Metric Dimension of the Möbius Ladder

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Abstract. It is claimed in [13] that the metric dimension of the Möbius ladder M_n is 3 when $n \not\equiv 2 \pmod 8$, but it is wrong; we give a counter example when $n \equiv 6 \pmod 8$. In this paper we not only give the correct metric dimension in this case but also solve the open problem regarding the metric dimension of M_n when $n \equiv 2 \pmod 8$. Moreover, we conclude that M_n has two sub families with constant metric dimensions.

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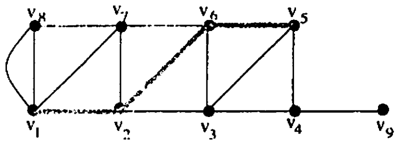
1. Introduction

The concepts of metric dimension and resolving set were introduced by Slater in [15, 16] and studied independently by Harary and Melter in [5]. Since then the resolving sets have been widely investigated, as you can see in [2, 3, 9, 14, 17, 18].

Applications of metric dimension to the navigation of robots in networks are discussed in [10], to chemistry in [4], and to image processing in [12].

A *graph* G is a pair $(V(G), E(G))$, where V is the set of vertices and E is the set of edges. A *path* from a vertex v to a vertex w is a sequence of vertices and edges that starts from v and stops at w. The number of edges in a path is the *length* of that path. A graph is said to be *connected* if there is a path between any two of its vertices. The *distance* $d(u, v)$ between two vertices u, v of a connected graph G is the length of a shortest path between them.



A connected graph with a highlighted shortest path from v_1 to v_5

Let $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered set of vertices of G and let v be a vertex of G. The representation $r(v|_W)$ of v with respect to W is the k-tuple $(d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$. If distinct vertices of G have distinct representations with respect to W, then W is called a *resolving set* for G [1]. A resolving set of minimum cardinality is called a *basis* of G; the number of elements in this basis is the *metric dimension* of G, $\dim(G)$. A family $\mathcal{G}$ of connected graphs is said to have constant metric dimension if it is independent of any choice of member of that family.

The metric dimension of wheel W_n is determined by Buczkowski et al. [1], of fan f_n by Caceres et al. [3, 2], and of Jahangir graph J_{2n} by Tomescu et al. [17]. Chartrand et al. [4] proved that the family of path P_n has the constant metric dimension 1. Javaid et al. proved in [9] that the plane graph antiprism $A_n, n \geq 5$ constitutes a family of regular graphs with constant metric dimension 3. The metric dimensions of some classes of plane graphs and convex polytopes have been studies in [8], of generalized Petersen graphs $P(n, 3)$ in [6], of some rotationally-symmetric graphs in [7].

This article particularly deals with *Möbius ladder* which is an infinite subclass of circulant graphs. Let n, m and $a_1, ..., a_m$ be positive integers, $1 \leq a_i \leq \lfloor \frac{n}{2} \rfloor$ and $a_i \neq a_j$ for all $1 \leq i < j \leq m$. An undirected graph with the set of vertices $V = \{v_1, ..., v_n\}$ and the set of edges $E = \{v_i v_{i+a_j} : 1 \leq i \leq n, 1 \leq j \leq m\}$, the indices being taken modulo n, is called a circulant graph and is denoted by $C_n(a_1, ..., a_m)$. In [19], C. Monica computed the metric dimension $C_n(1, 2)$ and gave a result as follows

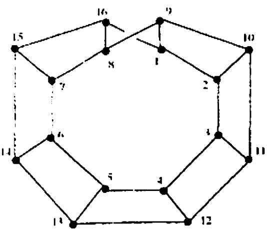
Theorem 1.1. *Let* $G = C_n(1, 2)$. *Then* $dim(G) = \begin{cases} = 3, & n \equiv 0, 2, 3 (mod\ 4) \\ \leq 4, & n \equiv 1 (mod\ 4) \end{cases}$

It is clear that upper bound for the metric dimension of M_n is 4 mode(4) when $n \equiv 1$. Later, Imran et al.[21] extended the above results to $C_n(1, 2, 3)$ and formulated the following result,

Theorem 1.2. *Let* $G = C_n(1, 2, 3)$. *Then* $dim(G) = \begin{cases} 4, & n \equiv 2, 3, 5 (mod\ 6) \\ \leq 5, & n \equiv 0 (mod\ 6) \\ \leq 6, & n \equiv 1 mod\ 6). \end{cases}$

In [20] C. Grior et al. generalized it $C_n(a_1, ..., a_j)$, where $j < \lfloor \frac{n}{2} \rfloor$ and deduced the lower bound to be $j + 1$.

The *Möbius ladder* M_n is a cubic circulant graph with even number of vertices formed from an n-cycle by adding edges connecting opposite pair of vertices in the cycle, except with two pairs which are connected with a twist. Here n is even and vertices are considered mode(n). Take a look at the following example.



Möbius ladder M_{16}

Remark 1.3. It can be easily deduced from [20] that $3 \leq dim(M_n) \leq 4$. To be more precise, we in this article find the exact value of $dim(M_n)$ for all $n \geq 8$. For the rest of this article n is considered even to keep the notations homogenized as in [13].

2. Main Theorem

It is claimed in [13] that *"The metric dimension of the Möbius ladder M_n is 3, when $n \geq 8$ and $n \not\equiv 2$ (mod 8),"* but there are two issues regarding this result:

Issue I. Although this result is true when $n \equiv 0, 4$ (mod 8), yet there are some technical mistakes in their proof:

1. Part a(i): The distances of W from the vertices v_1, v_2 and $v_{1+4k}, k \geq 2$, when $n \equiv 0$ (mod 8) and from the vertices v_1, v_2 and $v_{3+4k}, k \geq 2$, when $n \equiv 4$ (mod 8) are missing.
2. Part a(i): We receive different vectors just for one value of i. For instance if $i = k = 1$ and $W = \{v_1, v_2, v_5\}$, then we get $r(v_3|_W) = (2, 1, 2)$ and $r(v_3|_W) = (2, 1, 3)$, which does not make sense.
3. Part a(i): If $n = 8$ and $k = 1$, many inequalities in the formula are wrong. For instance, third inequality in the first formula becomes $2 \leq i \leq 1$, which again does not make sense.

Issue II. The result is wrong when $n \equiv 6$ (mod 8). For instance if we take $n = 14$ and $W = \{v_1, v_2, v_7\}$, then we receive $r(v_4|_W) = (3, 2, 3) = r(v_{10}|_W)$ which implies that the metric dimension is surely not 3.

We now make separate statements and give proofs, avoiding such mistakes, along with examples:

Proposition 2.1. *The metric dimension of the Möbius ladder M_n when $n \equiv 0 \pmod 8$ is 3 for all $n \geq 8$.*

Proof. Let us take $n = 8k, k \in \mathbb{Z}^+$ and $W = \{v_1, v_2, v_{4k+1}\}$. We show that W is a resolving set. For this, the representation of any vertex of M_n with respect to W is:

$$r(v_{2i+1}|w) = \begin{cases} (2i, |2i-1|, 2i+1), & 0 \leq i \leq k-1; \\ (2k, 2k-1, 2k), & i = k; \\ (4k-2i+1, 4k-2i+2, \\ 4k-2i), & k+1 \leq i \leq 2k-1; k \geq 2 \\ (2k-i+1, 2k-i+2, 2k-i), & i = 2k; \\ (2i-4k+1, 2i-4k, \\ 2i-4k), & 2k+1 \leq i \leq 3k-1; k \geq 2 \\ (2k, 2k, 2k), & i = 3k; \\ (8k-2i, 8k-2i+1, \\ 8k-2i+1), & 3k+1 \leq i \leq 4k-1; k \geq 2 \end{cases}$$

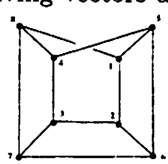
and

$$r(v_{2i}|w) = \begin{cases} (2i-1, 2i-2, 2i), & 1 \leq i \leq k; \\ (2k, 2k, 2k-1), & i = k+1; \\ (4k-2i+2, 4k-2i+3, 4k-2i+1), & k+2 \leq i \leq 2k; k \geq 2 \\ (2i-4k, 2i-4k-1, 2i-4k-1), & 2k+1 \leq i \leq 3k; \\ (8k-2i+1, 8k-2i+2, 8k-2i+2), & 3k+1 \leq i \leq 4k; \end{cases}$$

Observe that all these representations are distinct, implying that $dim(M_{2n}) \leq 3$. On the other hand, if we take $W = \{v_1, v_2\}$, then we get $r(v_{1+4k}|w) = (1, 2) = r(v_{8k}|w)$, which indicates that the dimension is not 2. So, $\dim(M_n) = 3$. □

Example. If $k = 1, n = 8$ and $W = \{v_1, v_2, v_5\}$, the resolving vectors are

$v_1 = (0, 1, 1), v_2 = (1, 0, 2), v_3 = (2, 1, 2), v_4 = (2, 2, 1), v_5 = (1, 2, 0), v_6 = (2, 1, 1), v_7 = (2, 2, 2)$, and $v_8 = (1, 2, 2)$.



M_8

Proposition 2.2. *The metric dimension of the Möbius ladder M_n when $n \equiv 4 \pmod 8$ is 3 for all $n \geq 8$.*

Proof. In this case, we write $n = 8k + 4, k \in \mathbb{Z}^+$ and claim that $W = \{v_1, v_2, v_{4k+3}\} \subset V(M_n)$ is a resolving set for M_n. The representations of vertices of M_n with respect to W are:

$$r(v_{2i+1}|W) = \begin{cases} (2i, |2i-1|, 2i+1), & 0 \leq i \leq k; \\ (2k+1, 2k+1, 2k), & i = k+1; \\ (4k-2i+3, 4k-2i+4, \\ 4k-2i+2), & k+2 \leq i \leq 2k+1; \\ (2i-4k-1, 2i-4k-2, \\ 2i-4k-2), & 2k+2 \leq i \leq 3k+1; \\ (8k-2i+4, 8k-2i+5, \\ 8k-2i+5), & 3k+2 \leq i \leq 4k+1; \end{cases}$$

and

$$r(v_{2i}|W) = \begin{cases} (2i-1, 2i-2, 2i), & 1 \leq i \leq k; \\ (2k+1, 2k, 2k+1), & i = k+1; \\ (4k-2i+4, 4k-2i+5, 4k-2i+3), & k+2 \leq i \leq 2k+1; \\ (2i-4k-2, 2i-4k-3, 2i-4k-3), & 2k+2 \leq i \leq 3k+1; \\ (2k+1, 2k+1, 2k+1), & i = 3k+2; \\ (8k-2i+5, 8k-2i+6, 8k-2i+6), & 3k+3 \leq i \leq 4k+2; \end{cases}$$

Since no two vertices have the same representations, $dim(M_{2n}) \leq 3$. Moreover, for $W = \{v_1, v_2\}$, we receive $r(v_{3+4k}|W) = r(v_{4+8k}|W) = (1,2)$, indicating that $dim(M_n) \neq 2$. □

Example. If $k = 1$.then $n = 12$ and $W = \{v_1, v_2, v_7\}$, and the resolving vectors are These are $v_1 = (0,1,1), v_2 = (1,0,2), v_3 = (2,1,3), v_4 = (3,2,3), v_5 = (3,3,2), v_6 = (2,3,1), v_7 = (1,2,0), v_8 = (2,1,1)$, $v_9 = (3,2,2)$, $v_{10} = (3,3,3)$, $v_{11} = (2,3,3)$, and $v_{12} = (1,2,2)$.

Proposition 2.3. *When $n \equiv 6 \pmod 8$, $\dim(M_n) = 4$ for all $n \geq 8$.*

Proof. In this case we take $W = \{v_1, v_2, v_{2k+3}, v_{4k+4}\}$ and claim that all distance vectors

$$r(v_{2i+1}|W) = \begin{cases} (2i, 2i-1, |2i-2k-2|, 2i+1), & 0 \leq i \leq k; \\ (2k+2, 2k+1, 2k-2i+2, 2k+1), & i = k+1; \\ (4k-2i+4, 4k-2i+5, \\ 2i-2k-2, 4k-2i+3), & k+2 \leq i \leq 2k+1; \\ (2i-4k-2, 4k-2i+3, \\ 6k-2i+6, 4k-2i+3), & 2k+2 \leq i \leq 3k+2; \\ (8k-2i+6, 8k-2i+7, \\ 6k-2i+4, 8k-2i+7), & 3k+3 \leq i \leq 4k+2; \end{cases}$$

and

$$r(v_{2i}|W) = \begin{cases} (2i-1, 2i-2, |2i-2k-3|, 2i), & 1 \le i \le k+1; \\ (4k-2i+5, 4k-2i+6, \\ 2i-2k-3, 4k-2i+4), & k+2 \le i \le 2k+2; \\ (2i-4k-3, 2i-4k-4, \\ 6k-2i+7, 2i-4k-4), & 2k+3 \le i \le 3k+2; \\ (8k-2i+7, 8k-2i+8, \\ 6k-2i+5, 8k-2i+8), & 3k+3 \le i \le 4k+3; \end{cases}$$

are distinct. However, if we take $W = \{v_1, v_2, v_{2k+3}\}$, then we get two similar resolving vectors: $r(v_{3+4k}|W) = r(v_{5+8k}|W) = (2, 3, 2k)$, confirming that the dimension is not 3. □

Example. If $k = 1, n = 14$, the resolving vectors with respect to $W = \{v_1, v_2, v_5, v_8\}$ become $v_1 = (0, 1, 4, 1), v_2 = (1, 0, 3, 2), v_3 = (2, 1, 2, 3), v_4 = (3, 2, 1, 4), v_5 = (4, 3, 0, 3), v_6 = (3, 4, 1, 2), v_7 = (2, 3, 2, 1), v_8 = (1, 2, 3, 0), v_9 = (2, 1, 4, 1), v_{10} = (3, 2, 3, 2), v_{11} = (4, 3, 2, 3), v_{12} = (3, 4, 1, 4), v_{13} = (2, 3, 2, 3)$, and $v_{14} = (1, 2, 3, 2)$.

Proposition 2.4. *When* $n \equiv 2 \ (mod\ 8), \dim(M_n) = 4$ *for all* $n \ge 8$.

Proof. Take $n = 8k + 2, k \in \mathbb{Z}^+$ and $W = \{v_1, v_2, v_{2k+2}, v_{4k+2}\} \subset V(M_n)$. The resolving vectors with respect to W are

$$r(v_{2i+1}|W) = \begin{cases} (2i, |2i-1|, |2i-2k-1|, 2i+1), & 0 \le i \le k; \\ (4k-2i+2, 4k-2i+3, \\ |2k-2i+1|, 4k-2i+1), & k+1 \le i \le 2k; \\ (|4k-2i|, |4k-2i+1|, \\ 6k-2i+3, |4k-2i+1|), & 2k+1 \le i \le 3k; \\ (8k-2i+2, 8k-2i+3, \\ |6k-2i+1|, 8k-2i+3), & 3k+1 \le i \le 4k; \end{cases}$$

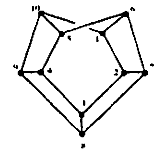
and

$$r(v_{2i}|W) = \begin{cases} (2i-1, 2i-2, |2i-2k-2|, 2i), & 1 \le i \le k; \\ (2k+1, 2k, 2i-2k-2, 2k), & i = k+1; \\ (4k-2i+3, 4k-2i+4, \\ 2i-2k-2, 4k-2i+2), & k+2 \le i \le 2k+1; \\ (2i-4k-1, |4k-2i+2|, \\ 6k-2i+4, |4k-2i+2|), & 2k+2 \le i \le 3k+1; \\ (8k-2i+3, 8k-2i+4, \\ |6k-2i+2|, 8k-2i+4), & 3k+2 \le i \le 4k+1; \end{cases}$$

which are all distinct, confirming that $\dim(M_n) \le 4$. On the other hand we show that $dim(M_n) \ge 4$ by proving that there is no resolving set with cardinality less or equal to 3. Suppose that $W = \{v_1, v_2, v_{2k+2}\}$. Then we receive $r(v_{2+4k}|W) = r(v_{2+8k}|W) = (1, 2, 2k)$, indicating that dimension is not 3. So, we must have $\dim(M_n) = 4$, and this completes the proof. □

Example. If $k = 1$, then $n = 10$ and resolving set is $W = \{v_1, v_2, v_3, v_4\}$.

The resolving vectors are: $v_1 = (0,1,3,1), v_2 = (1,0,2,2), v_3 = (2,1,1,3), v_4 = (3,2,0,2), v_5 = (2,3,1,1), v_6 = (1,2,2,0), v_7 = (2,1,3,1), v_8 = (3,2,2,2), v_9 = (2,3,1,3)$, and $v_{10} = (1,2,2,2)$.



M_{10}

Conclusion.

In this paper we presented complete and exact result of metric dimension of M_n providing the solution of an open problem posed in [13]. We conclude below with absolute answer to the metric dimension of the Möbius ladder for all possible cases:

Theorem 2.5. *For all $n \geq 8$*

$$\dim(M_n) = \begin{cases} 3, & when\ n \equiv 0\ (mod\ 4) \\ 4, & when\ n \equiv 2\ (mod\ 4) \end{cases}$$

As a subsidiary result we also conclude that this family does not have a constant metric dimension.

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