A refinement of Guo's theorem concerning divisibility properties of binomial coefficients*

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Abstract

Let $s(n,k) = \binom{6k}{3k}\binom{3k}{k}\binom{6(n-k)}{3(n-k)}\binom{3(n-k)}{n-k}/((2n-1)\binom{3n}{n})$. Recently, Guo confirmed a conjecture of Z.-W. Sun by showing that $s(n,k)$ is an integer for $k = 0, 1, \dots, n$. Let $d = (3n+2)/\gcd(3n+2, 2n-1)$. In this paper, we prove that $s(n,k)$ is a multiple of the odd part of d for $k = 0, 1, \dots, n$. Furthermore, if $\gcd(k,n) = 1$, then $s(n,k)$ is also a multiple of n. We also show that the 2-adic order of $s(n,k)$ is at least the sum of the digits in the binary expansion of $3n$.

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1 Introduction

In 2009, Bober [1] determined all cases such that

$$\frac{(a_1n)!\cdots(a_kn)!}{(b_1n)!\cdots(b_{k+1}n)!}\in\mathbb{Z},$$

where $a_s\neq b_t$ for all s,t, $\sum a_s=\sum b_t$ and $\gcd(a_1,\ldots,a_k,b_1,\ldots,b_{k+1})=1$.

Motivated by some new series for $1/\pi$ and the related congruences on sums of binomial coefficients, Z.-W. Sun [13] proposed the following interesting conjecture.

Conjecture 1.1. (See [13, Conjecture 4.2]) *For $n=0,1,2,\ldots$, define*

$$s_n:=\frac{1}{(2n-1)\binom{3n}{n}}\sum_{k=0}^{n}\binom{6k}{3k}\binom{3k}{k}\binom{6(n-k)}{3(n-k)}\binom{3(n-k)}{n-k}.$$

Then $s_n\in\mathbb{Z}$ for all n.

Later, Sun [14, 15] proved many results on the divisibility of binomial coefficients. Recently, Guo [5] gave a proof of Conjecture 1.1.

Theorem A. (See [5, Theorem 1.2]) *For $0\le k\le n$, we have*

$$s(n,k):=\frac{1}{(2n-1)\binom{3n}{n}}\binom{6k}{3k}\binom{3k}{k}\binom{6(n-k)}{3(n-k)}\binom{3(n-k)}{n-k}\in\mathbb{Z}.$$

For related results, one can refer to [2–4] and [6–11].

In this paper, we improve Theorem A by proving the following theorems.

Theorem 1.1. *Let n,k be integers with $n\ge 1$ and $0\le k\le n$, and let d be the odd part of $(3n+2)/\gcd(2n-1,3n+2)$. Then $s(n,k)\equiv 0 \pmod d$. Moreover, if $\gcd(n,k)=1$, then $s(n,k)\equiv 0 \pmod{dn}$.*

We write $p^k\|n$ if $p^k|n$ and $p^{k+1}\nmid n$, and use $\nu_p(n)$ to denote such k. Let $\alpha_p(m)$ denote the sum of the digits of m in the expansion of m in base p.

Theorem 1.2. *Let n,k be integers with $n\ge 1$ and $0\le k\le n$. Then $\nu_2(s(n,k))\ge\alpha_2(3n)$. Moreover, the equality holds for $k=0$ and $k=n$.*

Corollary 1.1. *Let n be a positive integer with $n \not\equiv 4 \pmod 7$ and $n \not\equiv 2 \pmod 8$. Then $s(n,k) \equiv 0 \pmod{3n+2}$ for $k = 0, 1, \ldots, n$.*

Throughout this paper, for a real number x, $\lfloor x \rfloor$ denotes the greatest integer not exceeding x and $\{x\}$ denotes the fractional part of x.

2 Preliminary lemmas

For the p-adic order of $n!$, it is well known that

$$\nu_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor . \tag{1}$$

Now we give some basic lemmas in the following.

Lemma 2.1. *(Legendre's theorem, see [12, pp.22-24].) For any positive integer n, we have*

$$\nu_p(n!) = \frac{n - \alpha_p(n)}{p-1}.$$

Lemma 2.2. *(See [5, Lemma 2.2].) Let $0 \le k \le n$ be integers. Then*

$$h(n,k) = \frac{(6k)!(6n-6k)!(2n)!}{(3k)!(3n-3k)!(3n)!(2k)!(2n-2k)!(2n-1)} \in \mathbb{Z}.$$

Lemma 2.3. *(See [16, Problem 6, p.30]) Let $\theta_1, \theta_2, \ldots, \theta_s$ be real numbers. Then*

$$\lfloor \theta_1 + \theta_2 + \cdots + \theta_s \rfloor \ge \lfloor \theta_1 \rfloor + \lfloor \theta_2 \rfloor + \cdots + \lfloor \theta_s \rfloor .$$

Lemma 2.4. *Let a, b be positive integers with $\gcd(a,b) = 1$. Then*

$$a!b! | (a+b-1)!.$$

Proof. Since $\gcd(a,b) = 1$, it follows that

$$\frac{(a+b)!}{a!b!} = \binom{a+b}{a} = \frac{a+b}{a}\binom{a+b-1}{a-1}$$

is an integer divisible by $a+b$. Hence, $a!b!|(a+b-1)!$. □

Lemma 2.5. *Let x and y be two real numbers. Then*

$$\begin{aligned} & \lfloor 6x \rfloor + \lfloor 6y - 6x \rfloor + \lfloor y \rfloor + \lfloor 2y \rfloor \\ \geq \quad & \lfloor 3x \rfloor + \lfloor 2x \rfloor + \lfloor x \rfloor + \lfloor 3y - 3x \rfloor + \lfloor 2y - 2x \rfloor + \lfloor y - x \rfloor + \lfloor 3y \rfloor . \end{aligned}$$

Proof. It is easy to see that

$$\lfloor 6x \rfloor + \lfloor 6y - 6x \rfloor \geq \lfloor 3x \rfloor + \lfloor 3y - 3x \rfloor + \lfloor 3y \rfloor, \tag{2}$$

$$\lfloor y \rfloor \geq \lfloor x \rfloor + \lfloor y - x \rfloor, \tag{3}$$

$$\lfloor 2y \rfloor \geq \lfloor 2x \rfloor + \lfloor 2y - 2x \rfloor \tag{4}$$

(Note that (2) also appears in the proof of [5, Lemma 2.4]). Combining the inequalities (2)-(4), we complete the proof. □

Lemma 2.6. *Let x be a real number such that $\{x\} \geq 1/6$. Then*

$$\lfloor 6x \rfloor \geq \lfloor x \rfloor + \lfloor 2x \rfloor + \lfloor 3x \rfloor + 1.$$

Lemma 2.7. *Let x be a real number such that $0 \leq \{x\} < 1/3$ or $1/2 \leq \{x\} < 2/3$. Then*

$$\lfloor x \rfloor + \lfloor 2x \rfloor = \lfloor 3x \rfloor .$$

Proofs of Lemmas 2.6 and 2.7 are easy and we leave them to the reader.

3 Proof of Theorem 1.1

Let $t(n,k) = (2n-1)s(n,k)$. Then

$$t(n,k) = \frac{(6k)!}{k!(2k)!(3k)!} \cdot \frac{(6n-6k)!}{(n-k)!(2n-2k)!(3n-3k)!} \cdot \frac{n!(2n)!}{(3n)!}.$$

Let d' denote the odd part of $3n+2$. Then $d = d'/\gcd(2n-1, d')$. Suppose that $d' | t(n,k)$. By $2n-1 | t(n,k)$, we have that $t(n,k)$ is divisible by the least common multiple of d' and $2n-1$. It follows that

$$\frac{d'}{\gcd(2n-1,d')} \Bigg| \frac{t(n,k)}{2n-1}, \quad \text{i.e.,} \quad d | s(n,k).$$

Hence, in order to prove the first part of Theorem 1.1, it suffices to prove $d'|t(n,k)$.

Let $p^r\|d'$, where p is a prime and $r \geq 1$. Then $p|3n+2$ and $p \geq 5$. Now we shall prove $p^r|t(n,k)$, i.e., $\nu_p(t(n,k)) \geq r$. Let

$$f(m,p^i) = \left\lfloor \frac{6m}{p^i} \right\rfloor - \left\lfloor \frac{m}{p^i} \right\rfloor - \left\lfloor \frac{2m}{p^i} \right\rfloor - \left\lfloor \frac{3m}{p^i} \right\rfloor$$

and

$$g(m,p^i) = \left\lfloor \frac{m}{p^i} \right\rfloor + \left\lfloor \frac{2m}{p^i} \right\rfloor - \left\lfloor \frac{3m}{p^i} \right\rfloor.$$

Then, by (1), we have

$$\nu_p(t(n,k)) = \sum_{i=1}^{\infty} \left(f(k,p^i) + f(n-k,p^i) + g(n,p^i) \right).$$

By Lemma 2.5, we have $f(k,p^i)+f(n-k,p^i)+g(n,p^i) \geq 0$ for all $i \geq 1$. If $f(k,p^i) + f(n-k,p^i) + g(n,p^i) \geq 1$ for all $i \leq r$, then $\nu_p(t(n,k)) \geq r$.

Now we shall prove $f(k,p^i) + f(n-k,p^i) \geq 1$ and $g(n,p^i) = 0$ for $i = 1, 2, \ldots, r$.

By $p^r\|3n+2$, we have $3n \equiv -2 \pmod{p^r}$. Take an integer $i \in \{1, 2, \ldots, r\}$. Then $3n \equiv p^i-2 \pmod{p^i}$. By Lemma 2.3, we have $f(k,p^i) \geq 0$ and $f(n-k,p^i) \geq 0$. Next we consider the following two cases.

Case 1. $p^i \equiv 2 \pmod 3$. It follows that $n \equiv (p^i-2)/3 \pmod{p^i}$, and then $\{n/p^i\} < 1/3$. By Lemma 2.7, we have $g(n,p^i) = 0$.

Now we show that $f(k,p^i) + f(n-k,p^i) \geq 1$. Suppose that $f(k,p^i) = f(n-k,p^i) = 0$. By Lemma 2.6, we have $\{k/p^i\} < 1/6$ and $\{(n-k)/p^i\} < 1/6$. Let

$$k \equiv k' \pmod{p^i}, \quad 0 \leq k' < p^i.$$

Then

$$\left\{ \frac{k}{p^i} \right\} = \frac{k'}{p^i} < \frac{1}{6}$$

and

$$\left\{ \frac{n-k}{p^i} \right\} = \left\{ \left\{ \frac{n}{p^i} \right\} - \left\{ \frac{k}{p^i} \right\} \right\} = \frac{p^i-2}{3p^i} - \frac{k'}{p^i} < \frac{1}{6}.$$

Thus, we have

$$\frac{p^i-4}{6} \le k' \le \frac{p^i}{6}. \tag{5}$$

Since $p^i \equiv 2 \pmod 3$ and p^i is odd, it follows that $p^i \equiv 5 \pmod 6$. Thus, there exist no integers in the interval $[(p^i-4)/6, p^i/6]$, which contradicts the equality (5).

Therefore, $f(k,p^i)+f(n-k,p^i) \ge 1$ in this case.

Case 2. $p^i \equiv 1 \pmod 3$. It follows that $n \equiv (2p^i-2)/3 \pmod{p^i}$, and then $1/2 < \{n/p^i\} < 2/3$. By Lemma 2.7, we have $g(n,p^i)=0$.

Since

$$\left\{\frac{k}{p^i}\right\}+\left\{\frac{n-k}{p^i}\right\} \ge \left\{\frac{n}{p^i}\right\} > \frac{1}{3},$$

it follows that either $\{k/p^i\} > 1/6$ or $\{(n-k)/p^i\} > 1/6$. By Lemma 2.6, we have either $f(k,p^i) \ge 1$ or $f(n-k,p^i) \ge 1$. That is, $f(k,p^i)+f(n-k,p^i) \ge 1$.

Hence, $p^r | t(n,k)$, and so $d' | t(n,k)$. Therefore, $s(n,k) \equiv 0 \pmod d$.

If $\gcd(k,n)=1$, then $\gcd(k,n-k)=1$. Noting that

$$s(n,k)=\frac{h(n,k)n!}{k!(n-k)!},$$

by Lemmas 2.2 and 2.4, we have $n|s(n,k)$. Since $\gcd(3n+2,n)=\gcd(n,2)$, $d|3n+2$ and d is odd, it follows that $\gcd(d,n)=1$.

Thus, $s(n,k) \equiv 0 \pmod{dn}$.

This completes the proof of Theorem 1.1.

4 Proof of Theorem 1.2

Since

$$\nu_2((6k)!) = 3k + \nu_2((3k)!),$$

$$\nu_2((6n-6k)!) = 3n-3k+\nu_2((3n-3k)!),$$

$$\nu_2(n!) \ge \nu_2(k!) + \nu_2((n-k)!) \tag{6}$$

and

$$\nu_2((2n)!) \geq \nu_2((2k)!) + \nu_2((2n-2k)!), \tag{7}$$

we have

$$\begin{aligned}
\nu_2(s(n,k)) &= \nu_2((6k)!) + \nu_2((6n-6k)!) + \nu_2(n!) + \nu_2((2n)!) - \nu_2(k!) \\
&\quad -\nu_2((2k)!) - \nu_2((3k)!) - \nu_2((n-k)!) - \nu_2((2n-2k)!) \\
&\quad -\nu_2((3n-3k)!) - \nu_2((3n)!) \\
&\geq 3n - \nu_2((3n)!).
\end{aligned}$$

By Lemma 2.1, we have $\nu_2((3n)!) = 3n - \alpha_2(3n)$.

Hence, $\nu_2(s(n,k)) \geq \alpha_2(3n)$. If $k = 0$ or $k = n$, then equalities in (6) and (7) hold. Thus, $\nu_2(s(n,k)) = \alpha_2(3n)$.

This completes the proof of Theorem 1.2.

5 Proof of Corollary 1.1

Noting that $\gcd(3n+2, 2n-1) = \gcd(n+3, 2n-1) = \gcd(n+3, 7)$ and $n \not\equiv 4 \pmod 7$, we have $\gcd(3n+2, 2n-1) = 1$. Let $3n+2 = 2^{\ell} n'$ with $2 \nmid n'$. By Theorem 1.1, we obatin $n' | s(n,k)$.

By Theorem 1.2, we have $\nu_2(s(n,k)) \geq \alpha_2(3n) \geq 2$. Since $n \not\equiv 2 \pmod 8$, if follows that $3n+2 \not\equiv 0 \pmod 8$. Hence $\nu_2(3n+2) \leq 2 \leq \nu_2(s(n,k))$. That is, $2^{\ell} | s(n,k)$.

Therefore, $s(n,k) \equiv 0 \pmod{3n+2}$.

This completes the proof of Corollary 1.1.

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