# Proof of a conjecture on the Catalan-Larcombe-French numbers

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**Abstract.** Let $P_n$ denote the $n$-th Catalan-Larcombe-French number. Recently, the 2-log-convexity of the Catalan-Larcombe-French sequence was proved by Sun and Wu. Moreover, they also conjectured that the quotient sequence $\{\frac{P_n}{P_{n-1}}\}_{n=0}^{\infty}$ of the Catalan-Larcombe-French sequence is log-concave. In this paper, this conjecture is confirmed by utilizing the upper and lower bounds for $\frac{P_n}{P_{n-1}}$ and finding a middle function $f(n)$.

**Keywords:** the Catalan-Larcombe-French number, log-concavity.

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## 1 Introduction

Consider the infinite sequence

$$\{P_n\}_{n=0}^{\infty} = \{P_0,\ P_1,\ P_2,\ P_3,\ P_4, \cdots\} = \{1,\ 8,\ 80,\ 896,\ 10816, \cdots\},$$

known as the Catalan-Larcombe-French sequence (Sequence No. A053175 in Sloane's database [6]). In their delightful paper [2], Larcombe and French developed a number of properties of $P_n$. The sequence satisfies the following recurrence relation:

$$P_n = \frac{8(3n^2-3n+1)}{n^2}P_{n-1} - \frac{128(n-1)^2}{n^2}P_{n-2}, \qquad (1.1)$$

for $n \geq 2$, with the initial values given by $P_0 = 1$ and $P_1 = 8$. For more details, see [1, 2, 3, 4, 5].

Recently, some combinatorial properties for $P_n$ have been proven. Zhao [11] studied the log-behavior of the Catalan-Larcombe-French sequence and proved that the sequence $\{P_n\}_{n=0}^{\infty}$ is log-balanced. Recall that an infinite sequence $\{a_n\}_{n=0}^{\infty}$ is said to be log-concave (respectively, log-convex) if for any positive integer $n$,

$$a_n^2 \geq a_{n-1}a_{n+1}, \qquad \text{(respectively, } a_n^2 \leq a_{n-1}a_{n+1}\text{)}.$$

Xia and Yao [8, 9, 10] proved that the sequences $\{\frac{P_{n+1}}{P_n}\}_{n=0}^{\infty}$ and $\{\sqrt[n]{P_n}\}_{n=1}^{\infty}$ are strictly increasing. Very recently, Sun and Wu [7] proved that the sequence $\{P_n\}_{n=0}^{\infty}$ is 2-log-convex. Furthermore, Sun and Wu [7] presented the following conjecture:

**Conjecture 1.1** *The quotient sequence $\{\frac{P_n}{P_{n-1}}\}_{n=1}^{\infty}$ of the Catalan-Larcombe-French sequence is log-concave, equivalently, for all $n \geq 2$,*

$$P_{n-2}P_n^3 \geq P_{n+1}P_{n-1}^3. \tag{1.2}$$

The aim of this paper is to prove Conjecture 1.1 by using the upper and lower bounds of $\frac{P_n}{P_{n-1}}$ and finding a middle function $f(n)$.

# 2 Proof of Conjecture 1.1

In order to confirm Conjecture 1.1, we first prove some lemmas.

**Lemma 2.1** *For $n \geq 48$,*

$$16 - \frac{16}{n} - \frac{18}{n^3} < \frac{P_n}{P_{n-1}}. \tag{2.1}$$

*Proof.* We prove this Lemma by induction on $n$. It is a routine to verify that (2.1) holds for $n = 48$. Assume that Lemma 2.1 is true for $n = m \geq 48$, that is,

$$16 - \frac{16}{m} - \frac{18}{m^3} < \frac{P_m}{P_{m-1}}. \tag{2.2}$$

In order to prove this lemma, it suffices to prove that this lemma is true for $n = m + 1$, namely,

$$16 - \frac{16}{m+1} - \frac{18}{(m+1)^3} < \frac{P_{m+1}}{P_m}. \tag{2.3}$$

Thanks to (1.1) and (2.2),

$$\begin{aligned}\frac{P_{m+1}}{P_m} &= \frac{8(3(m+1)^2-3(m+1)+1)}{(m+1)^2} - \frac{128m^2}{(m+1)^2}\frac{P_{m-1}}{P_m}\\ &> \frac{8(3(m+1)^2-3(m+1)+1)}{(m+1)^2} - \frac{128m^2}{(m+1)^2}\frac{1}{16-\frac{16}{m}-\frac{18}{m^3}}\\ &= \frac{8(16m^5-16m^3-35m^2-27m-9)}{(m+1)^2(2m-3)(4m^2+2m+3)}. \qquad (2.4)\end{aligned}$$

Thanks to (2.4),

$$\begin{aligned}&\frac{P_{m+1}}{P_m} - \left(16-\frac{16}{m+1}-\frac{18}{(m+1)^3}\right)\\ >&\frac{8(16m^5-16m^3-35m^2-27m-9)}{(m+1)^2(2m-3)(4m^2+2m+3)} - \left(16-\frac{16}{m+1}-\frac{18}{(m+1)^3}\right)\\ =&\frac{2(4m^3-176m^2-72m-117)}{(m+1)^2(2m-3)(4m^2+2m+3)} > 0, \qquad (2.5)\end{aligned}$$

which yields (2.3). This completes the proof of this lemma by induction. ∎

**Lemma 2.2** *For* $n \geq 48$,

$$\frac{P_{n+1}P_{n-1}}{P_n^2} < f(n), \qquad (2.6)$$

*where*

$$f(n) = \frac{(8n^3+4n^2-2n-9)(2n-1)^3}{(2n+1)3(8n^3-20n^2+14n-11)}. \qquad (2.7)$$

*Proof.* Set

$$a(n) = \frac{8(3n^2-3n+1)}{n^2}, \qquad (2.8)$$

and

$$b(n) = -\frac{128(n-1)^2}{n^2}. \qquad (2.9)$$

It is easy to verify that for $n \geq 48$,

$$\begin{aligned}&a^2(n+1)+4f(n)b(n+1)\\ =&\frac{64c(n)}{(n+1)^4(2n+1)^3(8n^3-20n^2+14n-11)} > 0, \qquad (2.10)\end{aligned}$$

where $c(n)$ is a polynomial in $n$. Moreover, it is easy to verify that for $n \geq 0$,

$$
\begin{aligned}
&2f(n)\left(16-\frac{16}{n}-\frac{18}{n^3}\right)-a(n+1)\\
=&\frac{4d(n)}{(2n+1)^3(8n^3-20n^2+14n-11)(n+1)^2n^3}>0 \qquad (2.11)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(2f(n)\left(16-\frac{16}{n}-\frac{18}{n^3}\right)-a(n+1)\right)^2-\left(a^2(n+1)+4f(n)b(n+1)\right)\\
=&\frac{16(2n-1)^3(8n^3+4n^2-2n-9)e(n)}{n^6(n+1)^2(2n+1)^6(8n^3-20n^2+14n-11)}>0, \qquad (2.12)
\end{aligned}
$$

where $d(n)$ and $e(n)$ are polynomials in $n$. It follows from (2.10), (2.11) and (2.12) that for $n \geq 0$,

$$16-\frac{16}{n}-\frac{18}{n^3}>\frac{a(n+1)+\sqrt{a^2(n+1)+4f(n)b(n+1)}}{2f(n)}. \qquad (2.13)$$

In view of (2.1) and (2.13),

$$\frac{P_n}{P_{n-1}}>\frac{a(n+1)+\sqrt{a^2(n+1)+4f(n)b(n+1)}}{2f(n)},$$

which implies that for $n \geq 48$,

$$f(n)\left(\frac{P_n}{P_{n-1}}\right)^2-a(n+1)\frac{P_n}{P_{n-1}}-b(n+1)>0. \qquad (2.14)$$

Thanks to (1.1),

$$f(n)P_n^2-P_{n-1}P_{n+1}=P_{n-1}^2\left(f(n)\left(\frac{P_n}{P_{n-1}}\right)^2-a(n+1)\frac{P_n}{P_{n-1}}-b(n+1)\right). \qquad (2.15)$$

Lemma 2.2 follows from (2.14) and (2.15). This completes the proof. ∎

**Lemma 2.3** *For* $n \geq 48$,

$$\frac{P_{n+1}P_{n-1}}{P_n^2}>f(n+1), \qquad (2.16)$$

*where* $f(n)$ *is defined by* (2.7).

*Proof.* Let $a(n)$ and $b(n)$ be defined by (2.8) and (2.9), respectively. It is easy to check that for $n \geq 48$,

$$\begin{aligned} & a^2(n+1) + 4f(n+1)b(n+1) \\ =& \frac{64g(n)}{(n+1)^4(2n+3)^3(8n^3+4n^2-2n-9)} > 0, \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} & 2f(n+1)\left(16 - \frac{16}{n} - \frac{18}{n^3}\right) - a(n+1) \\ =& \frac{4h(n)}{n^3(n+1)^2(2n+3)^3(8n^3+4n^2-2n-9)} > 0, \end{aligned} \tag{2.18}$$

where $g(n)$ and $f(n)$ are polynomials in $n$. By (2.17) and (2.18),

$$\frac{a(n+1) - \sqrt{a^2(n+1) + 4f(n+1)b(n+1)}}{2f(n+1)} < 16 - \frac{16}{n} - \frac{18}{n^3}. \tag{2.19}$$

Furthermore, it is easy to check that for $n \geq 0$,

$$\begin{aligned} & 2f(n+1)\left(16 - \frac{16}{n} - \frac{16}{n^3}\right) - a(n+1) \\ =& \frac{8k(n)}{n^3(n+1)^2(2n+3)^3(8n^3+4n^2-2n-9)} > 0, \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} & \left(a^2(n+1) + 4f(n+1)b(n+1)\right) \\ & \quad - \left(2f(n+1)\left(16 - \frac{16}{n} - \frac{16}{n^3}\right) - a(n+1)\right)^2 \\ =& \frac{512(2n+1)^3(8n^3+28n^2+30n+1)s(n)}{n^6(2n+3)^6(n+1)^2(8n^3+4n^2-2n-9)} > 0, \end{aligned} \tag{2.21}$$

where $k(n)$ and $s(n)$ are polynomials in $n$. Combining (2.17), (2.20) and (2.21) yields

$$16 - \frac{16}{n} - \frac{16}{n^3} < \frac{a(n+1) + \sqrt{a^2(n+1) + 4f(n+1)b(n+1)}}{2f(n+1)}. \tag{2.22}$$

Sun and Wu [7] proved that for $n \geq 6$,

$$\frac{P_n}{P_{n-1}} < 16 - \frac{16}{n} - \frac{16}{n^3}. \tag{2.23}$$

It follows from (2.19), (2.22) and (2.23) that for $n \geq 48$,

$$\frac{a(n+1) - \sqrt{a^2(n+1) + 4f(n+1)b(n+1)}}{2f(n+1)} < \frac{P_n}{P_{n-1}} < \frac{a(n+1) + \sqrt{a^2(n+1) + 4f(n+1)b(n+1)}}{2f(n+1)},$$

which yields

$$f(n+1)\left(\frac{P_n}{P_{n-1}}\right)^2 - a(n+1)\frac{P_n}{P_{n-1}} - b(n+1) < 0. \tag{2.24}$$

In view of (1.1),

$$\begin{aligned} &f(n+1)P_n^2 - P_{n-1}P_{n+1} \\ =&P_n^2\left(f(n+1)\left(\frac{P_n}{P_{n-1}}\right)^2 - a(n+1)\frac{P_{n-1}}{P_{n-1}} - b(n+1)\right). \end{aligned} \tag{2.25}$$

Lemma 2.3 follows from (2.24) and (2.25). This completes the proof. ∎

Now, we turn to prove Conjecture 1.1.

*Proof of Conjecture 1.1.* Replacing $n$ by $n-1$ in (2.16), we deduce that for $n \geq 48$,

$$\frac{P_n P_{n-2}}{P_{n-1}^2} > f(n). \tag{2.26}$$

In view of (2.6) and (2.26), we deduce that for $n \geq 48$,

$$\frac{P_n P_{n-2}}{P_{n-1}^2} > \frac{P_{n+1}P_{n-1}}{P_n^2}. \tag{2.27}$$

It is a routine to verify that (2.27) also holds for $2 \leq n \leq 47$. This completes the proof of Conjecture 1.1. ∎

# 3 Summary

By establishing the upper and lower bounds for $\frac{P_n}{P_{n-1}}$ and constructing a middle function $f(n)$, in this paper, we provide a proof of the log-concavity of the quotient sequence $\{\frac{P_n}{P_{n-1}}\}_{n=0}^{\infty}$ for Catalan-Larcombe-French sequence $\{P_n\}_{n=0}^{\infty}$, which confirms a conjecture presented by Sun and Wu [7]. Sun and Wu [7] also conjectured that the Catalan-Larcombe-French sequence $P_n$ is $\infty$-log-convex. Unfortunately, our method can not be used to prove the $\infty$-log-convexity of $P_n$. Therefore, it is interesting to find a proof for the $\infty$-log-convexity of $P_n$.

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