (r \cdot i(2^{r_2+r_3}) - 2\varepsilon_{i(2^{r_2+r_3})} + r_3(2^{r_3} - 1)2^{r_1+r_2} \\
&\quad + \sum_{k=1}^{2^{r_3}} \sum_{j=1}^{2^{r_2}-1} [r(2^{r_1+r_2} - k \cdot 2^{r_1}) - 2\, \varepsilon_{2^{r_1+r_2}-k\cdot 2^{r_1}}].
\end{aligned}
$$

4 Time Complexity

In computer science, the time complexity of an algorithm quantifies the amount of time taken by an algorithm to run as a function of the size of the input to the problem. An algorithm is said to take linear time, or $O(n)$ time, if its time complexity is $O(n)$. Informally, this means that for large enough input sizes the running time increases linearly with the size of the input.

Linear time is often viewed as a desirable attribute for an algorithm. Much research has been carried out into creating algorithms exhibiting (nearly) linear time or better. This research includes both software and hardware methods. In the case of hardware, some algorithms which, mathematically speaking, can never achieve linear time with standard computation models are able to run in linear time. There are several hardware technologies which exploit parallelism to provide this. An example is content-addressable memory. This concept of linear time is used in string matching algorithms such as the Boyer-Moore Algorithm and Ukkonen's Algorithm [22, 40].

In this Section, we compute the time complexity of finding the minimum wirelength of embedding hypercube into generalized book using the proof of Theorem 3.4. The algorithm is formally presented as follows.

Time Complexity Algorithm

Input : The r-dimensional hypercube Q^r and the generalized book $GB[2^{r_1}, 2^{r_2}, 2^{r_3}]$, where $r_1 + r_2 + r_3 = r$.

Algorithm : Wirelength Algorithm.

Output : The time taken to compute the minimum wirelength of embedding Q^r into $GB[2^{r_1}, 2^{r_2}, 2^{r_3}]$ is $O(n)$, $n = 2^r$, which is linear.

Method : We know that, Q^r contains 2^r vertices. For assigning the labels of 2^r vertices, we require 2^r time units. By Theorem 3.4, we have $2^{r_1} - 1 + 2^{r_3} - 1 + 2^{r_3}(2^{r_2} - 1)$ edge cuts. Since $(2^{r_3} - 1)(2^{r_2} - 1)$ edge cuts are similar and we need one time unit for each edge cut, we require $2^{r_1} - 1 + 2^{r_3} - 1 + 2^{r_2}$ time units. Then, we require $2^{r_1 - 1} + 2^{r_2} + 1$ time units for finding the congestion on edge cuts and the same time units to apply Partition Lemma.

$$
\begin{aligned}
\text{Hence the total time taken is} &= 2^r + 2^{r_1} - 1 + 2^{r_3} - 1 + 2^{r_2} \\
&\quad +2(2^{r_1-1} + 2^{r_2} + 1) \\
&= 2^r + 2^{r_1+1} + 3\ 2^{r_2} + 2^{r_3} \\
&= O(n).
\end{aligned}
$$

Hence, the time taken to compute the minimum wirelength of embedding Q^r into $GB[2^{r_1}, 2^{r_2}, 2^{r_3}]$ is $O(n)$-linear time, where $|V(Q^r)| = |V(GB[2^{r_1}, 2^{r_2}, 2^{r_3}])| = 2^r = n$.

5 Dilation of Embedding Hypercubes into Generalized Books

The dilation problem and the wirelength problem are different in the sense that an embedding that gives minimum dilation need not give minimum wirelength and vice-versa. In the literature there is no efficient method to compute exact dilation of graph embeddings [2, 18]. Recently, Paul Manuel, Indra Rajasingh and Sundara Rajan, obtained a strategy to compute a lower bound for dilation using exact wirelength and formulated the result as IPS Lemma [28].

Lemma 5.1. (IPS Lemma)[28] *Let G and H be graphs on same number of vertices. Let δ and WL be the dilation and wirelength of embedding graph*

G into graph H and let $f : G \to H$ be an embedding realizing δ. If d_{i_j} number of edges in G are of dilation i_j with respect to f, $1 \le j \le k$, then

$$\delta \geq \frac{WL - \sum_{j=1}^{k} i_j \, d_{i_j}}{|E| - \sum_{j=1}^{k} d_{i_j}}.$$

The following is a particular case.

Lemma 5.2. *Let δ and WL be the dilation and wirelength of embedding graph G into graph H. Then $\delta \geq \frac{WL}{|E(G)|}$.*

Next, we embed hypercube Q^r into generalized book $GB[4, 2, 2^{r-3}]$ with dilation 3 and prove that the lower bound for $GB[4, 2, 2^{r-3}]$ obtained using Lemma 5.2 is sharp. For proving this result we need the following result which is a particular case of Theorem 3.4.

Lemma 5.3. *The minimum wirelength of Q^r into $GB[4, 2, 2^{r-3}]$, $r > 2$ is given by*

$$WL(Q^r, GB[4, 2, 2^{r-3}]) \quad = \quad 2^{r-1}(3r - 5) - 8r + 24.$$

Dilation Algorithm A

Input : The r-dimensional hypercube Q^r and the generalized book $GB[4, 2, 2^{r-3}]$, $r \geq 6$.

Algorithm : The lexicographic embedding [3] of Q^r with the labeling 0 to $2^r - 1$ into $GB[4, 2, 2^{r-3}]$ is an assignment of labels to the nodes of $GB[4, 2, 2^{r-3}]$ as follows:

For $1 \leq i \leq 2^{r-3}$, $1 \leq j \leq 2$, the vertices in the j^{th} column of M_i of $GB[4, 2, 2^{r-3}]$ are labeled $8(i-1) + 4(j-1), 8(i-1) + 4(j-1) + 1, \ldots, 8(i-1) + 4(j-1) + 3$ from top to bottom.

Output : An embedding f of Q^r into $GB[4, 2, 2^{r-3}]$ given by $f(x) = x$ with minimum dilation. See Figure 4.

Proof of correctness : By Lemma 5.2 and Lemma 5.3, we get

$$\begin{aligned} \delta &\geq \frac{2^{r-1}(3r - 5) - 8r + 24}{r \, 2^{r-1}} \\ &= 2 + \frac{2^{r-1}(r - 5) - 8r + 24}{r \, 2^{r-1}} \\ &> 2. \end{aligned}$$

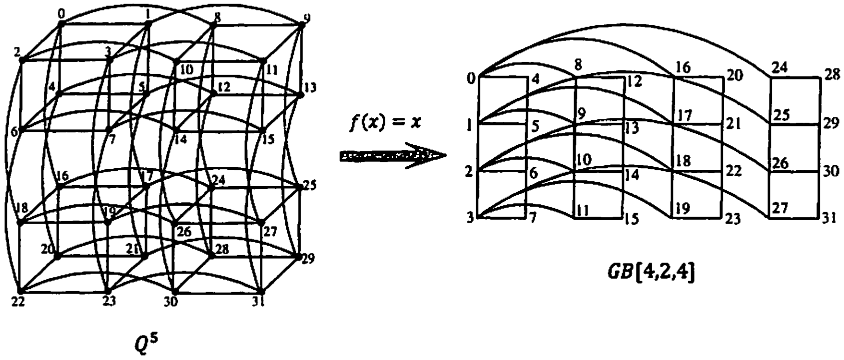

Figure 4: Embedding of Q^5 into $GB[4, 2, 4]$ with dilation 3

By Dilation Algorithm, any edge $e \in Q^r$ is mapped into a path of length at most 3 in $GB[4, 2, 2^{r-3}]$.

The proof of the following theorem is an easy consequence of Dilation Algorithm A.

Theorem 5.4. *The r-dimensional hypercube Q^r can be embedded into a generalized book $GB[4, 2, 2^{r-3}]$ with dilation 3, $r \geq 6$.*

Next, we embed hypercube Q^r into another class $GB[l]$ of books with dilation 2. $GB[l]$, $l = 2^{r-2} - 1, r > 2$ is a particular case of generalized book $GB[2^{r_1}, 2^{r_2}, 2^{r_3}]$, when grid becomes path.

In other words, we define $GB[l]$ as follows.

Definition 5.5. *Let $M[4 \times 2]$ be a 4×2 mesh with 4 rows and 2 columns. A graph which is obtained from l copies of M, say $M_1, M_2, \ldots, M_l$, by identifying the vertices of the 1st column of M_1 and the corresponding vertices of the 1st column of M_i for all $i = 2, 3, \ldots, l$.*

Remark 5.6. *$GB[l]$ has $4(l+1)$ vertices and $7l+3$ edges. The diameter $d(GB[l]) = 4$.*

The following Lemma is a particular case of Theorem 3.4.

Lemma 5.7. *For $l = 2^{r-2} - 1, r > 2$, the minimum wirelength of Q^r into $GB[l]$ is given by*

$$WL(Q^r, GB[l]) = 2^{r-1}(2r - 1) - 4(r - 2).$$

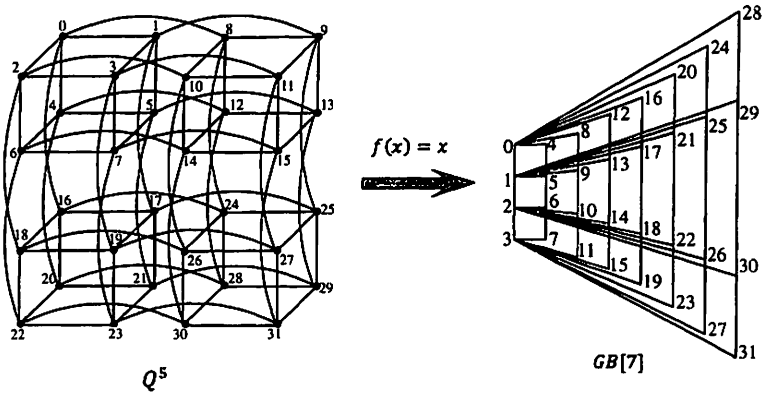

Figure 5: Embedding of Q^5 into $GB[7]$ with dilation 2

Dilation Algorithm B

Input : The r-dimensional hypercube Q^r and the book $GB[l]$, where $l = 2^{r-2} - 1, r > 2$.

Algorithm : The lexicographic embedding [3] of Q^r with the labeling 0 to $2^r - 1$ into $GB[l]$ is an assignment of labels to the nodes of $GB[l]$ as $0, 1, 2, \ldots, 2^r - 1$ from top to bottom.

Output : An embedding f of Q^r into $GB[l]$ given by $f(x) = x$ with minimum dilation. See Figure 5.

Proof of correctness : By Lemma 5.2 and Lemma 5.7, we get

$$\begin{aligned} \delta &\geq \frac{2^{r-1}(2r-1) - 4(r-2)}{r\,2^{r-1}} \\ &= 2 - \frac{2^{r-1} + 4r - 8}{r\,2^{r-1}} \\ &> 1 \text{ as } 0 < \frac{2^{r-1} + 4r - 8}{r\,2^{r-1}} < 1. \end{aligned}$$

By Dilation Algorithm, any edge $e \in Q^r$ is mapped into a path of length at most 2 in $GB[l]$.

The proof of the following theorem is an easy consequence of Dilation Algorithm B.

Theorem 5.8. *For $l = 2^{r-2} - 1, r > 2$, the r-dimensional hypercube Q^r can be embedded into a book $GB[l]$ with dilation 2.*

Remark 5.9. *Since Q^r is not a subgraph of $GB[l]$, the dilation of embedding Q^r into $GB[l]$ is >1. By Dilation Algorithm, Q^r can be embedded into $GB[l]$ with dilation 2. This is ano ther way to prove Theorem 5.8 without using Lemma 5.2.*

6 Concluding Remarks

Obtaining embeddings of r-dimensional hypercube Q^r into $GB[4, 2, 2^{r-3}]$ and $GB[l]$ with minimum dilation opens up the study of dilation problem which remains open problem for several architectures. Further, we compute the minimum wirelength of embedding r-dimensional hypercube into generalized book $GB[2^{r_1}, 2^{r_2}, 2^{r_3}]$. We provide an $O(n)$-linear time algorithm to compute minimum wirelength of embedding r-dimensional hypercube into generalized book $GB[2^{r_1}, 2^{r_2}, 2^{r_3}]$. Finding the dilation of embedding hypercube into $GB[2^{r_1}, 2^{r_2}, 2^{r_3}]$ is under investigation.

Using the techniques of this paper and combining the results of the papers [26, 28], we may obtain the following results.

Theorem 6.1. *The minimum wirelength of r-dimensional folded hypercube FQ^r into generalized book $GB[2^{r_1}, 2^{r_2}, 2^{r_3}]$ is given by*

$$\begin{aligned} WL(FQ^r, GB[2^{r_1}, 2^{r_2}, 2^{r_3}]) &= \sum_{i=1}^{2^{r_1}-1} ((r+1) \cdot i \cdot (2^{r_2+r_3}) - 2\varepsilon_{i \cdot 2^{r_2+r_3}}) \\ &\quad + (r_3+1)(2^{r_3}-1)2^{r_1+r_2} \\ &\quad + \sum_{k=1}^{2^{r_3}} \sum_{j=1}^{2^{r_2}-1} [(r+1)(2^{r_1+r_2} - k \cdot 2^{r_1}) \\ &\quad - 2\,\varepsilon_{2^{r_1+r_2} - k \cdot 2^{r_1}}], \end{aligned}$$

where ε_i denotes the number of edges in L_i.

Theorem 6.2. *The minimum wirelength of r-dimensional augmented cube AQ^r into generalized book $GB[2^{r_1}, 2^{r_2}, 2^{r_3}]$ is given by*

$$\begin{aligned} WL(AQ^r, GB[2^{r_1}, 2^{r_2}, 2^{r_3}]) &= \sum_{i=1}^{2^{r_1}-1} ((2r-1) \cdot i \cdot (2^{r_2+r_3}) - 2\varepsilon_{i \cdot 2^{r_2+r_3}}) \\ &\quad + (2^{r_3}-1)[(2r-1)(2^{r_1+r_2}) - 2\varepsilon_{2^{r_1+r_2}}] \\ &\quad + \sum_{k=1}^{2^{r_3}} \sum_{j=1}^{2^{r_2}-1} [(2r-1)(2^{r_1+r_2} - k \cdot 2^{r_1}) - 2\,\varepsilon_{2^{r_1+r_2} - k \cdot 2^{r_1}}], \end{aligned}$$

where ε_i denotes the number of edges in L_i.

Open Problem : Embedding variants of hypercube such as crossed cube, twisted cube, Möbius cube and Fibonacci cube into generalized book with minimum dilation and wirelength.

References

[1] Arockiaraj M, Manuel P, Rajasingh I, Rajan B. Wirelength of 1-fault hamiltonian graphs into wheels and fans. Inform Process Lett 2011; 111: 921-925.

[2] Bezrukov S.L. Embedding complete trees into the hypercube. Discrete Appl Math 2001; 110: 101-119.

[3] Bezrukov S.L, Chavez J.D, Harper L.H, Röttger M, Schroeder U.P. Embedding of hypercubes into grids. Lect Notes Comput Sc 1998; 693-701.

[4] Bezrukov S.L, Chavez J.D, Harper L.H, Röttger M, Schroeder U.P. The congestion of n-cube layout on a rectangular grid. Discrete Math 2000; 213: 13-19.

[5] Bezrukov S.L, Das S.K, Elsässer R, An edge-isoperimetric problem for powers of the Petersen graph, Ann Comb 2000; 4: 153-169.

[6] Boals A.J, Gupta A.K, Sherwani N.A. Incomplete hypercubes: Algorithms and embeddings. J Supercomput 1994; 8: 263-294.

[7] Caha R, Koubek V. Optimal embeddings of generalized ladders into hypercubes. Discrete Math 2001; 233: 65-83.

[8] Chavez J.D, Trapp R. The cyclic cutwidth of trees, Discrete Appl Math 1998; 87: 25-32.

[9] Chen B. On embedding rectangular grids in hypercubes. IEEE T Comput 1988; 37(10): 1285-1288.

[10] Chen H.-L, Tzeng N.-F, A boolean expression-based approach for maximum incomplete subcube identification in faulty hypercubes. IEEE T Parall Distr 1997; 8: 1171-1183.

[11] Chen W.K, Stallmann M.F.M. On embedding binary trees into hypercubes. J Parallel Distr Com 1995; 24: 132-138.

[12] Choudum S.A, Sunitha V. Augmented cubes, Networks 2002; 40(2): 71-84.

[13] Dvořák T. Dense sets and embedding binary trees into hypercubes. Discrete Appl Math 2007; 155(4): 506-514.

[14] Ellis J.A, Embedding rectangular grids into square grids. IEEE T Comput 1991; 40(1): 46-52.

[15] Fan J, Jia X. Embedding meshes into crossed cubes. Inform Sciences 2007; 177(15): 3151-3160.

[16] Fang J.-F, Lai K.-C. Embedding the incomplete hypercube in books, Inform Sciences 2005; 96: 1-6.

[17] Garey M.R, Johnson D.S, Computers and Intractability, A Guide to the Theory of NP-Completeness, Freeman, San Francisco 1979.

[18] Gupta A.K, Nelson D, Wang H. Efficient embeddings of ternary trees into hypercubes, J Parallel Distr Com 2003; 63(6): 619-629.

[19] Guu C.-J. The Circular Wirelength Problem for Hypercubes. Ph.D. dissertation, University of California, Riverside, 1997.

[20] Han Y, Fan J, Zhang S, Yang J, Qian P. Embedding meshes into locally twisted cubes, Inform Sciences 2010; 180(19): 3794-3805.

[21] Harper L.H, Global Methods for Combinatorial Isoperimetric Problems, Cambridge University Press, 2004.

[22] Horowitz E, Sahni S. Fundamentals of Computer Algorithms. Computer Science Press, New York, 1978.

[23] Katseff H. Incomplete Hypercubes. IEEE T Comput 1988; 37: 604-608.

[24] Lai P.-L, Tsai C.-H. Embedding of tori and grids into twisted cubes. Theor Comput Sci 2010; 411: 3763-3773.

[25] Lai Y.L, Williams K. A survey of solved problems and applications on bandwidth, edgesum, and profile of graphs. J Graph Theor 1999; 31: 75-94.

[26] Manuel P. Minimum average congestion of enhanced and augmented hypercube into complete binary tree. Discrete Appl Math 2010; 159(5): 360-366.

[27] Manuel P, Arockiaraj M, Rajasingh I, Rajan B. Embedding hypercubes into cylinders, snakes and caterpillars for minimizing wirelength. Discrete Appl Math 2011; 159(17): 2109-2116.

[28] Manuel P, Rajasingh I, Rajan R.S, Embedding variants of hypercubes with dilation 2. Journal of Interconnection Networks 2012; 13: 1-16.

[29] Manuel P, Rajasingh I, Rajan B, Mercy H. Exact wirelength of hypercube on a grid. Discrete Appl Math 2009; 157(7): 1486-1495.

[30] Opatrny J, Sotteau D. Embeddings of complete binary trees into grids and extended grids with total vertex-congestion 1, Discrete Appl Math 2000; 98(3): 237-254.

[31] Rajasingh I, Arockiaraj M, Rajan B, Manuel P, Minimum wirelength of hypercubes into n-dimensional grid networks. Inform Process Lett 2012; 112: 583-586.

[32] Rajasingh I, Rajan B, Rajan R.S. On embedding of m-sequencial k-ary trees into hypercubes. Applied Mathematics 2010; 1(6): 499-503.

[33] Rajasingh I, Rajan B, Rajan R.S. Embedding of hypercubes into necklace, windmill and snake graphs. Inform Process Lett 2012; 112: 509–515.

[34] Rajasingh I, Rajan B, Rajan R.S. Embedding of special classes of circulant networks, hypercubes and generalized Petersen graphs. Int J Comput Math 2012; 89(15): 1970-1978.

[35] Rajasingh I, Manuel P, Arockiaraj M, Rajan B. Embeddings of circulant networks. J Comb Optim 2013; 26(1): 135-151.

[36] Rajasingh I, Rajan R.S. Exact wirelength of embedding circulant networks into necklace and windmill Graphs, Ars Combinatoria, (in press).

[37] Rajasingh I, Quadras J, Manuel P, William A. Embedding of cycles and wheels into arbitrary trees. Networks 2004; 44: 173-178.

[38] Rottger M, Schroeder U.P, Efficient embeddings of grids into grids. Discrete Appl Math 2001; 108(1-2): 143-173.

[39] Saad Y, Schultz M.H. Topological properties of hypercubes. IEEE T Compu 1988; 37(7): 867-872.

[40] Sipser M, Introduction to the Theory of Computation, Course Technology Inc., Boston, 2006.

[41] Tsai C.-H. Embedding of meshes in Möbius cubes. Theor Comput Sci 2008; 401(1-3): 181-190.

[42] Vodopivec A. On embeddings of snarks in the torus. Discrete Math 2008; 308(10): 1847-1849.

[43] Xu J.M. Topological Structure and Analysis of Interconnection Networks. Kluwer Academic Publishers, 2001.

[44] Yang M.-C, Path embedding in star graphs, Appl Math Compu 2009; 207(2): 283-291.

[45] Yang X, Dong Q, Tan Y.Y. Embedding meshes/tori in faulty crossed cubes. Inform Process Lett 2010; 110: 559-564.