Strong Z_{4p}- Magic labeling

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Abstract

For any non-trivial abelian group A under addition, a graph G is said to be strong A-magic if there exists a labeling f of the edges of G with non zero elements of A such that the vertex labeling f^+ defined as $f^+(v) = \sum f(uv)$ taken over all edges uv incident at v is a constant [4], and the constant is same for all possible values of $|V(G)|$. A graph is said to be strong A-magic if it admits strong A -magic labeling. In this paper we consider $(modulo\ \ Z_4, +)$ as abelian group and we prove strong Z_4- magic labeling for various graphs and generalize strong Z_{4p}- magic labeling for those graphs. The graphs which admit strong Z_{4p} -magic labeling are called as strong Z_{4p}-magic graphs.

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1 Introduction

By a graph $G(V, E)$ we mean G is a finite, simple, undirected graph. The concept of magic labeling was introduced by Sedlacek in 1963. Kong, Lee and Sun [4] used the term magic labeling for the labeling of edges with non negative integers such that for each vertex v the sum of the labels of all edges incident at v is same for all v. In particular the edge labels need not be distinct.

For any non-trivial abelian group A under addition a graph G is said to be A-magic if there exists a labeling f of the edges of G with non zero elements of A such that, the vertex labeling f^+ defined as $f^+(v) = \sum f(uv)$ over all edges uv incident at v is a constant. If this constant is same for all the vertices of G, in all possible values of $|V(G)|$, then it is said to be strong A-magic. Throughout this paper, we choose Z_4 which is additive modulo 4 as the abelian group and we prove some graphs such as $P_m \times P_n$, $C_m \times C_n$, $MT(m,n)$ and $S'(C_n)$ are strong Z_4 -magic graphs. At the end, we prove that they are all strong Z_{4p}-magic graphs. Throughout this paper by a path P_n, we mean it is a path of length $n-1$, and by C_n, we mean it is a cycle of length n.

2 Main Results

Definition 2.1. *The cross product $G_1 \times G_2$ has its vertex set $V_1 \times V_2$ and two points $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent in $G_1 \times G_2$ whenever $u_1 = v_1$ and u_2 adjacent to v_2 or $u_2 = v_2$ and u_1 adjacent to v_1.*

Definition 2.2. *The product $P_m \times P_n$ is called a planner grid.*

Example 2.3.

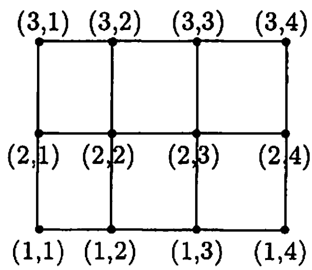

Fig.1 $P_4 \times P_3$

Theorem 2.4. *$P_m \times P_n$ is strong Z_4-magic for $n \geq 2$ and $m \geq 2$.*

Proof. Let $(1,1)(1,2)...(1,m); (2,1)(2,2)...(2,m); ...(n,1)(n,2)...(n,m)$ be the mn vertices of the grid.

Let (i,j) be the vertex where i denotes the row (counted from the bottom to the top) and j denotes the column (counted from left to right).

It has mn vertices and $(m(n-1)+n(m-1))$ edges.

Let $f : E(P_m \times P_n) \to Z_4 - \{0\}$ be defined as
For a fixed $i = 1$ and n

$f((i,j)(i,j+1)) = 1$ for $j = 1,2,...,(m-1)$
For fixed $j = 1$ and m

$f((i,j)(i+1,j)) = 3$ for $i = 1,2,...,(n-1)$
For a fixed $i = 2,3,...,(n-1)$

$f(i,j)(i,j+1) = 2,$ for $j = 1,2,...,(m-1)$
and for a fixed $j = 2,3,...,(m-1)$

$f(i,j)(i+1,j) = 2$ for $i = 1,2,...,(n-1)$
Now $f^+ : V(P_m \times P_n) \to Z_4$
It is easy to check that
$f^+(i,j) = 0 \ (mod\ 4), 1 \leq i \leq n, 1 \leq j \leq m$
For example

$$\begin{aligned} f^+(1,1) &= f((1,1)(1,2)) + f((1,1)(2,1)) \\ &\equiv (1+3) \ (mod\ 4) = 0. \end{aligned}$$

Hence, $f^+(v) = 0$ for all vertices of $P_m \times P_n, m \geq 2$ and $n \geq 2$.
Here, the magic constant is 0 for all possible values of m, n of $P_m \times P_n$.
Hence, $P_m \times P_n$ is strong Z_4-magic. $m \geq 2$ and $n \geq 2$.

Example 2.5. *Strong Z_4-magic labelings are shown for $P_5 \times P_3$ and $P_3 \times P_6$.*

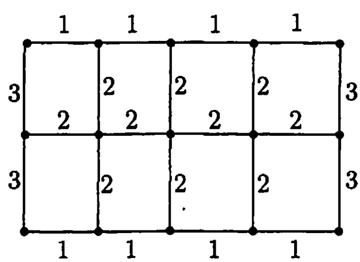

Fig. 2 $P_5 \times P_3$

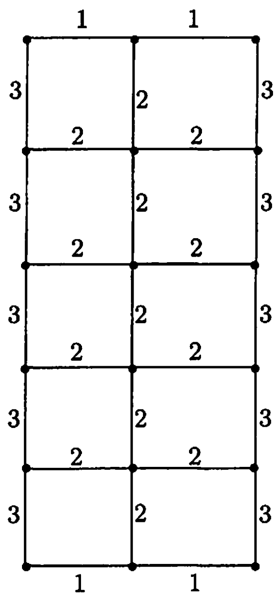

Fig. 3 $P_3 \times P_6$

Definition 2.6. *The product* $C_m \times C_n$ *is called a grid on cylinder.*

Theorem 2.7. $C_m \times C_n$ *is strong* Z_4*-magic for* $m \geq 3$ *and* $n \geq 3$.

Proof. Let G be $C_m \times C_n$ graph.
Then $V(G) = \left\{u_i^{(j)} | 1 \leq i \leq m \text{ and } 1 \leq j \leq n\right\}$
$E(G) = \left\{u_i^{(j)} u_{i+1}^{(j)} | 1 \leq i \leq m-1 \text{ and } 1 \leq j \leq n\right\} \cup \left\{u_m^{(j)} u_1^{(j)} | 1 \leq j \leq n\right\}$
$\cup \left\{u_i^{(j)} u_i^{(j+1)} | 1 \leq j \leq n-1 \text{ and } 1 \leq i \leq m\right\} \cup \left\{u_i^{(n)} u_i^{(1)} | 1 \leq i \leq m\right\}$

case 1 m is even and $n \geq 3$
Let $f : E(G) \rightarrow Z_4 - \{0\}$ be defined as

$$\begin{aligned} f\left(u_{2i-1}^{(j)} u_{2i}^{j}\right) &= 3, \quad 1 \leq i \leq m/2 \text{ and } 1 \leq j \leq n \\ f\left(u_{2i}^{(j)} u_{2i+1}^{(j)}\right) &= 1, \quad 1 \leq i \leq m/2 \ (u_{n+1} = u_1) \text{ and } 1 \leq j \leq n \\ f\left(u_i^{(j)} u_i^{(j+1)}\right) &= 1, \quad 1 \leq i \leq m \text{ and } 1 \leq j \leq n \ \left(u_i^{(n+1)} = u_i^{(1)}\right) \end{aligned}$$

Now $f^+ : V(G) \to Z_4$
By definition

$$\begin{aligned} f^+\left(u_i^{(j)}\right) &= f\left(u_{i-1}^{(j)}u_i^{(j)}\right) + f\left(u_i^{(j)}u_{i+1}^{(j)}\right) + f\left(u_i^{(j)}u_i^{(j+1)}\right) + f\left(u_i^{(j-1)}u_i^{(j)}\right), \\ &\quad 1 \le i \le m \text{ and } 1 \le j \le n \quad \left(u_i^{(0)} = u_i^{(n)} \text{ and } u_0^{(j)} = u_n^{(j)}\right) \\ f^+\left(u_i^{(j)}\right) &= (3+1+1+1)\ (mod\ 4) \equiv 6\ (mod\ 4) \\ &= 2, \quad 1 \le i \le m \text{ and } 1 \le j \le n \end{aligned}$$

case 2 m is odd, $n \ge 3$.
Let $f : E(G) \to Z_4 - \{0\}$ be defined as
$f\left(u_i^{(j)}u_{i+1}^{(j)}\right) = 2, \quad 1 \le i \le m$ and $1 \le j \le n.$ $\left(u_{m+1}^{(j)} = u_1^{(j)}\right)$
$f\left(u_i^{(j)}u_i^{(j+1)}\right) = 1, \quad 1 \le i \le m$ and $1 \le j \le n.$ $\left(u_i^{(n+1)} = u_i^{(1)}\right)$
$f^+ : V(G) \to Z_4$
We get

$$\begin{aligned} f^+\left(u_i^{(j)}\right) &= f\left(u_{i-1}^{(j)}u_i^{(j)}\right) + f\left(u_i^{(j)}u_{i+1}^{(j)}\right) + f\left(u_i^{(j)}u_i^{(j+1)}\right) + f\left(u_i^{(j-1)}u_i^{(j)}\right) \\ f^+\left(u_i^{(j)}\right) &\equiv (2+2+1+1)\ (mod\ 4) \equiv 6\ (mod\ 4) \\ &= 2, \quad 1 \le i \le m \text{ and } 1 \le j \le n. \end{aligned}$$

In both the cases $f^+(v)$ is the same constant $\forall\, v \in V(G)$ and the magic constant is 2 for all possible values of m, n. Therefore $C_m \times C_n$ is strong Z_4 - magic graph. □

Example 2.8. *Strong Z_4 - magic labelings with magic constant 2 are shown for the graphs $C_6 \times C_3$ and $C_5 \times C_4$.*

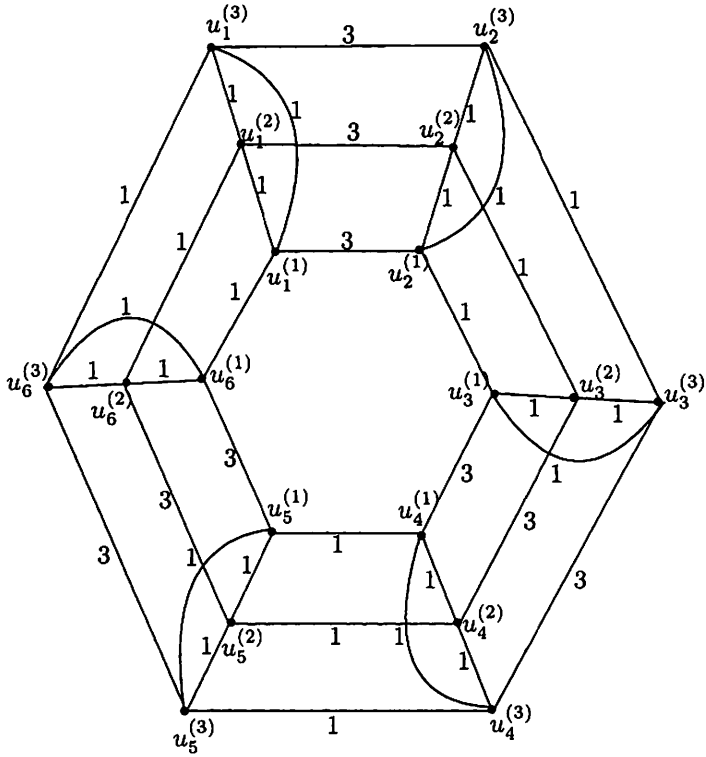

Fig. 4 $C_6 \times C_3$

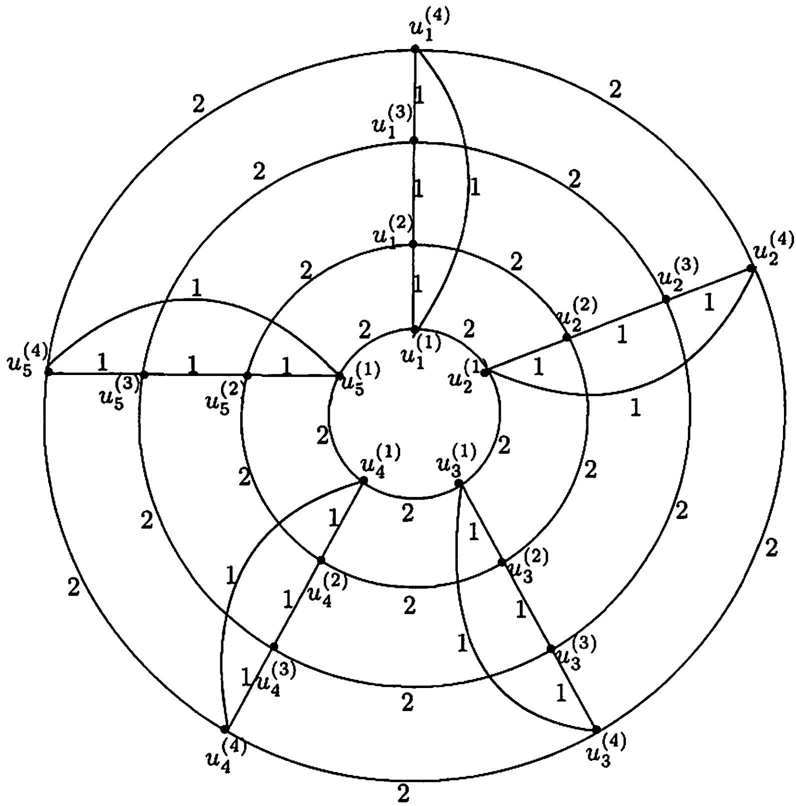

Fig. 5 $C_5 \times C_4$

Definition 2.9. *[5] Mongolian tent is a graph obtained from* $P_m \times P_n$ *by adding one extra vertex* u *above the grid and joining every vertex of the top row of* $P_m \times P_n$ *to the new vertex* u. *It is denoted as* $MT(m,n)$.

Example 2.10. *The Mongolian tent of* $MT(5,3)$ *is shown below.*

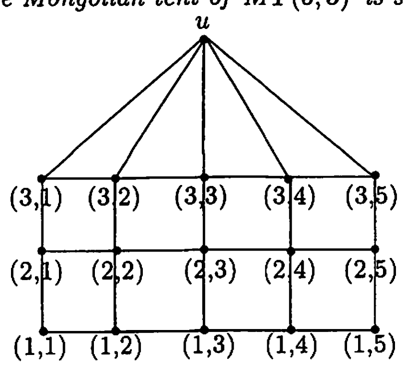

Fig. 6 $MT(5,3)$

Theorem 2.11. *Mongolian tent graph is strong Z_4-magic for $m \geq 2$ and $n \geq 2$*

Proof. Let G be $MT(m,n)$.
Let $|V(G)| = mn + 1$, $|E(G)| = 2mn - n$
As in the plannar graph, here each vertex is represented as (i,j) where i denotes the row (counted from the bottom to the top) and j denotes the column (counted from left to right).

The roof vertex is denoted as u.
Case 1 m be even and $n \geq 2$
Let $f : E(G) \to Z_4 - \{0\}$ be defined as
For a fixed $i = 1$
$f[(i,j)(i,j+1)] = 3$, for $j = 1, 2, ..., (m-1)$
For a fixed $i = n$
$f[(i,j)(i,j+1)] = 1$, for $j = 1, 3, ..., (m-1)$
$f[(i,j)(i,j+1)] = 3$, for $j = 2, 4, 6, ...(m-2)$
For a fixed $j = 1$ and m
$f[(i,j)(i+1,j)] = 1$, $i = 1, 2, 3, ..., (n-1)$
For $j = 2, 3, ..., (m-1)$
$f[(i,j)(i+1,j)] = 2$, $1 \leq i \leq (n-1)$
For $i = 2, 3, ..., (n-1)$
$f[(i,j)(i,j+1)] = 2$, $1 \leq j \leq (m-1)$
$f[(u)(n,j)] = 2$, $1 \leq j \leq m$.
Now, $f^+ : V(G) \to Z_4$.

$$\begin{aligned} f^+(1,1) &= f[(1,1)(1,2)] + f[(1,1)(2,1)] \\ &\equiv (3+1)\ (mod\ 4) \\ &= 0 \\ \text{similarly } f^+(1,m) &\equiv (3+1)\ (mod\ 4) \\ &= 0 \\ f^+(n,1) &= f[((n-1),1)(n,1)] + f[(n,1)(n,2)] + f[(u)(n,1)] \\ &\equiv (1+1+2)\ (mod\ 4) \\ &= 0 \\ f^+(n,m) &= f[(n-1,m)(n,m)] + f[(n,m-1)(n,m)] \\ &\quad + f[(u)(n,m)] \\ &\equiv (1+1+2)\ (mod\ 4) \\ &= 0 \end{aligned}$$

For a fixed $j = 2, 3, ..., (m-1)$

$$\begin{aligned}
f^+(i,j) &= [(i-1,j)(i,j)] + f[(i,j)(i+1,j)] + f[(i,j)(i,j+1)] \\
&\quad + f[(i,j-1)(i,j)] \\
&\equiv (2+2+2+2)\ (mod\ 4) \\
&= 0,\ \ 2 \leq i \leq n-1 \\
f^+(1,j) &= f[(1,j-1)(1,j)] + f[(1,j)(1,j+1)] + f[(1,j)(2,j)] \\
&\equiv (3+3+2)\ (mod\ 4) \\
&= 0,\ \ 2 \leq j \leq m-1 \\
f^+(i,1) &= f[(i,1)(i,2)] + f[(i-1,1)(i,1)] + f[(i,1)(i+1,1)] \\
&\equiv (2+1+1)\ (mod\ 4) \\
&= 0,\ \ 2 \leq i \leq n-1 \\
\text{similarly} f^+(i,m) &= f[(i,m)(i+1,m)] + f[(i-1,m)(i,m)] \\
&\quad + f[(i,m-1)(i,m)] \\
&\equiv (1+1+2)\ (mod\ 4) \\
&= 0,\ \ 2 \leq i \leq n-1 \\
f^+(n,j) &= f[(n,j)(n,j-1)] + f[(n,j)(n,j+1)] \\
&\quad + f[(n-1,j)(n,j)] + f[(u)(n,j)] \\
&\equiv (1+3+2+2)\ (mod\ 4) \\
&= 0,\ \ 2 \leq j \leq m-1 \\
f^+(u) &= \sum_{j=1}^{m} f[(u)(n,j)] \\
&\equiv (2+2+...+2)\ (mod\ 4) \\
&\equiv (m\ times\ 2)\ (mod\ 4) \\
&= 0
\end{aligned}$$

Hence, $f^+(v)$ is constant for all $v \in G$

Case 2 m is odd and $n \geq 2$.

Let $f : E(G) \to Z_4 - \{0\}$ be defined as

For $i = 1$

$f[(i,j)(i,j+1)] = 3,\ \ 1 \leq j \leq m-1$

For $i = n$

$f[(i,j)(i,j+1)] = 2,\ \ 1 \leq j \leq m-1$

For $j = 1$ and m

$f[(i,j)(i+1,j)] = 1,\ \ 1 \leq i \leq n-1$

For a fixed $j = 2, 3, ..., (m-1)$

$f[(i,j)(i+1,j)] = 2,\ \ 1 \leq i \leq n-1$

For a fixed $i = 2, 3, ..., (n-1)$
$f[(i,j)(i,j+1)] = 2, \quad 1 \le j \le m-1$
$f[(u)(n,1)] = 1 = f[(u)(n,m)]$
$f[(u)(n,j)] = 2, \quad 2 \le j \le m-1$
Now, $f^+ : V(G) \to Z_4$

$$\begin{aligned} f^+(1,1) &= f[(1,1)(1,2)] + f[(2,1)(1,1)] \\ &\equiv (3+1)\ (mod\ 4) \\ &= 0 \\ \text{Similarly} f^+(1,m) &\equiv (3+1)\ (mod\ 4) = 0 \\ f^+(n,1) &= f[(n-1,1)(n,1)] + f[(n,1)(n,2)] + f[(u)(n,1)] \\ &\equiv (1+2+1)\ (mod\ 4) \\ &= 0 \\ f^+(n,m) &= f[(n-1),m)(n,m)] + f(n,m-1)(n,m)] + f[(u)(n,m)] \\ &\equiv (1+2+1)\ (mod\ 4) = 0 \end{aligned}$$

For a fixed $j = 2, 3, ..., (m-1)$

$$\begin{aligned} f^+(i,j) &= f[(i-1,j)(i,j)] + f[(i,j)(i+1,j)] + f[(i,j)(i,j+1)] \\ &\quad + f[(i,j-1)(i,j)] \\ &\equiv (2+2+2+2)\ (mod\ 4) \\ &= 0, \quad 2 \le i \le n-1 \\ f^+(1,j) &= f[(1,j-1)(1,j)] + f[(1,j)(1,j+1)] + f[(2,j)(1,j)] \\ &\equiv (3+3+2)\ (mod\ 4) \\ &= 0, \quad 2 \le j \le m-1 \\ f^+(i,1) &= f[(i,1)(i,2)] + f[(i-1,1)(i,1)] + f[(i,1)(i+1,1)] \\ &\equiv (2+1+1)\ (mod\ 4) \\ &= 0, \quad 2 \le i \le n-1 \\ \text{Similarly } f^+(i,m) &= f[(i,m)(i+1,m)] + f[(i-1,m)(i,m)] \\ &\quad + f[(i,m-1)(i,m)] \\ &\equiv (1+1+2)\ (mod\ 4) \\ &= 0, \quad 2 \le i \le n-1 \\ f^+(n,j) &= f[(n,j-1)(n,j)] + f[(n,j)(n,j+1)] \\ &\quad + f[(n-1,j)(n,j)] + f[(u)(n,j)] \\ &\equiv (2+2+2+2)\ (mod\ 4) \\ &= 0, \quad 2 \le j \le m-1 \end{aligned}$$

$$
\begin{aligned}
f^{+}(u) &= \sum_{j=1}^{m} f[(u)(n,j)] \\
&\equiv (1 + (m-2)\ \textit{times}\ 2 + 1)\ (\textit{mod}\ 4) \\
&= 0\ (\textit{mod}\ 4) = 0
\end{aligned}
$$

In both the cases $f^{+}(v)$, $\forall v \epsilon V(G)$ is the same constant and the magic constant is 0 here. Therefore the mongolian tent graph is strong Z_4 - magic. □

Example 2.12. *Strong Z_4-magic labelings of $MT(6,5)$ and $M(7,4)$ are shown below.*

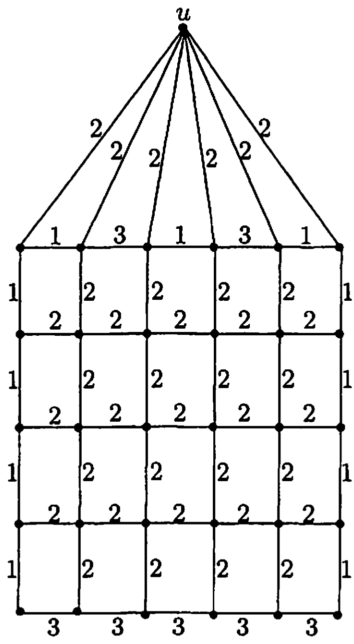

Fig. 7 Strong Z_4 - magic labeling of $MT(6,5)$

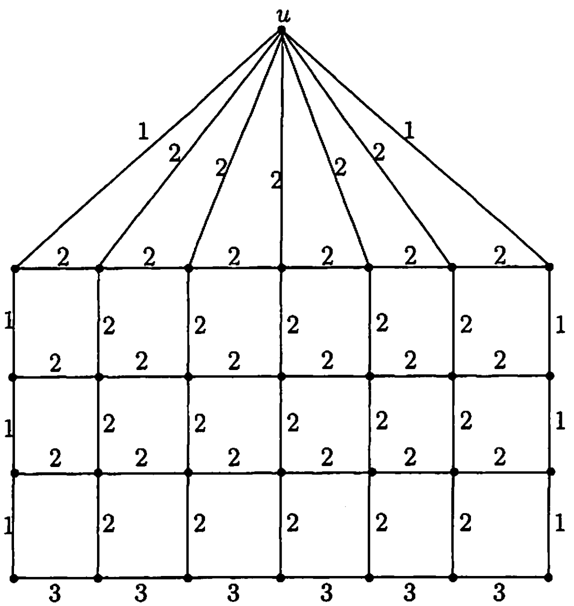

Fig. 8 Strong Z_4 - magic labeling of $MT(7,4)$

Definition 2.13. *Let G be a graph. For each point v of a graph G, take a new vertex v'. Join v' to those points of G which are adjacent to v. The graph, thus obtained is called the splitting graph of G. It is denoted as $S'(G)$.*

Theorem 2.14. *$S'(C_n)$ is strong Z_4-magic for $n \geq 3$.*

Proof. Let $V(S'(c_n)) = \{v_i | 1 \leq i \leq n\} \cup \{v_i' | 1 \leq i \leq n\}$ and
$E(S'(C_n)) = \{v_i v_{i+1} | 1 \leq i \leq n\} \cup \{v_{i-1} v_i' | 1 \leq i \leq n\} \cup \{v_i' v_{i+1} | 1 \leq i \leq n\}$
$[v_{n+1} = v_1 \text{ and } v_0 = v_n]$
Let $f : E(S'(C_n)) \to Z_4 - \{0\}$ be defined as
$f(v_i v_{i+1}) = 2, \quad 1 \leq i \leq n$
$f(v_{i-1} v_i') = 1$ and $f(v_i' v_{i+1}) = 1, \quad 1 \leq i \leq n$
Now, $f^+ : V(S'(C_n)) \to Z_4$

$$\begin{aligned} f^+(v_i) &= f(v_{i-1}v_i) + f(v_iv_{i+1}) + f(v_{i-1}'v_i) + f(v_iv_{i+1}') \\ &\equiv (2+2+1+1) \ (mod\ 4), \ 1 \leq i \leq n \\ &\equiv (6 \ mod\ 4) \\ &= 2, \ 1 \leq i \leq n \end{aligned}$$

$$
\begin{aligned}
f^+(v_i') &= f(v_{i-1}v_i') + f(v_i'v_{i+1}) \\
&\equiv (1+1)\ (mod\ 4) \quad 1 \le i \le n \\
&= 2
\end{aligned}
$$

Hence, $f^+(v)$, is constant for all $v \epsilon V(S'(C_n))$ clearly. Hence, $S'(C_n)$ admits strong Z_4-magic labeling and therefore $S'(C_n)$ is strong Z_4-magic graph. □

Example 2.15. *Strong Z_4- magic labelings of $S'(C_5)$ and $S'(C_8)$ are shown below.*

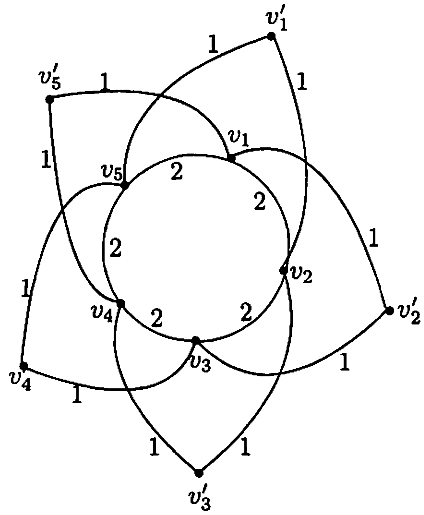

Fig. 9 Strong Z_4 - magic labeling of $S'(C_5)$)

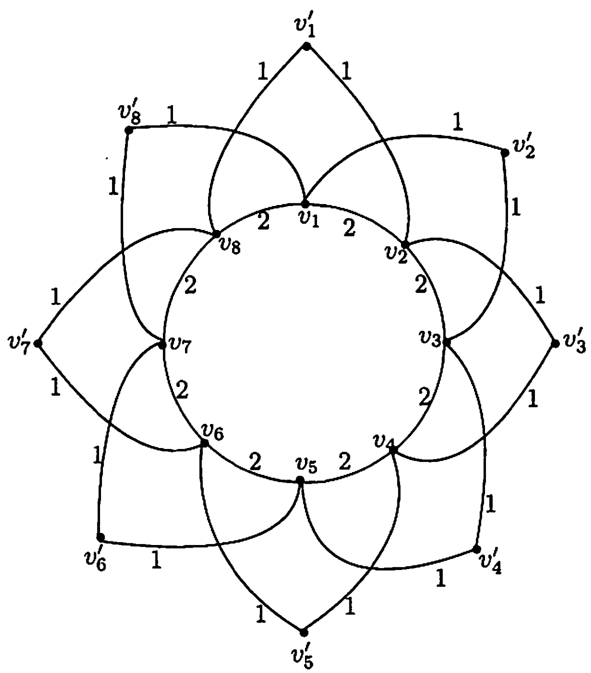

Fig. 10 Strong Z_4 - magic labeling of $S'(C_8)$

Observation 2.16. *In all the above theorems, if we multiply the edge labeling by a positive integer p, the vertex labeling remains to be a constant and this magic constant is equal to p times the original magic constant value we obtained. Hence all the above graphs admit strong Z_{4p}-magic labeling. Hence the graphs $P_m \times P_n, C_m \times C_n, MT(m,n)$ and $S'(C_n)$ are all strong Z_{4p}-magic graphs.*

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