The correlation between the f-chromatic class and the g_c-chromatic class of a simple graph *

Xia Zhang

School of Mathematical Sciences, Shandong Normal University
Jinan, Shandong, P.R. China, 250014
E-mail: pandarhz@sina.com

Abstract In the previous researches on classification problems, there are some similar results obtained between f-coloring and g_c-coloring. In this article, the author shows that there always are coincident classification results for a regular simple graph G when the f-core and the g_c-core of G are same and $f(v) = g(v)$ for each vertex v in the f-core (the g_c-core) of G. However, it is not always coincident for nonregular simple graphs under the same conditions. In addition, the author obtains some new results on the classification problem of f-colorings for regular graphs. Based on the coincident correlation mentioned above, new results on the classification problem of g_c-colorings for regular graphs are deduced.

Keywords edge-coloring, edge-covering-coloring, classification of graphs, chromatic index
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1 Introduction

In this article, all graphs considered are undirected with a finite nonempty vertex set and a finite edge set. An *empty graph* is a graph with an empty edge set. For a graph G, we associate it with two integer functions $g : V(G) \longrightarrow N$ and $f : V(G) \longrightarrow Z^+$ which satisfy that $g(v) \leq f(v)$ for every $v \in V(G)$. Let C be a color set. A *(g, f)-coloring* of G is an edge-coloring satisfying that, for each vertex $v \in V(G)$ and each $c \in C$, there are $h(v)$ adjacent edges colored with c, where $g(v) \leq h(v) \leq f(v)$. The (g, f)-coloring is a generalization of the proper edge-coloring ($g \equiv 0, f \equiv 1$) and the edge covering coloring ($g \equiv 1, f(v) = d(v)$). When $g \equiv 0$, (g, f)-coloring is called *f-coloring*; when $f(v) = d(v)$ for all

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$v \in V(G)$, (g, f)-coloring is called *g_c-coloring*. A graph G has always an f-coloring for any positive integer function $f : V(G) \longrightarrow Z^+$. However, for a nonnegative integer function $g : V(G) \longrightarrow N$, if there exists a vertex $v \in V(G)$ with $g(v) > d(v)$, then G has no g_c-coloring. For example, an empty graph H has an f-coloring with k colors for any positive k and, H has a g_c-coloring if and only if $g \equiv 0$. If a graph G has an f-coloring with k colors, say η, then η is also an f-coloring with s colors for any positive integer $s > k$. (In η, each of at least $s - k$ colors appears 0 times at each $v \in V(G)$.) In contrast, if graph G has a g_c-coloring with $k > 1$ colors, say ζ, then G has also a g_c-coloring with t colors for any positive integer $t < k$ (by recoloring the i-edges in ζ with the color t, where $t+1 \le i \le k$). So it is trivial to determine the maximum number of colors for a graph G to have an f-coloring or the minimum number of colors for G to have a g_c-coloring. In this article, we focus on the minimum number of colors needed to f-color G, which is called the *f-chromatic index* of G and denoted by $\chi'_f(G)$, and the maximum number of colors needed to g_c-color G, which is called the *g_c-chromatic index* of G and denoted by $\chi'_{g_c}(G)$.

Since Holyer [3] proved that the proper edge-coloring problem is NP-complete (even if the restriction to the cubic graphs), the f-coloring problem and the g_c-coloring problem are NP-complete, too.

Let

$$\Delta_f(G) = \max_{v \in V(G)} \{\lceil \frac{d(v)}{f(v)} \rceil\}, \quad \delta_g(G) = \min_{v \in V(G)} \{\lfloor \frac{d(v)}{g(v)} \rfloor\},$$

$$V_{\Delta_f} = \{v \in V(G) : d(v) = f(v)\Delta_f(G)\},$$

$$V_{\delta_g} = \{v \in V(G) : d(v) = g(v)\delta_g(G)\}.$$

Note that, for a nonnegative integer function $g : V(G) \longrightarrow N$, if there exists a vertex $v \in V(G)$ with $g(v) > d(v)$, then $\delta_g(G) = 0$, i.e. G has no g_c-coloring. In addition, for a vertex v with $g(v) = 0$, $\lfloor \frac{d(v)}{g(v)} \rfloor = +\infty$. So a graph G has $\delta_g(G) = +\infty$ if and only if each vertex $v \in V(G)$ has $g(v) = 0$. Regardless of these two trivial cases, we focus our attention on the cases that $0 < \delta_g(G) < +\infty$ in the rest of the article.

We call the subgraphs induced by V_{Δ_f}, V_{δ_g} in G the *f-core* of G, the *g_c-core* of G, respectively, and denote them by G_{Δ_f}, G_{δ_g}, respectively. For simple graphs, the following results have been known.

Theorem 1.1 *[2] Let G be a simple graph. Then*

$$\Delta_f(G) \le \chi'_f(G) \le \Delta_f(G) + 1.$$

Theorem 1.2 *[8] Let G be a simple graph. Then*

$$\delta_g(G) - 1 \le \chi'_{g_c}(G) \le \delta_g(G).$$

When $f(v) = 1$ for all $v \in V(G)$, Theorem 1.1 is the theorem of Vizing [9], i.e. $\Delta(G) \le \chi'(G) \le \Delta(G) + 1$; when $g(v) = 1$ for all $v \in V(G)$, Theorem 1.2 is the theorem of Gupta [1], i.e. $\delta(G) - 1 \le \chi'_c(G) \le \delta(G)$. For f-colorings, G is called of *f-class* 1 if $\chi'_f(G) = \Delta_f(G)$, of *f-class* 2 if $\chi'_f(G) = \Delta_f(G) + 1$;

for g_c-colorings, G is called of *g_c-class* 1 if $\chi'_{g_c}(G) = \delta_g(G)$, and of *$g_c$-class* 2 if $\chi'_{g_c}(G) = \delta_g(G) - 1$.

The research methods between f-colorings and g_c-colorings are different. In the previous researches, there are many similar results for these two classification problems [2, 4, 5, 7, 8] [10]-[19]. For examples, any bipartite simple graph is of both f-class 1 and g_c-class 1; a simple graph associated with a positive and even function f is of both f-class 1 and f_c-class 1. What will be used in the article are listed below.

Let $V(G) = \{v_1, \ldots, v_n\}$, $G_i = G - \{v_1, v_2, \ldots, v_i\}$ $(1 \le i \le n-1)$, $G_0 = G$ and $S \subseteq V(G)$. We call a graph G is *S-peelable* if all vertices of G can be removed iteratively by using the following operation: Remove a vertex v_i, which has at most one neighbour $u \in S$ in G_{i-1} and $d_{G_{i-1}}(u) = d_G(u)$.

Theorem 1.3 *Let G be a simple graph.*

1. *[16] If G is V_{Δ_f}-peelable, then G is of f-class* 1.
2. *[7] If G is V_{δ_g}-peelable, then G is of g_c-class* 1.

The following result could be deduced immediately.

Corollary 1.1 *Let G be a simple graph.*

1. *[14] If $V_{\Delta_f} = \emptyset$, then G is of f-class* 1.
2. *[10] If $V_{\delta_g} = \emptyset$, then G is of g_c-class* 1.

Remark 1. The results above on g_c-colorings are based on an integer function $g : V(G) \longrightarrow Z^+$. When $g : V(G) \longrightarrow N$ and $V_0 = \{v \in V(G) : g(v) = 0\} \neq \emptyset$, we can construct an auxiliary graph G' from G as follows: for each $v \in V_0$, stick a new complete graph $H_v = K_{2\delta_g(G)+2}$ at v in such a way that identify v with an arbitrary vertex of H_v. Define a function $h : V(G') \longrightarrow Z^+$ in such a way that

$$\begin{cases} h(v) = g(v), & v \in V(G) \setminus V_0; \\ h(v) = 2, & \text{otherwise.} \end{cases}$$

It is easy to see that $\delta_g(G) = \delta_h(G')$, $V_{\delta_g}(G) = V_{\delta_h}(G')$ and $\chi'_{g_c}(G) = \chi'_{h_c}(G')$. So the results on g_c-colorings between Theorem 1.2 and Corollary 1.1 are still true for a function $g : V(G) \longrightarrow N$.

Based on the results above, Liu and Zhang [6] posed the following problem:

Problem 1.1 *(G. Liu and X. Zhang [6]) What kinds of simple graphs G always have coincident classification results between f-coloring and g_c-coloring when $V^* = V_{\Delta_f} = V_{\delta_g}$ and $f(v) = g(v)$ for each $v \in V^*$?*

If, for a graph G, there are always coincident results between the classification problem on f-colorings and the one on g_c-colorings when $G_{\Delta_f} = G_{\delta_g}$, then some new results for G on the classification problem on g_c-colorings could be deduced from the ones on f-colorings, and vice versa. Relatively, the research on g_c-coloring is going slowly. This is one motive for researching the correlation between the f-chromatic class and the g_c-chromatic class of a simple graph.

For convenience, we always define that $f_H(v) = f_G(v)$ and $g_H(v) = g_G(v)$ for each $v \in V(H)$ and for any $H \subset G$. For f-colorings, if a graph G has an f-coloring with k colors, then any subgraph H of G has one (by restricting an f-coloring to the subgraph H). However, that is not the case for g_c-colorings even if $d_H(v) \geq kg(v)$ for each $v \in V(H)$. For example, let F be a graph constructed from two cycles $C_1 = u_1u_2u_3$, $C_2 = v_1v_2v_3$ by connecting u_1, v_1 with an edge, where $g \equiv 1$. Clearly F has a g_c-coloring with 2 colors, but either of C_1, C_2 has no g_c-coloring with 2 colors. For some special classes of simple graphs, we can verify that a graph G is of f-class 1 in such a way that extend an f-coloring with $\Delta_f(G)$ colors of subgraphs of G to an f-coloring with $\Delta_f(G)$ colors of G (see [16, 17]). A natural question is: how to determine some g_c-class 1 graphs according to edge-colorings of their subgraphs? To avoid facing the case that some subgraph H of a graph G has no g_c-coloring with $\delta_g(G)$ colors when G has such one, the author defines an auxiliary graph H^+, called by the *degree-restoration subgraph* of G, which is constructed from H as follows: add some new pendant edges to each $v \in V(H)$ so that $d_{H^+}(v) = d_G(v)$. In addition, define $g(v) = 0$ for each end with degree one of the new pendant edges in H^+. It is easy to see that, when G has a g_c-coloring with k colors, then any degree-restoration subgraph H^+ of G has such one. So we can verify some g_c-class 1 graphs G by extending a g_c-coloring with $\delta_g(G)$ colors of a degree-restoration subgraph of G to a g_c-coloring with $\delta_g(G)$ colors of G.

In Section 2, we prove that there are always coincident classification results for regular simple graphs when $G_{\Delta_f} = G_{\delta_g}$ and $f(v) = g(v)$ for each $v \in V_{\Delta_f}(= V_{\delta_g})$. We give a new result on the classification problem of f-colorings for regular graphs. Based on the coincident correlation above, a new result on the classification problem of g_c-colorings for regular graphs is deduced. In Section 3, we show that it is not always coincident for nonregular simple graphs under the same conditions. However, with an extra constraint, there are coincident classification results.

2 Regular graphs

We call the two nonnegative integer functions f, g a pair of *related-functions* on G, if $V^* = V_{\Delta_f} = V_{\delta_g}$ and $f(v) = g(v) > 0$ for each $v \in V^*$. The following result is easily to verify.

Lemma 2.1 *If f, g is a pair of related-functions on G, then $\Delta_f(G) = \delta_g(G)$ and $g(v) < \frac{d(v)}{\delta_g(G)} = \frac{d(v)}{\Delta_f(G)} < f(v)$ for each $v \in V(G) \setminus V^*$.*

For a graph G associated a positive integer function $f : V(G) \longrightarrow Z^+$, we can associate G a nonnegative integer function g :

$$\begin{cases} g(v) = f(v), & v \in V_{\Delta_f}; \\ 0 \leq g(v) < \frac{d(v)}{\Delta_f(G)}, & v \in V(G) \setminus V_{\Delta_f}. \end{cases} \tag{1}$$

Conversely, for a graph G associated a nonnegative integer function $g : V(G) \longrightarrow N$, we can associate G a positive integer function f :

$$\begin{cases} f(v) = g(v), & v \in V_{\delta_g}; \\ f(v) > \frac{d(v)}{\delta_g(G)} \geq 0, & v \in V(G) \setminus V_{\delta_g}. \end{cases} \quad (2)$$

Since we concentrate ourselves on the nontrivial cases that $0 < \delta_g(G) < \infty$, there are $V^* = V_{\Delta_f} = V_{\delta_g}$ and $f(v) = g(v) > 0$ for each $v \in V^*$ for either of (1) and (2), i.e. f and g are a pair of function-related functions on G for either of (1) and (2). And, for a graph G, its pair of related-functions is not unique.

When G is given an edge-coloring with the colors in $C = \{c_1, \ldots, c_k\}$, an edge colored with $c \in C$ is called a *c-edge*. Let $c(v)$ denote the number of c-edges incident with $v \in V(G)$ for $c \in C$, $\sigma_g(v) = |\{c \in C : c(v) \geq g(v)\}|$ and $\sigma^f(v) = |\{c \in C : c(v) \leq f(v)\}|$. We call a walk $W = v_0 f_1 v_1 \ldots v_{t-1} f_t v_t$ an *ab-alternating walk* starting at v_0 and ending at v_t if W satisfies all of the following conditions:

(1) $f_i = v_{i-1}v_i$, $1 \leq i \leq t$, and $f_i \neq f_j$ $(i \neq j)$;

(2) W is colored with a and b alternately and the first edge $f_1 = v_0v_1$ is an a-edge;

(3) $a(v_0) > b(v_0)$ when $v_0 \neq v_t$, $a(v_0) > b(v_0) + 1$ when $v_0 = v_t$;

(4) $b(v_t) > a(v_t)$ when t is even, $a(v_t) > b(v_t)$ when t is odd.

Note that, when $v_0 = v_t$, t must be odd for an ab-alternating walk W by the definition above. *Switching* an ab-alternating walk W means exchanging the colors a and b of W.

Theorem 2.1 *Let G be a d-regular simple graph and f, g be a pair of related-functions on G. Then G is of f-class 1 if and only if G is of g_c-class 1.*

Proof. Let $k = \Delta_f(G) = \delta_g(G)$, $C = \{c_1, \ldots, c_k\}$, $V^* = V_{\Delta_f} = V_{\delta_g}$ and $f^* = f(v) = g(v)$ for each $v \in V^*$. If $V^* = \emptyset$, then G is of f-class 1 and g_c-class 1 by Corollary 1.1. Next, assume that $V^* \neq \emptyset$. Then $d = kf^*$ and $g(v) < f^* < f(v)$ for each $v \in V(G) \setminus V^*$ by Lemma 2.1.

If G is of f-class 1, then we can get an f-coloring η of G with k colors in C. Clearly, for each $1 \leq i \leq k$, there are $c_i(v) = f^*$ for each $v \in V^*$ and $c_i(v) \leq f(v)$ for each $v \in V(G) \setminus V^*$. If $c_i(v) \geq g(v)$ for each $v \in V(G) \setminus V^*$ and each $1 \leq i \leq k$, then η is a g_c-coloring of G with k colors, which means that G is of g_c-class 1. Otherwise, there exists a vertex $u \in V(G) \setminus V^*$ and a color $\alpha \in C$ such that $\alpha(u) \leq g(u) - 1$. Then $\alpha(u) \leq f^* - 2$. Since $d = kf^*$, there must exist another color $\beta \in C$ such that $\beta(u) \geq f^* + 1$. Construct a $\beta\alpha$-alternating walk W starting at u and switch W. If W does not end at u, switching W makes $\alpha(u)$ increase by 1 and $\beta(u)$ decrease by 1; if W ends at u, switching W makes $\alpha(u)$ increase by 2 and $\beta(u)$ decrease by 2. In either case, we still have $\beta(u) \geq f^* - 1 \geq g(u)$ and $\sigma_g(v)$ does not decrease for each v with $\sigma_g(v) = k$. Use this operation until $c_i(v) \geq g(v)$ for each $v \in V(G) \setminus V^*$ and each $1 \leq i \leq k$, which is a g_c-coloring of G with k colors. So G is of g_c-class 1.

If G is of g_c-class 1, then we can get a g_c-coloring ζ of G with k colors in C. Then, for each $1 \le i \le k$, there are $c_i(v) = f^*$ for each $v \in V^*$ and $c_i(v) \ge g(v)$ for each $v \in V(G) \setminus V^*$. If $c_i(v) \le f(v)$ for each $v \in V(G) \setminus V^*$ and each $1 \le i \le k$, then ζ is an f-coloring of G with k colors and G is of f-class 1. Otherwise, there exists a vertex $u \in V(G) \setminus V^*$ and a color $\alpha \in C$ such that $\alpha(u) \ge f(u) + 1$. Then $\alpha(u) \ge f^* + 2$. Thus there must exist another color $\beta \in C$ such that $\beta(u) \le f^* - 1$. Construct an $\alpha\beta$-alternating walk W starting at u and switch W. Similar to the discussion above, switching W makes $\alpha(u)$ decrease by 1 or 2 and $\beta(u)$ increase by 1 or 2, respectively. Whether W ends at u or not, we still have $\beta(u) \le f^* + 1 \le f(u)$ and $\sigma^f(v)$ does not decrease for each v with $\sigma^f(v) = k$ after switching W. Use this operation until $c_i(v) \le f(v)$ for each $v \in V(G) \setminus V^*$ and each $1 \le i \le k$, which is an f-coloring of G with k colors. So G is of f-class 1. ∎

Based on the result above, some results on classification problem for regular graphs can be deduced from each other between f-coloring and g_c-coloring. (See [4, 18].)

According to Corollary 1.1, when $V_{\Delta_f} = V_{\delta_g} = \emptyset$, G is both f-class 1 and g_c-class 1. Next we focus on the cases with $V_{\Delta_f} \ne \emptyset$ for f-colorings and the ones with $V_{\delta_g} \ne \emptyset$ for g_c-colorings.

Let G be a graph and $A \subsetneq V(G)$. The *degree-restoration subgraph* of G based on A, denoted by $(G[A])^+$, is an auxiliary graph constructed from $G[A]$ as follows: add some new pendant edges to each $v \in A$ so that $d_{(G[A])^+}(v) = d_G(v)$. In addition, we assign $g(v) = 0$ for each end with degree one of the new pendant edges.

In the rest of the article, we always define that $f_H(v) = f_G(v)$ for each $v \in V(H)$ and for any $H \subset G$; and $g_H(v) = g_G(v)$ for each $v \in V(H) \cap V(G)$ and for any degree restoration subgraph H of G.

Theorem 2.2 *Let G be a d-regular simple graph and $V_{\Delta_f} \ne \emptyset$. G is of f-class 1 if and only if the f-core of G has an f-coloring with $\Delta_f(G)$ colors.*

Proof. The necessity is obvious. Now we prove the sufficiency. If $V(G) = V_{\Delta_f}$, we are done. Let $|V(G)| = n$, $k = \Delta_f(G)$, $f^* = \frac{d}{k}$ and $C = \{c_1, \dots, c_k\}$. Assume that $V(G) \setminus V_{\Delta_f} = \{w_1, w_2, \dots, w_t\}$, $1 \le t \le n-1$, and G_{Δ_f} has an f-coloring η with k colors in C. Denote $G_i = G - \{w_1, w_2, \dots, w_i\}$ $(i = 1, 2, \dots, t)$. Basing on η, we can obtain an f-coloring of G_{t-1} with k colors in C as follows. If $d_{G_{t-1}}(w_t) = 0$, we are done. Otherwise, for each edge of $\{w_t u \in G_{t-1}\}$, we color $w_t u$ with a color in $M(u) = \{c \in C : c(u) < f(u)\}$. Clearly, $c(v) \le f(v)$ for each $c \in C$ and each $v \ne w_t$ in G_{t-1}. If $c(w_t) \le f(w_t)$ for each $c \in C$, this is an f-coloring of G_{t-1} with k colors in C. Otherwise, there exists a color $\beta \in C$ such that $\beta(w_t) > f(w_t)$. Since $w_t \notin V_{\Delta_f}$, $f(w_t) \ge f^* + 1$. This means that $\beta(w_t) \ge f^* + 2$. Then there must exist another color $\gamma \in C$ such that $\gamma(w_t) \le f^* - 1$. Construct a $\beta\gamma$-alternating walk W starting at w_t. Switching W makes $\beta(w_t)$ decrease by 1 or 2 and $\gamma(w_t)$ increase by 1 or 2, respectively. Note that there is still $\gamma(w_t) \le f^* + 1 \le f(w_t)$ and $\sigma^f(v)$ does not decrease for each v with $\sigma^f(v) = k$ after switching W. Use the operation until $c(w_t) \le f(w_t)$ for each $c \in C$. If $t = 1$, we are done. Otherwise, basing on an f-coloring η_i of

G_{t-i} with k colors in C, we can get an f-coloring of G_{t-i-1} with k colors in C with same operations, for each $1 \le i \le t-1$. (Note that $G_0 = G$.) So, G is of f-class 1.

∎

Corollary 2.1 *Let G be a d-regular simple graph, $V_{\Delta_f} \neq \emptyset$ and $S = \{v \in V_{\Delta_f} : N(v) \subseteq V_{\Delta_f}\}$. If the f-core of G is S-peelable, then G is of f-class* 1.

Proof. Obviously, $\Delta_f(G_{\Delta_f}) \le \Delta_f(G)$ and $S = V_{\Delta_f}(G_{\Delta_f})$. By Theorem 1.3 (1), G_{Δ_f} has an f-coloring η with $\Delta_f(G_{\Delta_f})$ colors. Of course, η is an f-coloring of G_{Δ_f} with $\Delta_f(G)$ colors. So G is of f-class 1 according to Theorem 2.2. ∎

Theorem 2.3 *Let G be a d-regular simple graph and $V_{\delta_g} \neq \emptyset$. G is of g_c-class 1 if and only if the degree restoration subgraph of G based on the g_c-core has a g_c-coloring with $\delta_g(G)$ colors.*

Proof. The necessity is clear. We prove the sufficiency. It is true when $V_{\delta_g} = V(G)$. Next assume that $V(G) \setminus V_{\delta_g} \neq \emptyset$. Let $k = \delta_f(G)$, $g^* = \frac{d}{k}$ and $C = \{c_1, \dots, c_k\}$. Define a function f on $V(G)$ in such a way that g, f are a pair of related-functions of G. Let ζ be a g_c-coloring with $\delta_g(G)$ colors in C of the degree restoration subgraph of G based on the g_c-core. Restrict ζ to the g_c-core of G and denote the edge-coloring of G_{δ_g} by η. Since $c(v) = g^*$ for each $c \in C$ and each $v \in V_{\delta_g}$ in ζ, we have $c(v) \le g^* = f(v)$ for each $c \in C$ and each $v \in V_{\delta_g}$ in η, i.e. η is an f-coloring of G_{δ_g} $(= G_{\Delta_f})$ with $\Delta_f(G) = \delta_g(G)$ colors. Thus G is of f-class 1 by Theorem 2.2. According to Theorem 2.1, G is of g_c-class 1. ∎

Corollary 2.2 *Let G be a d-regular simple graph, $V_{\delta_g} \neq \emptyset$ and $S = \{v \in V_{\delta_g} : N(v) \subseteq V_{\delta_g}\}$. If the g_c-core of G is S-peelable, then G is of g_c-class* 1.

Proof. Define a function f on $V(G)$ in such a way that g, f are a pair of related-functions of G. Then $V_{\Delta_f} = V_{\delta_g}$ and $S = \{v \in V_{\delta_g} : N(v) \subseteq V_{\delta_g}\} = \{v \in V_{\Delta_f} : N(v) \subseteq V_{\Delta_f}\}$. So the condition that the g_c-core of G is S-peelable is equivalent to the condition that the f-core of G is S-peelable. By Corollary 2.1, G is of f-class 1. Also, by Theorem 2.1, G is of g_c-class 1. ∎

For regular graphs, Corollary 2.1, Corollary 2.2 are strictly stronger than Theorem 1.3 (1), (2), respectively. See the example in Fig. 1. For a subset $T = V(G) \setminus \{w, z\}$, the graph G in Fig. 1 (1) is not T-peelable because each vertex of G is adjacent to at least 4 vertices of T. When $g(w) = g(z) = 0$, $f(w) = f(z) = 2$ and $g(v) = f(v) = 1$ for each $v \in V(G) \setminus \{w, z\}$, as indicated in Fig. 1 (2), $S = \{v \in V_{\Delta_f} : N(v) \subseteq V_{\Delta_f}\} = \{v \in V_{\delta_g} : N(v) \subseteq V_{\delta_g}\} = \{p, q\}$. Clearly, the f-core (g_c-core) of G is S-peelable. So G is of f-class 1 and of g_c-class 1 by Corollary 2.1 and Corollary 2.2.

Furthermore, by reassigning functions f, g, we can give an example for a graph G of f-class 2 (of g_c-class 2). When $g(p) = g(y) = 0$, $f(p) = f(y) = 2$

and $g(v) = f(v) = 1$ for each $v \in V(G) \setminus \{p, y\}$, the f-core (g_c-core) of G contains two components A and B. Noting that $S = \{v \in V_{\Delta_f} : N(v) \subseteq V_{\Delta_f}\} = \{v \in V_{\delta_g} : N(v) \subseteq V_{\delta_g}\} = \{r, s, t\}$, the f-core (g_c-core) of G is not S-peelable. For the components B, $|E(B)| = 19 > \Delta_f(B) \times \lfloor \frac{f(V(B))}{2} \rfloor = 6 \times \lfloor \frac{7}{2} \rfloor = 18$. By Theorem 3.1, B has no f-coloring with $\Delta_f(B) = \Delta_f(G)$ colors. Therefore G is of f-class 2 and of g_c-class 2 when $g(p) = g(y) = 0$, $f(p) = f(y) = 2$ and $g(v) = f(v) = 1$ for each $v \in V(G) \setminus \{p, y\}$ by Theorem 2.2 and Theorem 2.1.

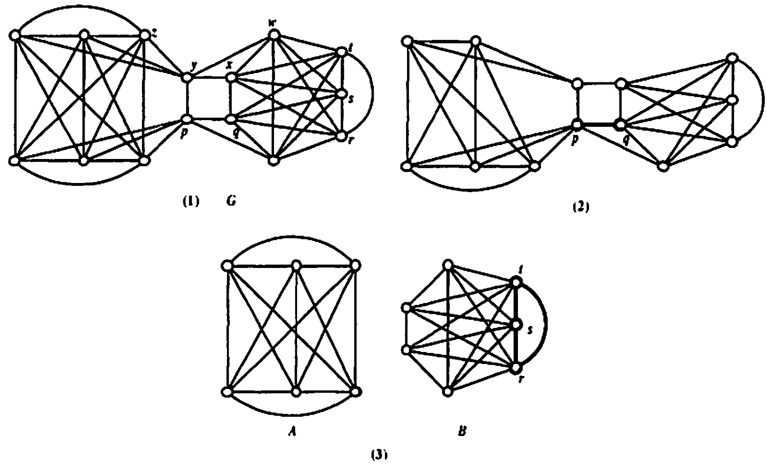


Fig. 1. (1) A regular graph G.
(2) The f-core (g_c-core) of G when $g(w) = g(z) = 0$, $f(w) = f(z) = 2$ and $g(v) = f(v) = 1$ for each $v \in V(G) \setminus \{w, z\}$.
(3) The f-core (g_c-core) of G when $g(p) = g(y) = 0$, $f(p) = f(y) = 2$ and $g(v) = f(v) = 1$ for each $v \in V(G) \setminus \{p, y\}$.

3 Nonregular graphs

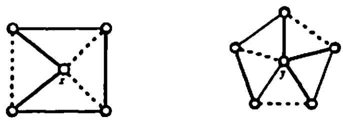
Let $h(V) = \sum_{v \in V(G)} h(v)$. The following results have symmetrical forms.

Theorem 3.1 *[5] Let G be a simple graph with m edges. Then G is of f-class 2 if $m > \Delta_f(G) \lfloor \frac{f(V)}{2} \rfloor$.*

Theorem 3.2 *Let G be a simple graph with m edges. Then G is of g_c-class 2 if $m < \delta_g(G) \lceil \frac{g(V)}{2} \rceil$.*

Proof. If G is of g_c-class 1, then, for any g_c-coloring of G with $\delta_g(G)$ colors, there are at least $\lceil \frac{g(V)}{2} \rceil$ edges colored with a same color. This contradicts with that $m < \delta_g(G) \lceil \frac{g(V)}{2} \rceil$. ∎

For a nonregular graph G with a pair of related-functions g, f, there is not always coincident classification results between f-coloring and g_c-coloring. In Fig. 2, the graph G_1 is of f-class 1 and the graph G_2 is of g_c-class 1 as indicated. However, G_1 is of g_c-class 2 because $|E(G_1)| = 8 < \delta_g(G_1)\lceil\frac{g(V)}{2}\rceil = 3 \times 3 = 9$; G_2 is of f-class 2 because $|E(G_2)| = 10 > \Delta_f(G_2)\lfloor\frac{f(V)}{2}\rfloor = 3 \times 3 = 9$.



(1) (2)

Fig. 2. (1) Graph G_1 with an f-coloring with 3 colors, where $g(x) = 1$, $f(x) = 2$ and $g(v) = f(v) = 1$ for each $v \in V(G_1) \setminus \{x\}$.
(2) Graph G_2 with a g_c-coloring with 3 colors, where $g(y) = 1$, $f(y) = 2$ and $g(v) = f(v) = 1$ for each $v \in V(G_2) \setminus \{y\}$.

In Fig. 2(2), $(G_2)_{\Delta_f} = G - y$ has an f-coloring with 3 colors. However, G_2 is of f-class 2. It means that the result in Theorem 2.2 is not true for a nonregular graph. Similarly, by the example in Fig. 2(1), we know that Theorem 2.3 is not true for a nonregular graph. Here we give two results, the former of which is similar to Theorem 2.2 and the latter of which is similar to Theorem 2.3, under an extra constrain respectively. Especially, the two examples in Fig. 2 show us that the bounds in the degree conditions of the following two theorems are sharp.

Theorem 3.3 *Let G be a simple graph. Suppose that $V_{\Delta_f} \neq \emptyset$ and $d(v) \leq \Delta_f(G)(f(v) - 1) + 1$ for each $v \in V(G) \setminus V_{\Delta_f}$. Then G is of f-class 1 if and only if the f-core of G has an f-coloring with $\Delta_f(G)$ colors.*

Proof. The proof is similar to the one of Theorem 2.2. Clearly, we only need prove the sufficiency. If $V(G) = V_{\Delta_f}$, we are done. Let $|V(G)| = n$, $k = \Delta_f(G)$ and $C = \{c_1, \ldots, c_k\}$. Assume that $V(G) \setminus V_{\Delta_f} = \{w_1, w_2, \ldots, w_t\}$, $1 \leq t \leq n - 1$, and G_{Δ_f} has an f-coloring η with k colors in C. Denote $G_i = G - \{w_1, w_2, \ldots, w_i\}$ $(i = 1, 2, \ldots, t)$. Basing on η, we can obtain an f-coloring of G_{t-1} with k colors in C as follows. If $d_{G_{t-1}}(w_t) = 0$, we are done. Otherwise, for each edge of $\{w_t u \in G_{t-1}\}$, we color $w_t u$ with a color in $M(u) = \{c \in C : c(u) < f(u)\}$. Clearly, $c(v) \leq f(v)$ for each $c \in C$ and each $v \neq w_t$ in G_{t-1}. If $c(w_t) \leq f(w_t)$ for each $c \in C$, this is an f-coloring of G_{t-1} with k colors in C. Otherwise, there exists a color $\beta \in C$ such that $\beta(w_t) \geq f(w_t) + 1$. Since $d(w_t) \leq k(f(w_t) - 1) + 1$, there must exist another color $\gamma \in C$ such that $\gamma(w_t) \leq f(w_t) - 2$. Construct a $\beta\gamma$-alternating walk W starting at w_t. Switching W makes $\beta(w_t)$ decrease by 1 or 2 and $\gamma(w_t)$ increase by 1 or 2, respectively. Note that there is still $\gamma(w_t) \leq f(w_t)$ and $\sigma^f(v)$ does not decrease for each v with $\sigma^f(v) = k$ after switching W. Use the operation until $c(w_t) \leq f(w_t)$ for each $c \in C$. If $t = 1$, we are done. Otherwise, basing on an f-coloring η_i of G_{t-i} with k colors in C, we can get an f-coloring of G_{t-i-1} with k colors in C with same operations, for each $1 \leq i \leq t - 1$. (Note that $G_0 = G$.) So, G is of f-class 1. ∎

Theorem 3.4 *Let G be a simple graph. Suppose that $V_{\delta_g} \neq \emptyset$ and $d(v) \geq \delta_g(G)(g(v)+1)-1$ for each $v \in V(G) \setminus V_{\delta_g}$. Then G is of g_c-class 1 if and only if the degree restoration subgraph of G based on the g_c-core has a g_c-coloring with $\delta_g(G)$ colors.*

Proof. We only need prove the sufficiency. If $V(G) = V_{\delta_g}$, we are done. Let $|V(G)| = n$, $k = \delta_g(G)$ and $C = \{c_1, \ldots, c_k\}$. Assume that $V(G) \setminus V_{\delta_g} = \{w_1, w_2, \ldots, w_t\}$, $1 \leq t \leq n-1$, and $(G_{\delta_g})^+$, the degree restoration subgraph of G based on G_{δ_g}, has a g_c-coloring η with k colors in C. Denote $G_i = G - \{w_1, w_2, \ldots, w_i\}$ $(i = 1, 2, \ldots, t)$. Basing on η and $(G_{\delta_g})^+$, we can obtain a g_c-coloring of $(G_{t-1})^+$ with k colors in C as follows: For each edge of $\{w_t u \in G_{t-1}\}$, we add the edge $w_t u$ to $(G_{\delta_g})^+$, color $w_t u$ with a color of a new pendent edge uu', where $uu' \in E((G_{\delta_g})^+) \setminus E(G_{\delta_g})$, and then remove the edge uu'. If necessary, we add some new pendent edges at w_t to obtain the graph $(G_{t-1})^+$ and color the pendent edges with a color in C. Clearly, $c(v) \geq g(v)$ for each $c \in C$ and each $v \neq w_t$ in $(G_{t-1})^+$. If $c(w_t) \geq g(w_t)$ for each $c \in C$, this is a g_c-coloring of $(G_{t-1})^+$ with k colors in C. Otherwise, there exists a color $\beta \in C$ such that $\beta(w_t) \leq g(w_t) - 1$. Since $d(w_t) \geq k(g(w_t)+1) - 1$, there must exist another color $\gamma \in C$ such that $\gamma(w_t) \geq g(w_t) + 2$. Construct a $\gamma\beta$-alternating walk W starting at w_t. Switching W makes $\gamma(w_t)$ decrease by 1 or 2 and $\beta(w_t)$ increase by 1 or 2, respectively. Note that there is still $\gamma(w_t) \geq g(w_t)$ and $\sigma_g(v)$ does not decrease for each v with $\sigma_g(v) = k$ after switching W. Use the operation until $c(w_t) \geq g(w_t)$ for each $c \in C$. If $t = 1$, we are done. Otherwise, basing on a g_c-coloring η_i of $(G_{t-i})^+$ with k colors in C, we can get a g_c-coloring of $(G_{t-i-1})^+$ with k colors in C with same operations, for each $1 \leq i \leq t-1$. (Note that $(G_0)^+ = G$.) So, G is of g_c-class 1. ∎

When f, g are a pair of related-functions on G, the f-core of G has an f-coloring with $\Delta_f(G)$ colors if and only if the degree restoration subgraph of G based on the g_c-core of G (which is same to the f-core of G in this case) has a g_c-coloring with $\delta_g(G)(=\Delta_f(G))$ colors. So we can obtain a coincident classification result between f-coloring and g_c-coloring for simple graphs by Theorem 3.3, Theorem 3.4 and Corollary 1.1.

Theorem 3.5 *Let G be a simple graph, f, g be a pair of related-functions on G and $k = \Delta_f(G) = \delta_g(G)$. When $k(g(v)+1) - 1 \leq d(v) \leq k(f(v)-1)+1$ for each $v \in V(G) \setminus V^*$, G is of f-class 1 if and only if G is of g_c-class 1.*

Remark 2. The degree conditions that $d(v) \leq k(f(v)-1)+1$ for each $v \in V(G) \setminus V_{\Delta_f}$ is not sufficient for a simple graph G to be of f-class 1, and the degree condition that $d(v) \geq k(g(v)+1)-1$ for each $v \in V(G) \setminus V_{\delta_g}$ is not sufficient for a simple graph G to be of g_c-class 1. See the example in Fig. 3. Clearly, $k = \Delta_f(G) = \delta_g(G) = 3$, $V_{\Delta_f} = V_{\delta_g} = V(G) \setminus \{x\}$ and $3(g(x)+1) - 1 \leq d(x) \leq 3(f(x)-1)+1$. It is well known that any component of $G - x$ has no f-coloring with 3 colors. This means that $G - x$, the f-core of G, has no f-coloring with 3 colors and $(G-x)^+$, the degree restoration subgraph of G based on the g_c-core of G, has no g_c-coloring with 3 colors. So G is of f-class 2 and g_c-class 2.

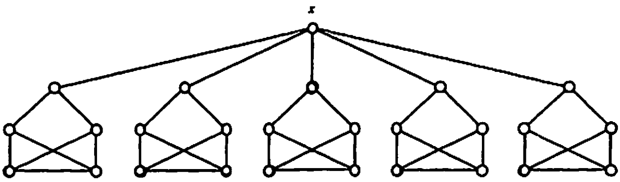


Fig. 3. A graph G, where $g(x) = 1$, $f(x) = 3$ and $g(v) = f(v) = 1$ for each $v \in V(G) \setminus \{x\}$.

By Theorem 3.3, the following result is easy to be verified with a proof similar to the one of Corollary 2.1.

Corollary 3.1 *Let G be a simple graph, $V_{\Delta_f} \neq \emptyset$, $d(v) \leq \Delta_f(G)(f(v) - 1) + 1$ for each $v \in V(G) \setminus V_{\Delta_f}$ and $S = \{v \in V_{\Delta_f} : N(v) \subseteq V_{\Delta_f}\}$. If the f-core of G is S-peelable, then G is of f-class 1.*

By Theorem 3.4, the following result is ready to be verified.

Corollary 3.2 *Let G be a simple graph, $V_{\delta_g} \neq \emptyset$, $d(v) \geq \delta_g(G)(g(v) + 1) - 1$ for each $v \in V(G) \setminus V_{\delta_g}$ and $S = \{v \in V_{\delta_g} : N(v) \subseteq V_{\delta_g}\}$. If the g_c-core of G is S-peelable, then G is of g_c-class 1.*

Proof. Let $C = \{c_1, c_2, \ldots, c_{\delta_g(G)}\}$ be a color set. Clearly, we can get a g-coloring η (not g_c-coloring) of G_{δ_g} with the $\delta_g(G)$ colors in C because the g_c-core of G is S-peelable. Basing on η, we can obtain a g_c-coloring of $(G_{\delta_g})^+$, the degree restoration subgraph of G based on G_{δ_g}, with the $\delta_g(G)$ colors in C as follows: For each pendent edge uu' in $E((G_{\delta_g})^+) \setminus E(G_{\delta_g})$ with $u \in V_{\delta_g}$, color uu' with a color in $\{c \in C : c(u) < g(u)\}$. (Note that $g(u') = 0$ for each end vertex with degree one in these pendent edges in $E((G_{\delta_g})^+) \setminus E(G_{\delta_g})$.) By Theorem 3.4, G is of g_c-class 1. ∎

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