Rainbow connection of the join of two paths*

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Abstract: An edge-colored graph G is (strong) rainbow connected if any two vertices are connected by a (geodesic) path whose edges have distinct colors. The (strong) rainbow connection number of a connected graph G, denoted by $(src(G))$ $rc(G)$, is the smallest number of colors that are needed in order to make G (strong) rainbow connected. The join $P_m \vee P_n$ of P_m and P_n is the graph consisting of $P_m \cup P_n$ and all edges between every vertex of P_m and every vertex of P_n, where P_m (resp. P_n) is a path of m (resp. n) vertices. In this paper, the precise values of $rc(P_m \vee P_n)$ and $src(P_m \vee P_n)$ are given for any positive integers m and n.

Keywords: Edge-coloring; Rainbow connection number; Strong rainbow connection number; Join

1 Introduction and notations

We follow Bondy and Murty [1] for graph-theoretical terminology and notation not defined here. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A sequence $P_n = v_1 v_2 \dots v_n$ of vertices in a graph G such that any two consecutive vertices are adjacent is a *path* between v_1 and v_n, a v_1-v_n path for short. The *length* of a path is the number of edges it contains. A graph G is *connected* if for any two vertices u and v, there exists a u-v path. The *distance* $d(u,v) = d_G(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest u-v path. The maximum distance among all pairs of vertices of G is the *diameter* of G and is denoted by $diam(G)$. If G and H are vertex-disjoint graphs, then the *join* $G \vee H$ of G and H is the graph consisting of $G \cup H$ and all edges between every vertex of G and every vertex of H. The *Cartesian product*

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$X_1 \times X_2 \times \ldots \times X_m$ of nonempty set X_i for $i = 1, 2, \ldots, m$ is the set of all ordered m-tuple $(x_1, x_2, \ldots, x_m)$, where $x_i \in X_i$, that is, $X_1 \times X_2 \times \ldots \times X_m = \{(x_1, x_2, \ldots, x_m) : x_i \in X_i \text{ for any } i = 1, 2, \ldots, m\}$. And we simply write X^m for Cartesian product of the m sets X.

Let G be a nontrivial connected graph on which an edge-coloring c: $E(G) \to \{1, 2, \ldots, k\}$, $k \in \mathbb{N}$, is defined, where adjacent edges may be colored the same. A path is a *rainbow path* if no two edges of it are colored the same. An edge-coloring graph is said to be *rainbow connected* if every pair of vertices is connected by at least one rainbow path. An edge-coloring c under which G is rainbow connected is called a *rainbow coloring* of G. If k colors are used, then c is a *rainbow k-coloring.* The *rainbow connection number* of a connected graph G, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow connected. For any two vertices u and v of G, a *rainbow u-v geodesic* in G is a rainbow u-v path of length $d(u, v)$. The graph G is *strongly rainbow connected* if there exists a rainbow u-v geodesic for any two vertices u and v in G. In this case, the coloring c is called a *strong rainbow coloring* of G. If k colors are used, then c is a *strong rainbow k-coloring.* The *strong rainbow connection number* of a connected graph G, denoted by $src(G)$, is the smallest number of colors that are needed in order to make G strongly rainbow connected.

The concept of rainbow coloring was introduced by Chartrand et al. [4]. For a general graph G, there are lots of bounds on rainbow connection number (see, for example, [2, 3, 5, 6, 7, 8, 9, 10]). However, there are very few results about strong rainbow connection number (see [4]). In [4], Chartrand et al. computed the precise (strong) rainbow connection number of serval special graph classes including Petersen graphs, complete graphs, cycles, trees, wheel graphs, complete bipartite graphs and complete multipartite graphs. In this paper, motivated by the results of the special graph classes in [4], we will compute the precise (strong) rainbow connection numbers of join $P_m \vee P_n$ for any positive integers m and n.

2 Main results

In [4], the authors derived the following result.

Theorem 2.1 ([4]). *Let G be a nontrivial connected graph. Then*

(a) $src(G) = 1$ if and only if G is a complete graph.

(b) $rc(G) = 2$ if and only if $src(G) = 2$.

Theorem 2.2. *For any positive integers m and n with $m \leq n$,*

$$src(P_m \vee P_n) = \begin{cases} 2, & \text{if } n = 3, \\ \lceil \sqrt[m]{\frac{n}{3}} \rceil, & \text{otherwise.} \end{cases}$$

Proof. Since $P_1 \vee P_1, P_1 \vee P_2$ and $P_2 \vee P_2$ are complete graphs, $src(P_1 \vee P_1) = src(P_1 \vee P_2) = src(P_2 \vee P_2) = 1$ by Theorem 2.1, implying that $src(P_m \vee P_n) = 1 = \lceil \sqrt[m]{\frac{n}{3}} \rceil$ for $m \leq n \leq 2$. Let $P_m = v_1 v_2 \ldots v_m$ and $P_n = u_1 u_2 \ldots u_n$.

Assume that $n = 3$. Since $P_m \vee P_3$ is not complete, where $m \in \{1,2,3\}$, $scr(P_m \vee P_3) \geq 2$ by Theorem 2.1. In order to show $scr(P_m \vee P_3) \leq 2$, we now provide a strong rainbow 2-coloring c: $E(P_m \vee P_3) \to \{1,2\}$ of $P_m \vee P_3$ such that $c(u_iu_{i+1}) = c(v_iv_{i+1}) = i$ and $c(u_iv_j) = 1$ for all i and j, implying that $scr(P_m \vee P_3) \leq 2$. As a result, we have $scr(P_m \vee P_3) = 2$. Hence we may assume that $n \geq 4$. Let $\lceil \sqrt[m]{\frac{n}{3}} \rceil = k$. Then it is easy to see that $k \geq 2$. Note that $k - 1 < \sqrt[m]{\frac{n}{3}} \leq k$. Therefore, $3(k-1)^m + 1 \leq n \leq 3k^m$.

First, we claim that $src(P_m \vee P_n) \geq k$. Suppose, to the contrary, that $src(P_m \vee P_n) \leq k-1$. Then there exists a strong rainbow $(k-1)$-coloring c: $E(P_m \vee P_n) \to \{1,2,\ldots,k-1\}$ of $P_m \vee P_n$. For $1 \leq i \leq n$, we can associate an ordered m-tuple $code(u_i) = (a_{i1}, a_{i2}, \ldots, a_{im})$ called the color code of u_i, where $a_{ij} = c(u_iv_j)$ for $1 \leq j \leq m$. Since $1 \leq a_{ij} \leq k-1$ for all i and j, the number of distinct color codes of the vertices of P_n is at most $(k-1)^m$. However, since $n \geq 3(k-1)^m + 1$, there exists $S \subseteq V(P_n)$ such that $|S| \geq 4$ and all vertices in S have the same color code. This implies that there exist at least two vertices $u_a, u_b \in S$ such that $|a-b| \geq 3$ and $c(u_av_j) = c(u_bv_j)$ for $1 \leq j \leq m$. Thus, there is no rainbow u_a-u_b geodesic in $P_m \vee P_n$, a contradiction to our assumption that c is a strong rainbow $(k-1)$-coloring.

Next, in order to show that $src(P_m \vee P_n) \leq k$, we will provide a strong rainbow k-coloring c of $P_m \vee P_n$. Let $A = \{1,2,\ldots,k\}$ and $B = \{1,2,\ldots,k-1\}$. Let A^m and B^m be Cartesian products of the m sets A and m sets B, respectively. We observe that $|A^m| = k^m$ and $|B^m| = (k-1)^m$. Hence $3|B^m| + 1 \leq n \leq 3|A^m|$. Let the color code $code(u_i) = (a_{i1}, a_{i2}, \ldots, a_{im}) \in A^m$ for $1 \leq i \leq n$, where $a_{ij} = c(u_iv_j)$ for $1 \leq j \leq m$. We consider two cases as follows.

Case 1. $k \geq 3$, or $k = 2$ and $m \leq 3$, or $k = 2$ and $3 \cdot 2^{m-1} + 1 \leq n \leq 3 \cdot 2^m$.

Let $code(u_{3i+1}) = code(u_{3i+2}) = code(u_{3i+3}) \in A^m$ for $i \in \{0,1,2,\ldots\}$ such that $code(u_i) \in B^m$ for $1 \leq i \leq 3(k-1)^m$. Note that $n \leq 3k^m$. Therefore, we may assume that $code(u_{3i+1}) \neq code(u_{3j+1})$ for $j > i \geq 0$. This implies that $code(u_i) \neq code(u_j)$ for all i and j with $|i-j| \geq 3$. We now provide a strong rainbow k-coloring c: $E(P_m \vee P_n) \to \{1,2,\ldots,k\}$ of $P_m \vee P_n$ defined by

$$c(e) = \begin{cases} 1, & \text{if } e = u_iu_{i+1} \text{ for odd integer } i, \text{ or if } e = v_iv_{i+1} \text{ for odd integer } i, \\ 2, & \text{if } e = u_iu_{i+1} \text{ for even integer } i, \text{ or if } e = v_iv_{i+1} \text{ for even integer } i, \\ a_{ij}, & \text{if } e = u_iv_j \text{ for all } i \text{ and } j. \end{cases}$$

Now we show that c is a strong rainbow k-coloring of $P_m \vee P_n$. It is easy to see that u_iv_j is a rainbow u_i-v_j geodesic for $i \in \{1,2,\ldots,n\}$ and $j \in \{1,2,\ldots,m\}$. Let $1 \leq a < b \leq n$. If $b = a+1$ (resp. $b = a+2$), then u_au_b (resp. $u_au_{a+1}u_b$) is a rainbow u_a-u_b geodesic. Hence we may assume that $b \geq a+3$. Recalling that $code(u_a) \neq code(u_b)$, there exists some l with $1 \leq l \leq m$ such that $code(u_a)$ and $code(u_b)$ have different l-th coordinates. Therefore, $c(u_av_l) \neq c(u_bv_l)$, implying that $u_av_lu_b$ is a rainbow u_a-u_b geodesic.

It remains to show that there exists a rainbow v_a-v_b geodesic for $1 \leq a < b \leq m$. If $b = a+1$ (resp. $b = a+2$), then v_av_b (resp. $v_av_{a+1}u_b$) is a rainbow v_a-v_b geodesic. This implies that if $m \leq 3$, then clearly there exists a rainbow v_a-v_b geodesic. Hence we may assume that $b \geq a+3$. Note that $code(u_{3i+1}) = code(u_{3i+2}) = code(u_{3i+3}) \in A^m$ for $i \in \{0,1,2,\ldots\}$, where $code(u_{3i+1}) \neq code(u_{3j+1})$ for $j > i \geq 0$, and $code(u_i) \in B^m$ for $1 \leq i \leq 3(k-1)^m$. Therefore, if $k \geq 3$, then there exists some $l \in \{1,2,\ldots,3(k-1)^m\}$ such that $c(u_l v_a) \neq c(u_l v_b)$, implying that $v_a u_l v_b$ is a rainbow v_a-v_b geodesic. If $k=2$ and $3 \cdot 2^{m-1}+1 \leq n \leq 3 \cdot 2^m$, then there exists some $l \in \{1,2,\ldots,n\}$ such that $c(u_l v_a) \neq c(u_l v_b)$ (otherwise, $n \leq 3 \cdot 2^{m-1}$ since $code(u_{3i+1}) = code(u_{3i+2}) = code(u_{3i+3}) \in A^m$ for any $i \in \{0,1,\ldots\}$, a contradiction), implying that $v_a u_l v_b$ is a rainbow v_a-v_b geodesic. Then it follows that c is a strong rainbow k-coloring of $P_m \vee P_n$.

Case 2. $k=2$ and $4 \leq m \leq n \leq 3 \cdot 2^{m-1}$.

For $1 \leq i,j \leq m$, let $a_{ij}=1$ if $i=j$ and $a_{ij}=2$ otherwise. And let $code(u_{m+3i+1}) = code(u_{m+3i+2}) = code(u_{m+3i+3})$ for any $i \in \{0,1,2,\ldots\}$ and let $code(u_{m+3i+1}) \neq code(u_{m+3j+1})$ for $j > i \geq 0$. Note that for $m \geq 4$,

$$m + \lceil \frac{n-m}{3} \rceil \leq m + \lceil \frac{3 \cdot 2^{m-1} - m}{3} \rceil = 2^{m-1} + m + \lceil \frac{-m}{3} \rceil < 2^{m-1} + m < 2^m.$$

Therefore, we may assume that $code(u_{m+3i+1}) \neq code(u_j)$ for $i \in \{0,1,2,\ldots\}$ and $j \in \{1,2,\ldots,m\}$. This implies that $code(u_i) \neq code(u_j)$ for all i and j with $|i-j| \geq 3$. We now provide a strong rainbow 2-coloring c: $E(P_m \vee P_n) \to \{1,2\}$ of $P_m \vee P_n$ defined by

$$c(e) = \begin{cases} 1, & \text{if } e = u_i u_{i+1} \text{ for odd integer } i, \text{ or if } e = v_i v_{i+1} \text{ for odd integer } i, \\ 2, & \text{if } e = u_i u_{i+1} \text{ for even integer } i, \text{ or if } e = v_i v_{i+1} \text{ for even integer } i, \\ a_{ij}, & \text{if } e = u_i v_j \text{ for all } i \text{ and } j. \end{cases}$$

Now we show that c is a strong rainbow 2-coloring of $P_m \vee P_n$. Clearly $u_i v_j$ is a rainbow u_i-v_j geodesic for $1 \leq i \leq n$ and $1 \leq j \leq m$. Let $1 \leq a < b \leq m$. If $b = a+1$, then clearly $v_a v_b$ is a rainbow v_a-v_b geodesic; otherwise, $v_a u_a v_b$ is a rainbow v_a-v_b geodesic since $a_{ii}=1$ and $a_{ij}=2$ for $1 \leq i < j \leq m$.

It remains to show that there exists a rainbow u_a-u_b geodesic for $1 \leq a < b \leq n$. If $b=a+1$ (resp. $b=a+2$), then clearly $u_a u_b$ (resp. $u_a u_{a+1} u_b$) is a rainbow u_a-u_b geodesic. Hence we may assume that $b \geq a+3$. Recalling that $code(u_a) \neq code(u_b)$, there exists some $l \in \{1,2,\ldots,m\}$ such that $code(u_a)$ and $code(u_b)$ have different l-th coordinates. This implies that $c(u_a v_l) \neq c(u_b v_l)$ and hence $u_a v_l u_b$ is a rainbow u_a-u_b geodesic. Then it follows that c is a strong rainbow 2-coloring of $P_m \vee P_n$. □

Theorem 2.3. *For any integers m and n with $m \leq n$,*

$$rc(P_m \vee P_n) = \begin{cases} 1, & \text{if } n \leq 2, \\ 2, & \text{if } 3 \leq n \leq 3 \cdot 2^m, \\ 3, & \text{otherwise.} \end{cases}$$

Proof. By Theorem 2.2, we have $rc(P_m \vee P_n) \leq src(P_m \vee P_n) = \lceil \sqrt[m]{\frac{n}{3}} \rceil = 1$ for $n \leq 2$ and hence $rc(P_m \vee P_n) = 1$. Since $src(P_m \vee P_n) = 2$ for $3 \leq n \leq 3 \cdot 2^m$

by Theorem 2.2, it follows from Theorem 2.1 that $rc(P_m \vee P_n) = 2$. Note that $src(P_m \vee P_n) = \lceil \sqrt[m]{\frac{n}{3}} \rceil \geq 3$ for $n \geq 3 \cdot 2^m + 1$ by Theorem 2.2 and $rc(P_m \vee P_n) \geq diam(P_m \vee P_n) = 2$. Hence by Theorem 2.1, $rc(P_m \vee P_n) \geq 3$ for $n \geq 3 \cdot 2^m + 1$. We now show that $rc(P_m \vee P_n) \leq 3$ for $n \geq 3 \cdot 2^m + 1$. Let $P_m = v_1 v_2 \dots v_m$ and $P_n = u_1 u_2 \dots u_n$. In this case, we now provide a rainbow 3-coloring c: $E(P_m \vee P_n) \to \{1,2,3\}$ of $P_m \vee P_n$ defined by

$$c(u_i v_j) = \begin{cases} 1, & \text{if } i = 3k+1 \text{ for } k \geq 0 \text{ and } 1 \leq j \leq m, \\ 2, & \text{if } i = 3k+2 \text{ for } k \geq 0 \text{ and } 1 \leq j \leq m, \\ 3, & \text{if } i = 3k+3 \text{ for } k \geq 0 \text{ and } 1 \leq j \leq m, \end{cases}$$

$$c(u_i u_{i+1}) = \begin{cases} 3, & \text{if } i = 3k+1 \text{ for } k \geq 0, \\ 1, & \text{if } i = 3k+2 \text{ for } k \geq 0, \\ 2, & \text{if } i = 3k+3 \text{ for } k \geq 0, \end{cases}$$

and $c(v_j v_{j+1}) = 1$ for $1 \leq j \leq m-1$. Now we show that c is a rainbow 3-coloring of $P_m \vee P_n$. It is easy to see that $u_i v_j$ is a rainbow u_i-v_j path for all i and j. And we observe that $v_i u_1 u_2 v_j$ is a rainbow v_i-v_j path for $1 \leq i < j \leq m$. Let $1 \leq i < j \leq n$. If $j \leq i+3$, then $u_i u_{i+1} \dots u_j$ is a rainbow u_i-u_j path; and if $j \geq i+4$, then $u_i v_1 u_j$ or $u_i u_{i+1} v_1 u_j$ is a rainbow u_i-u_j path. As a result, c is a rainbow 3-coloring of $P_m \vee P_n$ and hence $rc(P_m \vee P_n) \leq 3$. Therefore, we have $rc(P_m \vee P_n) = 3$ for $n \geq 3 \cdot 2^m + 1$. □

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