Constructions of pairs of Tutte-equivalent graphs *

Helin Gong [1,2] † Metrose Metsidik [3]

1. Department of Fundamental Courses, Zhejiang Industry Polytechnic College Shaoxing, Zhejiang 312000, China

2. Guangxi Colleges and Universities Key Laboratory of Mathematical and Statistical Model, Guangxi Normal University, Guangxi 541004, China

3. School of Mathematical Science, Xiamen University Xiamen, Fujian 361005, China

Abstract: Two graphs are said to be Tutte-equivalent if their Tutte polynomials are equal. In this paper, we provide several different constructions for Tutte-equivalent graphs including some that are not self-complementary but Tutte-equivalent to their complements (the Akiyama–Harary problem) and some 'large' Tutte-equivalent graphs obtained from 'small' Tutte-equivalent graphs by 2-sum operations.

Keywords: Tutte polynomial; Tutte-equivalence; self-complementary graph; the Akiyama–Harary problem

1 Introduction

Tutte polynomial, due to Tutte [16], is a two-variable polynomial satisfying a fundamental universal property with respect to the deletion-contraction reduction of a graph. Tutte polynomial encodes a substantial amount of interesting information about the graph, and plays an important role in several areas of sciences such as combinatorics, computer science, statistical physics and knot theory. For a thorough survey on the properties and applications of the Tutte polynomial, we refer the reader to [3].

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†E-mail: helingong@126.com

Which graphs are determined by their chromatic polynomial, a specialization of the Tutte polynomial? This question was raised by Read [13] who asked for a necessary and sufficient condition for two graphs to be chromatically equivalent, namely, to have the same chromatic polynomial. In 1976, Chao and Whitehead [4] defined a graph to be chromatically unique if no other non-isomorphic graph shares its chromatic polynomial. Since then, the problem of finding chromatically unique graphs has been extensively explored and it is still under investigation. We recommend [11] for more references and [8] for a monograph. Analogously, two graphs G and H are said to be *codichromatic* [17] or *Tutte-equivalent* [5, 6] if they have the same Tutte polynomial. Noy in [5] further defined that a graph G is *Tutte-unique* if any graph that has the same Tutte polynomial as G is isomorphic to G. Clearly, Tutte-equivalence implies chromatic equivalence. However, the converse is not always true and an example is given in Fig. 1. In addition, if a graph is chromatically unique then it must be Tutte-unique. The problem of finding Tutte-unique graphs has also drawn much attention in recent years. See [5–7, 9]. It is worth mentioning that Bollobás, Pebody and Riordan [2] conjectured that almost all graphs are Tutte-unique. The Akiyama–Harary

Fig. 1: Two graphs [6] that are chromatically equivalent but not Tutte-equivalent.

problem asks that 'can a non self-complementary graph has the same chromatic polynomial as its complement'. By the aid of a pair of graphs with same subgraph sequences, Azarija [1] gave a positive answer to a strong version of the Akiyama–Harary problem: whether there exist non self-complementary graphs having the same Tutte polynomial as their complements. In this paper, we shall extend the construction in [1] and provide more constructions for pairs of Tutte-equivalent graphs which are complements of each other.

Let G be a graph with no loop or coloop and $\mathbf{H} = \{H_e : e \in E(G)\}$ (each H_e is a connected graph with at least two vertices). The $\mathbf{H}$-replacement of G, denoted by $\widehat{G[\mathbf{H}]}$, is defined as the graph obtained by replacing each edge e of G by the corresponding graph H_e, that is, deleting the edge e in G and identifying pairs of vertices $\{h_e^1, e_u\}$ and $\{h_e^2, e_v\}$, where h_e^1 and h_e^2 are two distinct vertices (as two special vertices) which belong to H_e, e_u and e_v are ends of e. If every edge of G is replaced by the same graph H with two fixed special vertices u and v (i.e. $H_e = H$), we write $\widehat{G[H]}$ for $\widehat{G[\mathbf{H}]}$. Woodall [18] derived two mutually dual Tutte polynomial expansions of the graph $\widehat{G[\mathbf{H}]}$ in terms of the parameters of the graph H_e and flow (or tension) polynomials of 'small' graphs coming from G. 'How to apply these two complicated expansions into practice' attracted Woodall's concern and motivates our interest. In this paper, by means of the idea of the previously mentioned replacements, we also shall provide a general method for constructing infinite families of Tutte-equivalent graphs.

2 Tutte polynomial

Throughout this paper, unless otherwise specified, the graphs we consider may have multiple edges and loops. Let $G = (V, E)$ be a graph. The order and the size of G are the number of vertices and the number of edges of G, respectively. The complete graph and the path of order n in this paper are denoted by K_n and P_n, respectively. Given a vertex $v \in V(G)$, the *open neighborhood* of vertex v is denoted $N(v)$. For $X \subset E$, we denote by $\langle X \rangle$ the spanning subgraph of G with vertex set V and edge set X. We denote by $c(X)$ the number of components of $\langle X \rangle$. Whenever a graph G is mentioned, if no ambiguity arises, we always assume n, m and c to be the order, size and number of components of G, respectively.

Definition 2.1. *Tutte polynomial $T(G; x, y)$ of the graph $G = (V, E)$ is a two-variable polynomial recursively defined as follows:*

$$T(G;x,y) = \begin{cases} yT(G-e;x,y) & \text{if } e \text{ is a loop} \\ xT(G/e;x,y) & \text{if } e \text{ is a coloop, i.e. cut-edge} \\ T(G-e;x,y) + T(G/e;x,y) & \text{if } e \text{ is neither a loop nor a coloop} \end{cases}$$

with the initial condition $T(G; x, y) = 1$ for $E = \emptyset$.

Tutte polynomial is independent of the edge order of deletion-contraction operations. One way of seeing this is through the rank-size generating function [3].

$$T(G;x,y) = \sum_{X \subseteq E} (x-1)^{\rho(E)-\rho(X)}(y-1)^{\gamma(X)},$$

where the sum runs over all edge subsets $X \subseteq E$, and $\rho(X) = n - c(X)$ is the rank of $\langle X \rangle$ and $\gamma(X) = |X| - n + c(X)$ is the nullity of $\langle X \rangle$, in particular, $\rho(E) = n - c$, $\gamma(E) = m - n + c$.

Lemma 2.2 ([3]). *If G and H are graphs, then*

$$T(G \cup H; x, y) = T(G * H; x, y) = T(G; x, y)T(H; x, y), \tag{1}$$

where $G \cup H$ is the disjoint union of G and H, and $G * H$ is the graph obtained by identifying a vertex of G and a vertex of H into a single vertex.

Tutte polynomial, in a strong sense, contains every graph invariant that can be computed by deletion-contraction operations. Many polynomial invariants are evaluations of the Tutte polynomial along particular lines in the (X, Y) plane. For instance, the chromatic polynomial and the flow polynomial of a graph are evaluations of the Tutte polynomial along particular lines $y = 0$ and $x = 0$ in the (X, Y) plane, respectively. See e.g. [3] for details.

(i) The chromatic polynomial

$$\chi(G;\lambda) = (-1)^{\rho(G)}\lambda^c T(G; 1-\lambda, 0) = \sum_{X \subset E} (-1)^{|X|}\lambda^{c(X)}. \tag{2}$$

(ii) The flow polynomial

$$F(G;\lambda) = (-1)^{\gamma(G)}T(G;0,1-\lambda) = \sum_{X\subset E}(-1)^{m-|X|}\lambda^{c(X)+|X|-n}. \qquad (3)$$

3 The subgraph sequence of a graph

Given a graph $G = (V, E)$, let H be a spanning subgraph of G with k connected components of order $h_1, h_2, \cdots, h_k (0 < h_1 \leq h_2 \leq \cdots \leq h_k)$. Azarija [1] called the tuple $(|E(H)|; h_1, \cdots, h_k)$ a subgraph description of H and further defined the subgraph sequence $s(G)$ of G as the lexicographically sorted tuple of subgraph descriptions for all spanning subgraphs of G. Obviously, two graphs are Tutte-equivalent if they have the same subgraph sequence. However, the converse is not true. For example, $T(P_4; x, y) = T(K_{1,3}; x, y)$ but $s(P_4) \neq s(K_{1,3})$ [1]: $s(P_4) = [(0; 1, 1, 1, 1), (1; 1, 1, 2), (1; 1, 1, 2), (1; 1, 1, 2), (2; 1, 3), (2; 1, 3), (2; 2, 2), (3; 4)]$, $s(K_{1,3}) = [(0; 1, 1, 1, 1), (1; 1, 1, 2), (1; 1, 1, 2), (1; 1, 1, 2), (2; 1, 3), (2; 1, 3), (2; 1, 3), (3; 4)]$. In Fig. 2 we give another pair of graphs with the same Tutte polynomial since their geometric dual graphs are isomorphic. However, by an exhaustive enumeration, we obtain that the multiplicity of the element $(4; 1, 1, 4)$ in their subgraph sequences (i.e. the number of spanning subgraphs with exactly 4 edges and two 1-vertex components, one 4-vertex component) are 5 and 7, respectively. Again, we also note several pairs of graphs given in Fig. 3 who are derived by deleting (or contracting) their respective bold edges are isomorphic or Tutte equivalent.

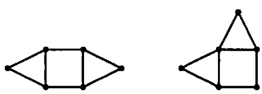
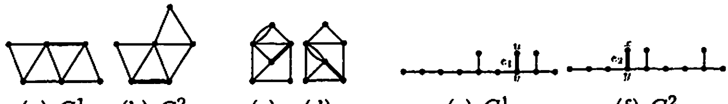


Fig. 2: Two Tutte-equivalent graphs with the same dual but different subgraph sequences.

That is, for $i = 1, 2, 3$, there exist an edge $e_1 \in E(G_i^1)$ (bold) and an edge $e_2 \in E(G_i^2)$ (bold) satisfying the following conditions: (1) $G_i^1 - e_1 \cong G_i^2 - e_2$;(2) $G_i^1/e_1 \cong G_i^2/e_2$, thus $T(G_i^1; x, y) = T(G_i^2; x, y)$. The second pair appeared in [17] and the third pair in [15]. However, their corresponding subgraph sequences are different which can be verified by a Sage program in [1]. Indeed, any simple graphs with the same subgraph sequence have order at least 8, see [15] for more details.

A simple graph G is *self-complementary* (s.c.) if it is isomorphic to its complement $\overline{G}$. Whether there exist non self-complementary graphs having the same Tutte polynomial as their complements is regarded as a strong version of the Akiyama–Harary problem in [1]. To give a positive answer to this problem, by using a brute force search, Azarija in [1] gave a pair of non-isomorphic graphs G_0 and $\overline{G_0}$ as shown in Fig. 4 which share the same subgraph sequence.



(a) G_1^1 (b) G_1^2 (c) G_2^1 (d) G_2^2 (e) G_3^1 (f) G_3^2

Fig. 3: Tutte-equivalent graphs with different subgraph sequences.

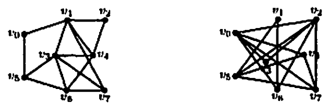


Fig. 4: Non self-complementary graph G_0 and its complement $\overline{G_0}$.

The *join* of graphs G_1 and G_2, denoted by $G = G_1 \vee G_2$, is the disjoint union $G_1 \cup G_2$ with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 together with all the edges joining vertices in V_1 and vertices in V_2. Moreover, let $S \subseteq V(G_2)$, we use $G_1 \vee_S G_2$ to denote the graph obtained from $G_1 \cup G_2$ by joining every vertex in S to every vertex of G_1. To construct more pairs of higher-order graphs with the same subgraph sequence, the following lemma will be repeatedly used in the sequel. Its proof is no more than an observation and similar to that of Lemma 4 in [1].

Lemma 3.1. *Let G, H, K, R be graphs such that $s(G) = s(H)$ and $s(K) = s(R)$. Then $s(G \vee K) = s(H \vee R)$ and moreover, for any vertex subset S of a graph Q, $s(G \vee_S Q) = s(H \vee_S Q)$.*

Proof. Let $\Gamma(G)$ be the set of spanning subgraphs of the graph G. Note that any spanning subgraph of $G \vee K$ is uniquely determined by a spanning subgraph G' of G, a spanning sugraph K' of K and some edges with one end-vertex in G' and the other in K', and the same holds for $H \vee R$. Since $s(G) = s(H)$ ($s(K) = s(R)$, respectively), there is a bijection φ (ϕ, respectively) from $\Gamma(G)$ ($\Gamma(K)$, respectively) to $\Gamma(H)$ ($\Gamma(R)$, respectively) preserving the subgraph description. By aid of φ and ϕ, it is easy to establish a bijection between $\Gamma(G \vee K)$ and $\Gamma(H \vee R)$ preserving the subgraph description. It follows that $s(G \vee K) = s(H \vee R)$. The second statement can be shown similarly. □

The families of Tutte-equivalent graphs presented in the following corollary have already been constructed in a pretty similar way by Bollobás, Pebody and Riordan [2] (Theorem 10, Remark 3), and actually they also give a more general construction that yields not only pairs, but exponentially large families of graphs, all sharing the same Tutte polynomial.

Corollary 3.2. *Let G and H be a pair of graphs with $s(G) = s(H)$ and the same connectivity 2. Then both $G \vee K_n$ and $H \vee K_n$ have the same connectivity $n + 2$ and have the same Tutte polynomial.*

Proof. By Lemma 3.1, $s(G \vee K_n) = s(H \vee K_n)$. Hence $T(G \vee K_n; x, y) = T(H \vee K_n; x, y)$. We use $\kappa(G)$ to denote the connectivity of a graph G. It is easy to show $\kappa(G \vee K_n) = \kappa(H \vee K_n) = n+2$ since $\kappa(G_1 \vee G_2) = \min\{|V(G_1)| + \kappa(G_2), |V(G_2)| + \kappa(G_1)\}$. This is based on two observations: (1) If $G_1 \vee G_2$ is a complete graph, $\kappa(G_1 \vee G_2) = |V(G_1)| + |V(G_2)| - 1 = |V(G_1)| + \kappa(G_2) = |V(G_2)| + \kappa(G_1)$; (2) If not, then, for any cutset C of $G_1 \vee G_2$, either $V(G_1) \subseteq C$ or $V(G_2) \subseteq C$ (otherwise, $G_1 \vee G_2 - C$ is connected). Therefore, $C_i \cup V(G_{\{1,2\} \setminus \{i\}})$ is a cutset of $G_1 \vee G_2$ whenever C_i is a minimum cutset in G_i $(i = 1, 2)$, which implies the result. □

4 Construction of Tutte-equivalent graphs

4.1 Graphs which are Tutte-equivalent to their complements

Let $G_i (i = 1, 2, 3, 4)$ be graphs with vertex sets $V_i (i = 1, 2, 3, 4)$, respectively. We temporarily use $[G_1, G_2, G_3, G_4]$ to denote the graph obtained from the disjoint union $\cup_{i=1}^{4} G_i$ by joining every vertex in V_i to every vertex in V_{i+1} for $i = 1, 2, 3$. Moreover, let G be a graph with vertex set V. $R(G; G_1, G_2, G_3, G_4)$ is defined as the graph obtained from $G \cup [G_1, G_2, G_3, G_4]$ by adding all possible edges between the vertex set V and the vertex subset $V_1 \cup V_4$ of $[G_1, G_2, G_3, G_4]$, i.e. $R(G; G_1, G_2, G_3, G_4) = G \vee_{V_1 \cup V_4} [G_1, G_2, G_3, G_4]$. Particularly, when $G_1 = H, G_2 = \overline{H}, G_3 = \overline{H}$ and $G_4 = H$ (H is arbitrary given graph), we denote $R(G; H, \overline{H}, \overline{H}, H)$ by $R(G; H)$ as shown in Fig. 5.

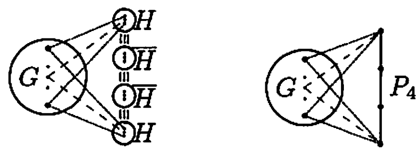


Fig. 5: $R(G; H)$ (left) and $R(G; K_1)$ (right), here '≡' indicates the graph join operation.

Theorem 4.1. *If $s(G) = s(\overline{G})$, then $R(G; H)$ is Tutte-equivalent to its complement $\overline{R(G; H)}$. In the case of $G = G_0$ and $\overline{G} = \overline{G_0}$ (see Fig. 4), $R(G_0; H)$ is not isomorphic to $\overline{R(G_0; H)}$.*

Proof. It is easy to see $\overline{R(G; H)} \cong R(\overline{G}; H)$. By Lemma 3.1, $s(R(G; H)) = s(\overline{R(G; H)})$. Accordingly, $T(R(G; H); x, y) = T(\overline{R(G; H)}; x, y)$. Now let $G = G_0$, $\overline{G} = \overline{G_0}$. Observe that v_1 and v_3 (the only two vertices with maximum degree 5 in G_0) have no common neighbor of degree 2, while v_0 and v_2 (the only two vertices with maximum degree 5 in $\overline{G_0}$) have one common neighbor v_3 of degree 2. Hence, the number of unordered pairs of vertices in $R(G_0; H)$ such that they have the same degree $2|V(H)|+5$ and only one common neighbor with degree $2|V(H)|+2$ is less than that of unordered pairs of vertices in $\overline{R(G_0;H)}$ with same requirements. Hence, $R(G_0;H) \not\cong \overline{R(G_0;H)}$. □

Corollary 4.2 ([1]). *If $s(G)=s(\overline{G})$, then the graph $R(G;K_1)$ shown in Fig. 5 is Tutte-equivalent to its complement $\overline{G'}$.*

Before we go on, we explain the definition of the *bipartite self-complementary* (b.s.c. [12]) graph. Let $B=B(X\cup Y,E)$ be a bipartite simple graph with disjoint vertex sets X,Y and edge set E, where each edge in E joins a vertex in X to a vertex in Y. The bipartite complement $\overline{\overline{B}}$ of B with respect to the partition $X\cup Y$ is the bipartite graph $B(X\cup Y,E')$, where E' contains all edges not in E that join a vertex in X to a vertex in Y. If B is connected then the partition $X\cup Y$ is unique and there is only one bipartite complement. B is a b.s.c. graph if there is an isomorphism φ from B to $\overline{\overline{B}}$ such that $\varphi(X)=Y$ and $\varphi(Y)=X$.

Definition 4.3. *A graph G is said to be of type (X,Y) if G has a vertex set partition $V(G)=X\cup Y$ such that*

$$G=\langle X\rangle\cup\langle X,Y\rangle\cup\langle Y\rangle,$$
$$\langle X\rangle\cong\overline{\langle Y\rangle},$$
$$\langle X,Y\rangle\cong\overline{\overline{\langle X,Y\rangle}}, \text{ here } \langle X,Y\rangle \text{ is a b.s.c graph,}$$

where $\langle X\rangle$, $\langle Y\rangle$ are the induced subgraph of G with vertex sets X, Y, respectively, and $\langle X,Y\rangle$ is the maximal subgraph of G with vertex set $X\cup Y$ where every edge joins a vertex in X to a vertex in Y.

Theorem 4.4. *Let $s(G)=s(\overline{G})$ and H be a graph of type (X,Y). Then $G\vee_X H$ and $\overline{G\vee_X H}$ are Tutte-equivalent.*

Proof. As can be seen from Fig. 6, $\overline{G\vee_X H}\cong\overline{G}\vee_X H$ because $\overline{\langle X\rangle}\cong\langle Y\rangle$, $\langle X\rangle\cong\overline{\langle Y\rangle}$ and $\overline{\overline{\langle X,Y\rangle}}\overset{b.s.c}{\cong}\langle X,Y\rangle$. By assumption $s(G)=s(\overline{G})$, it follows by Lemma 3.1 that $s(G\vee_X H)=s(\overline{G\vee_X H})$. Hence they have the same Tutte polynomial. □

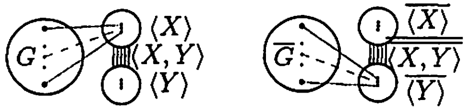


Fig. 6: $G\vee_X H$ (left) and its complement $\overline{G\vee_X H}$ (right).

Following [12]. Let G be an s.c.graph, one can construct a b.s.c graph by the following way: Let $G=(V,E)$ be a s.c. graph with a complementing permutation ϕ, i.e. an isomorphism from G to $\overline{G}$ which can be viewed as an element of the symmetric group S_n if $V(G)=\{1,2,\cdots,n\}$.

If $|V(G)|$ is odd, then G has an unique vertex (say x) fixed by ϕ (guaranteed by Theorem 1 in [10]), so R. Molina [12] considered the following unique decomposition of G into edge disjoint subgraphs:

$$G = \langle x, X\rangle \cup \langle X\rangle \cup \langle X, Y\rangle \cup \langle Y\rangle,$$

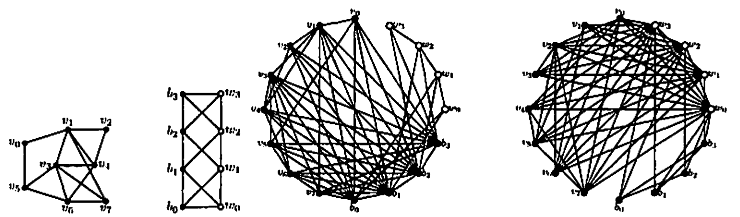
where $X = N(x)$, $Y = V\backslash(N(x) \cup \{x\})$. Since ϕ is a map from G to $\overline{G}$ that fixes x, we see that $\phi(\langle X\rangle) = \overline{\langle Y\rangle}$ and $\phi(\langle X, Y\rangle) = \overline{\langle X, Y\rangle}$. Thus $\langle X\rangle$ and $\langle Y\rangle$ are complements of each other and $\langle X, Y\rangle$ is a b.s.c. graph.

If $|V(G)|$ is even, a similar decomposition for G also exists. Assume that the vertices of G are labeled in such a way that the numbers in any cycle of ϕ appear in increasing order. Let X be the set of even numbered vertices and Y the set of odd numbered vertices. Then

$$G = \langle X\rangle \cup \langle X, Y\rangle \cup \langle Y\rangle.$$

Likewise, $\langle X\rangle$ and $\langle Y\rangle$ are complements of each other, and $\langle X, Y\rangle$ is a b.s.c. graph. In this case, unlike the case $|V(G)|$ odd, the subgraphs $\langle X\rangle$ and $\langle X, Y\rangle$ are not determined up to isomorphism. All b.s.c. graphs of order 8 and 12 are shown in [12].

The graph H_0 given in Fig. 7 (a) is of type (X, Y) with $X = \{\text{black vertices}\}$, and $G_0 \vee_X H_0$ is Tutte-equivalent to its complement, which is verified by Sage [14]). It is still open to determine if there is a graph G not having the same degree sequence as $\overline{G}$ yet the same Tutte polynomial.



(a) G_0 and H_0 (b) $G_0 \vee_X H_0$ and $\overline{G_0 \vee_X H_0}$

Fig. 7: $G_0 \vee_X H_0$ and $\overline{G_0 \vee_X H_0}$ are not isomorphic but Tutte-equivalent.

4.2 Tutte-equivalent graphs constructed by 2-sum operations

In the following section, we remind the reader that the definition of a 2-sum graph operation appears on section 1. To begin with, we review two dual expansions of the Tutte polynomial.

Lemma 4.5 (Theorem 2.1, [18]). *Let $N(G;q) = \frac{1}{q^c}\chi(G;q)$. Then*

$$T(G;x,y) = \frac{1}{(y-1)^{n-c}} \sum_{Y\subseteq E} N(G/Y;q)y^{|Y|}$$
$$= \frac{1}{(x-1)^{m-n+c}} \sum_{Y\subseteq E} F(G-Y;q)x^{|Y|},$$

where $q = (x-1)(y-1)$ and $G-Y$ (respectively G/Y) denote the graph obtained from G by deleting (respectively contracting) all edges in Y.

Let x, y and q be parameters such that $(x-1)(y-1) = q$. Let $\mathbf{H} = \{H_e : e \in E(G)\}$(each H_e is a connected graph with two special vertices h_e^1 and h_e^2). Let $H_{/e}$ denote the graph obtained from H_e by identifying h_e^1 and h_e^2, and define

$$N(G;q,y) := (y-1)^{n-c}T(G;x,y), \tag{4}$$
$$F(G;q,x) := (x-1)^{m+n-c}T(G;x,y), \tag{5}$$
$$\xi_e := (q-1)^{-1}(F(H_{/e},q,x) - F(H_e,q,x)), \tag{6}$$
$$\eta_e := (q-1)^{-1}(N(H_e,q,y) - N(H_{/e},q,y)), \tag{7}$$
$$x_e := \xi_e^{-1}F(H_e,q,x), \tag{8}$$
$$y_e := \eta_e^{-1}N(H_{/e},q,y). \tag{9}$$

In addition,

$$N(G;q,y) = \sum_{Y\subseteq E} N(G/Y;q)y^{|Y|},$$
$$F(G;q,x) = \sum_{Y\subseteq E} F(G-Y;q)x^{|Y|}.$$

See [18] for details.

Theorem 4.6 (Theorem 4.1, [18]).

$$T(\widehat{G[H]};x,y) = \frac{\prod_{e\in E}\eta_e(y_e-1)}{q^{\gamma(G)}(y-1)^{\rho(\widehat{G[H]})}} \sum_{Y\subseteq E} F(G-Y;q)\prod_{e\in Y} x_e \tag{10}$$
$$= \frac{\prod_{e\in E}\xi_e(x_e-1)}{q^{\rho(G)}(x-1)^{\gamma(\widehat{G[H]})}} \sum_{Y\subseteq E} N(G/Y;q)\prod_{e\in Y} y_e.$$

For the rest of the paper, we assume that all H_e's in $\mathbf{H}$ are the same. Whenever Theorem 4.6 is applied, the graph G is limited to a 2-connected graph.

Theorem 4.7. *Let G, G', R and R' be graphs such that $s(G) = s(G')$ and $s(R) = s(R')$. Let K be a graph and $\{u,v\}$ be two distinct vertices in K. Let $H = R \vee K$, $H' = R' \vee K$. If $\widehat{G[H]}$ (respectively $\widehat{G'[H']}$) is the graph obtained* *from G (respectively G′) by replacing each edge of G by H (respectively each edge of G′ by H′) with respect to $\{u,v\}$ (two special vertices of H or H′), then $\widehat{G[H]}$ is Tutte-equivalent to $\widehat{G'[H']}$.*

Proof. We shall use (10) to prove $T(\widehat{G[H]};x,y) = T(\widehat{G'[H']};x,y)$. First, we show $\gamma(G) = \gamma(G')$ and $\rho(\widehat{G[H]}) = \rho(\widehat{G'[H']})$. $s(G) = s(G')$ implies $|E(G)| = |E(G')| = m$ and $|V(G)| = |V(G')| = n$. Hence, $\gamma(G) = \gamma(G') = m+n-1$. By Lemma 3.1, $s(R \vee K) = s(R' \vee K)$ since $s(R) = s(R')$. This implies $|V(R \vee K)| = |V(R' \vee K)|$. Observe that

$$\rho(\widehat{G[H]}) = |V(\widehat{G[H]})| - 1 = |V(G)| + |E(G)|(|V(R \vee K)| - 2) - 1,$$
$$\rho(\widehat{G'[H']}) = |V(\widehat{G'[H']})| - 1 = |V(G')| + |E(G')|(|V(R' \vee K)| - 2) - 1.$$

Thus, $\rho(\widehat{G[H]}) = \rho(\widehat{G'[H']})$. Secondly, according to the previous discussion, we have

$$T(H;x,y) = T(H';x,y) \text{ (by Lemma 3.1)},$$
$$T(H+uv;x,y) = T(H'+uv;x,y) \text{ (by Lemma 3.1)}.$$

Moreover, by the deletion-contraction operation,

$$T(H_{/xy};x,y) = T(H'_{/xy};x,y).$$

Then by eqs. (4)-(9), we also easily obtain that

$$\xi_e = \frac{F(H_{/xy};q,x) - F(H,q,x)}{q-1} = \frac{F(H'_{/xy};q,x) - F(H';q,x)}{q-1} = \xi'_e, \tag{11}$$

$$\eta_e = \frac{N(H;q,y) - N(H_{/xy},q,y)}{q-1} = \frac{N(H';q,y) - N(H'_{/xy};q,y)}{q-1} = \eta'_e, \tag{12}$$

$$x_e = \xi_e^{-1} F(H;q,x) = \xi_e'^{-1} F(H';q,x) = x'_e, \tag{13}$$

$$y_e = \eta_e^{-1} N(H_{/xy};q,y) = \eta_e'^{-1} N(H'_{/xy};q,y) = y'_e. \tag{14}$$

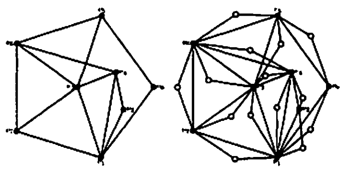
Finally, by $s(G) = s(G')$, we meant that the tuple in $s(G)$ which belongs to a spanning subgraph Y of G corresponds to the tuple in $s(G')$ which belongs to a spanning subgraph Y' of G'. By eq.(3), it is readily shown that

$$\begin{aligned}
\sum_{\{Y \subset G:|Y|=k\}} F(G-Y;q) &= \sum_{\{Y \subset G:|Y|=k\}} \sum_{X \subset E(G-Y)} (-1)^{m-k-|X|} q^{c(X)+|X|-n} \\
&= \sum_{\{X \subset G:|X| \le |E(G)|-k\}} (-1)^{m-k-|X|} q^{c(X)+|X|-n} \\
&= \sum_{\{X' \subset G':|X'| \le |E(G')|-k\}} (-1)^{m-k-|X'|} q^{c(X')+|X'|-n} \\
&= \sum_{\{Y' \subset G':|Y'|=k\}} \sum_{X' \subset E(G'-Y')} (-1)^{m-k-|X'|} q^{c(X')+|X'|-n} \\
&= \sum_{\{Y' \subset G':|Y'|=k\}} F(G'-Y';q).
\end{aligned}$$

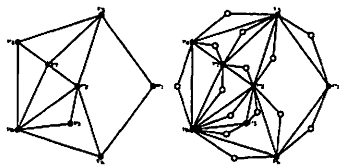
Now by Theorem 4.6, we conclude that

$$\begin{aligned}
T(\widehat{G[H]};x,y) &= \frac{\left(\eta_e(y_e-1)\right)^{|E|}}{q^{\gamma(G)}(y-1)^{\rho(\widehat{G[H]})}} \sum_{Y\subseteq E(G)} F(G-Y;q)x_e^{|Y|} \\
&= \frac{\left(\eta'_e(y'_e-1)\right)^{|E'|}}{q^{\gamma(G')}(y-1)^{\rho(\widehat{G'[H']})}} \sum_{Y'\subseteq E(G')} F(G'-Y';q)x_e'^{|Y'|} \\
&= T(\widehat{G'[H']};x,y),
\end{aligned}$$

where $q = (x-1)(y-1)$ and $\xi_e(\xi'_e), \eta_e(\eta'_e), x_e(x'_e), y_e(y'_e)$ as shown in eqs. (11),(12),(13),(14). □



(a) G_0 and $\widehat{G_0[K_3]}$



(b) $\overline{G_0}$ and $\widehat{\overline{G_0}[K_3]}$

Fig. 8: $\widehat{G_0[K_3]}$ and $\widehat{\overline{G_0}[K_3]}$ are not isomorphic but Tutte-equivalent (verified by Sage).

From Theorem 4.7, the special case when $R = K = H$ is given by:

Corollary 4.8. *Let G and G' be two graphs with $s(G) = s(G')$. Let $\widehat{G[H]}$ (respectively $\widehat{G'[H]}$) be the graph obtained from G (respectively G') by replacing each edge of G (respectively G') by the connected graph H with two special vertices. Then $\widehat{G[H]}$ is Tutte-equivalent to $\widehat{G'[H]}$.*

An example is given in Fig. 8 ($G = G_0$, $G' = \overline{G_0}$, $H = K_3$).

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