Toughness for the existence of k-Hamiltonian $[a, b]$-factors *†

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Abstract

Let a, b and k be three nonnegative integers with $a \geq 2$ and $b \geq a(k+1)+2$. A graph G is called a k-Hamiltonian graph if $G-U$ contains a Hamiltonian cycle for every subset $U \subseteq V(G)$ with $|U| = k$. An $[a, b]$-factor F of G is called a Hamiltonian $[a, b]$-factor if F contains a Hamiltonian cycle. If $G-U$ has a Hamiltonian $[a, b]$-factor for every subset $U \subseteq V(G)$ with $|U| = k$, then we say that G admits a k-Hamiltonian $[a, b]$-factor. Suppose that G is a k-Hamiltonian graph of order n with $n \geq a+k+2$. In this paper, it is proved that G includes a k-Hamiltonian $[a, b]$-factor if $\delta(G) \geq a+k$ and $t(G) \geq a-1+\frac{(a-1)(k+1)}{b-2}$.

Keywords: toughness, k-Hamiltonian graph, k-Hamiltonian $[a, b]$-factor.

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1 Introduction

We consider finite undirected graphs which do not include loops and multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex x of a graph G is defined as the number of edges which are incident to x, and is denoted by $d_G(x)$. The neighborhood $N_G(x)$ of a vertex x is defined as $N_G(x) = \{y \in V(G) : yx \in E(G)\}$. More generally $N_G(S) = \cup_{x\in S}N_G(x)$ for a subset $S \subseteq V(G)$. For a subset $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S, and write $G - S$ for $G[V(G) \setminus S]$. We use $\delta(G)$ and $\Delta(G)$ to denote the minimum degree and the maximum degree of G, respectively. We denote by $\omega(G)$ the number of connected components of G. A vertex subset S of G is called an independent set (a covering set) of G if each edge of G is incident with at most (at least) one vertex of S. It is not very difficult to deduce that a vertex subset S of G is an independent set of G if and only if $V(G) \setminus S$ is a covering set of G. The toughness $t(G)$ of a graph G was first defined by Chvátal in [2] as follows.

$$t(G) = \min\{\frac{|S|}{\omega(G-S)} : S \subseteq V(G), \omega(G-S) \geq 2\},$$

if G is not complete; otherwise, $t(G) = +\infty$.

Let a and b be two nonnegative integers with $1 \leq a \leq b$. Then a spanning subgraph F of G is called an $[a, b]$-factor if F satisfies $a \leq d_F(x) \leq b$ for each $x \in V(G)$. In particular, an $[r, r]$-factor is an r-factor. If for any $U \subseteq V(G)$ with $|U| = k$, $G-U$ admits a Hamiltonian cycle, then we say that G is a k-Hamiltonian graph. An $[a, b]$-factor is Hamiltonian if it includes a Hamiltonian cycle. If for any subset $U \subseteq V(G)$ with $|U| = k$, $G - U$ contains a Hamiltonian $[a, b]$-factor, then we say that G has a k-Hamiltonian $[a, b]$-factor. A k-Hamiltonian $[r, r]$-factor is a k-Hamiltonian r-factor. In particular, a 0-Hamiltonian graph is a Hamiltonian graph; a 0-Hamiltonian $[a, b]$-factor is said to be a Hamiltonian $[a, b]$-factor. Hamiltonian factors in graphs attract a great deal of attention [1, 4, 8, 9, 10].

The relationships between toughness and graph factors are investigated in [3, 5]. In this paper, we study k-Hamiltonian $[a, b]$-factors in graphs and obtain a toughness condition for graphs to have k-Hamiltonian $[a, b]$-factors. Our main result is the following theorem.

Theorem 1 *Let a, b, k be nonnegative integers with $a \geq 2$ and $b \geq a(k + 1) + 2$, and let G a k-Hamiltonian graph of order n with $n \geq a + k + 2$. If*

$\delta(G) \geq a+k$ *and* $t(G) \geq a-1+\frac{(a-1)(k+1)}{b-2}$, *then* G *admits a* k*-Hamiltonian* $[a,b]$*-factor.*

2 The Proof of Theorem 1

The proof of Theorem 1 relies heavily on the following lemmas. Lemma 2.1 is a well-known necessary and sufficient for a graph to have an $[a,b]$-factor, which is a special case of Lovász's (g,f)-factor theorem [7].

Lemma 2.1 (*Lovász [7]*). *Let* G *be a graph, and let* a *and* b *be two non-negative integers with* $a < b$. *Then* G *contains an* $[a,b]$*-factor if and only if for each subset* S *of* $V(G)$,

$$a|T| - d_{G-S}(T) \leq b|S|,$$

where $T = \{x : x \in V(G)\backslash S, d_{G-S}(x) \leq a-1\}$ *and* $d_{G-S}(T) = \sum_{x\in T} d_{G-S}(x)$.

Lemma 2.2 (*Katerinis [5]*). *If a graph* G *is not complete, then* $t(G) \leq \frac{1}{2}\delta(G)$.

Lemma 2.3 (*Liu and Zhang [6]*). *Let* G *be a graph and let* $H = G[T]$ *such that* $d_{G-S}(x) = a-1$ *for each* $x \in V(H)$ *and no component of* H *is isomorphic to* K_a *where* $T \subseteq V(G)$ *and* $a \geq 2$. *Then there exist an independent set* I *and the covering set* $C = V(H) - I$ *of* H *such that*

$$|V(H)| \leq (a - \frac{1}{a+1})|I|$$

and

$$|C| \leq (a-1-\frac{1}{a+1})|I|.$$

Lemma 2.4 (*Liu and Zhang [6]*). *Let* G *be a graph. Set* $H = G[T]$ *with* $\delta(H) \geq 1$ *and* $1 \leq d_G(x) \leq a-1$ *for each* $x \in V(H)$, *where* $T \subseteq V(G)$ *and* $a \geq 2$ *is an integer. Let* $T_1, T_2, \cdots, T_{a-1}$ *be a partition of* $V(H)$ *satisfying* $d_G(x) = j$ *for* $\forall x \in T_j$ *(where* T_j *may be empty sets),* $j = 1, 2, \cdots, a-1$. *Suppose that each component of* H *has at least one vertex of degree no more than* $a-2$ *in* G. *Then there exist a maximal independent set* I *and a covering set* $C = V(H) - I$ *of* H *satisfying*

$$\sum_{j=1}^{a-1}(a-j)c_j \leq \sum_{j=1}^{a-1}(a-2)(a-j)i_j,$$

where $i_j = |I \cap T_j|$, $c_j = |C \cap T_j|$, $j = 1, 2, \cdots, a-1$.

Lemma 2.5 *(Zhou [10]). Let a and b be two integers with $2 \le a < b$, and let G be a graph of order n with $n \ge a+2$. If G is complete, then G includes a Hamiltonian $[a,b]$-factor.*

Proof of Theorem 1. For any $U \subseteq V(G)$ with $|U| = k$, we write $G' = G - U$. In terms of the hypothesis of Theorem 1 and the definition of k-Hamiltonian graph, G' admits a Hamiltonian cycle C. Let $H = G' - E(C)$. Note that $V(H) = V(G') = V(G) \setminus U$ and $\delta(H) = \delta(G') - 2 \ge \delta(G) - k - 2$.

It is obvious that C is a k-Hamiltonian $[2,b]$-factor of G, and so Theorem 1 holds for $a = 2$. In the following, we may assume that $a \ge 3$.

If G is complete, then G' also is complete. According to Lemma 2.5, G' includes a Hamiltonian $[a,b]$-factor, and so G has a k-Hamiltonian $[a,b]$-factor. Hence, we may assume that G is not a complete graph. Obviously, G contains the desired factor if and only if H admits an $[a-2,b-2]$-factor. By way of contradiction, we assume that H has no $[a-2,b-2]$-factor. Then by Lemma 2.1, there exists some subset S' of $V(H)$ such that

$$(a-2)|T| - d_{H-S'}(T) > (b-2)|S'|, \quad (1)$$

where $T = \{x : x \in V(H) \setminus S', d_{H-S'}(x) \le a-3\}$. Note that $H = G' - E(C) = G - U - E(C)$. Thus, we obtain

$$d_{H-S'}(x) \ge d_{G'-S'}(x) - 2 = d_{G-U-S'}(x) - 2$$

for any $x \in T$. Let $S = S' \cup U$. Thus, we have

$$d_{G-S}(x) \le d_{H-S'}(x) + 2 \le (a-3) + 2 = a - 1, \quad (2)$$

for any $x \in T$. In terms of (1), (2), $|U| = k$ and $S = S' \cup U$, we obtain

$$a|T| - d_{G-S}(T) > (b-2)|S| - (b-2)k. \quad (3)$$

Claim 1. $|S| \ge k+1$.

Proof. Assume that $|S| \le k$. Note that $\delta(G) \ge a+k$. We have $d_{G-S}(x) \ge d_G(x) - |S| \ge \delta(G) - |S| \ge a$ for each $x \in T$. This contradicts (2). The proof of Claim 1 is complete. □

Claim 2. $(b-2)|S| - (b-2)k \ge \frac{(b-2)|S|}{k+1}$.

Proof. In view of Claim 1, we obtain

$$\begin{aligned} (b-2)|S| - (b-2)k &= \frac{(b-2)|S|}{k+1} + (b - 2 - \frac{b-2}{k+1})|S| - (b-2)k \\ &\ge \frac{(b-2)|S|}{k+1} + (b - 2 - \frac{b-2}{k+1})(k+1) - (b-2)k \\ &= \frac{(b-2)|S|}{k+1}. \end{aligned}$$

This completes the proof of Claim 2. □

It follows from (3) and Claim 2 that

$$a|T| - d_{G-S}(T) > \frac{(b-2)|S|}{k+1}. \quad (4)$$

Let m be the number of the components of $R' = G[T]$ which are isomorphic to K_a and set $T_0 = \{x : x \in V(R'), d_{G-S}(x) = 0\}$. Let R be the subgraph obtained from $R' - T_0$ by deleting those m components isomorphic to K_a. We shall consider two cases by the value of $|V(R)|$ and derive a contradiction in each case.

Case 1. $|V(R)| \geq 1$.

It is obvious that $\delta(R) \geq 1$. Set $R = R_1 \cup R_2$, where R_1 is the union of components of R which satisfy $d_{G-S}(x) = a - 1$ for each $x \in V(R_1)$ and $R_2 = R - R_1$. According to Lemma 2.3, there exist a maximal independent set I_1 and the covering set $C_1 = V(R_1) - I_1$ in R_1 satisfying

$$|V(R_1)| \leq (a - \frac{1}{a+1})|I_1| \quad (5)$$

and

$$|C_1| \leq (a - 1 - \frac{1}{a+1})|I_1|. \quad (6)$$

Note that $\delta(R_2) \geq 1$ and $\Delta(R_2) \leq a - 1$. We write $T_j = \{x : x \in V(R_2), d_{G-S}(x) = j\}$ for $j = 1, 2, \cdots, a-1$. In terms of the definitions of R and R_2, it is obvious that each component of R_2 has at least one vertex of degree no more than $a - 2$ in $G - S$. Using Lemma 2.4, R_2 has a maximal independent set I_2 and the covering set $C_2 = V(R_2) - I_2$ satisfying

$$\sum_{j=1}^{a-1}(a-j)c_j \leq \sum_{j=1}^{a-1}(a-2)(a-j)i_j, \quad (7)$$

where $i_j = |I_2 \cap T_j|$, $c_j = |C_2 \cap T_j|$, $j = 1, 2, \cdots, a-1$. Set $W = G - (S \cup T)$, $Q = S \cup C_1 \cup C_2 \cup (N_G(I_2) \cap V(W))$. Then since $|C_2| + |N_G(I_2) \cap V(W)| \leq \sum_{j=1}^{a-1} j i_j$, we obtain

$$|Q| \leq |S| + |C_1| + \sum_{j=1}^{a-1} j i_j \quad (8)$$

and

$$\omega(G - Q) \geq |T_0| + m + |I_1| + \sum_{j=1}^{a-1} i_j. \quad (9)$$

Claim 3. $|Q| \geq t(G)\omega(G-Q)$.

Proof. Claim 3 is true for $\omega(G-Q)=0$. If $\omega(G-Q)=1$, then it follows from Lemma 2.2 that

$$|Q| \geq d_{G-S}(x)+|S| \geq d_G(x) \geq \delta(G) \geq 2t(G) \geq t(G)\omega(G-Q)$$

for any $x \in T$. If $\omega(G-Q) \geq 2$, then from the definition of $t(G)$, we obtain

$$|Q| \geq t(G)\omega(G-Q).$$

This completes the proof of Claim 3. □

In terms of (8), (9) and Claim 3, we obtain

$$|S|+|C_1|+\sum_{j=1}^{a-1} j i_j \geq t(G)(|T_0|+m+|I_1|+\sum_{j=1}^{a-1} i_j). \tag{10}$$

According to (4), we have

$$a|T_0|+am+|V(R_1)|+\sum_{j=1}^{a-1}(a-j)i_j+\sum_{j=1}^{a-1}(a-j)c_j > \frac{(b-2)|S|}{k+1}. \tag{11}$$

It follows from (10) and (11) that

$$a|T_0|+am+|V(R_1)|+\sum_{j=1}^{a-1}(a-j)i_j+\sum_{j=1}^{a-1}(a-j)c_j$$

$$> \frac{b-2}{k+1}(t(G)(|T_0|+m+|I_1|+\sum_{j=1}^{a-1} i_j)-|C_1|-\sum_{j=1}^{a-1} j i_j),$$

that is,

$$\sum_{j=1}^{a-1}(a-j)c_j+|V(R_1)|+\frac{b-2}{k+1}|C_1| > \sum_{j=1}^{a-1}(\frac{(b-2)t(G)}{k+1}-\frac{(b-2)j}{k+1}-a+j)i_j$$

$$+\frac{(b-2)t(G)}{k+1}|I_1|+(\frac{(b-2)t(G)}{k+1}-a)(|T_0|+m).$$

By the conditions of Theorem 1, it is easy to see that $\frac{(b-2)t(G)}{k+1}-a>0$. Thus, we have

$$\sum_{j=1}^{a-1}(a-j)c_j+|V(R_1)|+\frac{b-2}{k+1}|C_1| > \sum_{j=1}^{a-1}(\frac{(b-2)t(G)}{k+1}-\frac{(b-2)j}{k+1}-a+j)i_j$$

$$+\frac{(b-2)t(G)}{k+1}|I_1|. \tag{12}$$

In terms of (5), (6), (7) and (12), we obtain

$$\sum_{j=1}^{a-1}(a-2)(a-j)i_j+((a-\frac{1}{a+1})+\frac{b-2}{k+1}(a-1-\frac{1}{a+1}))|I_1|$$

$$>\sum_{j=1}^{a-1}(\frac{(b-2)t(G)}{k+1}-\frac{(b-2)j}{k+1}-a+j)i_j+\frac{(b-2)t(G)}{k+1}|I_1|. \tag{13}$$

It follows from (13) that at least one of the following two cases must hold.

Subcase 1.1. There exists at least one j satisfying

$$(a-2)(a-j)>\frac{(b-2)t(G)}{k+1}-\frac{(b-2)j}{k+1}-a+j,$$

which implies

$$\frac{(b-2)t(G)}{k+1}<a(a-1)-(a-1)j+\frac{(b-2)j}{k+1}. \tag{14}$$

Note that $b\geq a(k+1)+2$. Thus, we have

$$\frac{b-2}{k+1}>a-1. \tag{15}$$

Combining $j\leq a-1$ with (14) and (15), we obtain

$$\frac{(b-2)t(G)}{k+1}<a(a-1)-(a-1)(a-1)+\frac{(b-2)(a-1)}{k+1}=a-1+\frac{(b-2)(a-1)}{k+1},$$

that is,

$$t(G)<a-1+\frac{(a-1)(k+1)}{b-2},$$

which contradicts $t(G)\geq a-1+\frac{(a-1)(k+1)}{b-2}$.

Subcase 1.2. $a-\frac{1}{a+1}+\frac{b-2}{k+1}(a-1-\frac{1}{a+1})>\frac{(b-2)t(G)}{k+1}$.

In terms of $t(G)\geq a-1+\frac{(a-1)(k+1)}{b-2}$, we have

$$\frac{(b-2)t(G)}{k+1}\geq\frac{b-2}{k+1}\cdot(a-1+\frac{(a-1)(k+1)}{b-2})=a-1+\frac{(a-1)(b-2)}{k+1}.$$

Thus, we obtain

$$a-\frac{1}{a+1}+\frac{b-2}{k+1}(a-1-\frac{1}{a+1})>a-1+\frac{(a-1)(b-2)}{k+1},$$

that is,

$$a(k+1)-(b-2)>0,$$

which contradicts $b\geq a(k+1)+2$.

Case 2. $|V(R)|=0$.

It follows from (4) that

$$a|T_0|+am>\frac{(b-2)|S|}{k+1},$$

which implies

$$|S|<\frac{a(k+1)(|T_0|+m)}{b-2}. \tag{16}$$

According to (16), $b\geq a(k+1)+2$ and Claim 1, we have

$$|T_0|+m>\frac{b-2}{a}\geq k+1\geq 1. \tag{17}$$

Note that $\omega(G-S)\geq|T_0|+m$. Then using (17), we obtain $\omega(G-S)\geq|T_0|+m>1$. In view of (16) and the definition of $t(G)$, we have

$$a-1+\frac{(a-1)(k+1)}{b-2}\leq t(G)\leq\frac{|S|}{\omega(G-S)}\leq\frac{|S|}{|T_0|+m}<\frac{a(k+1)}{b-2},$$

that is,

$$a-1<\frac{k+1}{b-2}.$$

Combining this with $b\geq a(k+1)+2$ and $a\geq 2$, we obtain

$$1\leq a-1<\frac{1}{a}\leq\frac{1}{2},$$

which is a contradiction. Theorem 1 is proved. □

Finally, we present the following problem.

Problem. Let a,b,k be nonnegative integers with $a\geq 2$ and $b\geq a(k+1)+2$, and let G a k-Hamiltonian graph of order n with $n\geq a+k+2$ and $\delta(G)\geq a+k$. For any little positive real ϵ, $t(G)\geq a-1+\frac{(a-1)(k+1)}{b-2}-\epsilon$. Does G contain a k-Hamiltonian $[a,b]$-factor?

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