Diagonal transformations in pentangulations on the sphere

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Abstract

A *pentangulation* is a simple plane graph such that each face is bounded by a cycle of length 5. We consider two *diagonal transformations* in pentangulations, called $\mathcal{A}$ and $\mathcal{B}$. In this paper, we shall prove that any two pentangulations with the same number of vertices can be transformed into each other by $\mathcal{A}$ and $\mathcal{B}$. In particular, if they are not isomorphic to a special pentangulation, then we do not need $\mathcal{B}$.

1 Introduction

An *n-angulation G* is a map of a 2-connected simple graph on the sphere such that each face of G is bounded by a cycle of length n, where $n \geq 3$ is an integer. In particular, for $n = 3, 4, 5, 6$, we call n-angulations *triangulations*, *quadrangulations*, *pentangulations* and *hexangulations*, respectively.

In triangulations on the sphere, flipping an edge as shown in Figure 1 is called a *diagonal flip*. When this transformation breaks the simpleness of graphs, we don't apply it. In the literature, Wagner [6] proved that any two triangulations on the sphere with the same number of vertices can be transformed into each other by diagonal flips. For related topics, see a survey [5].

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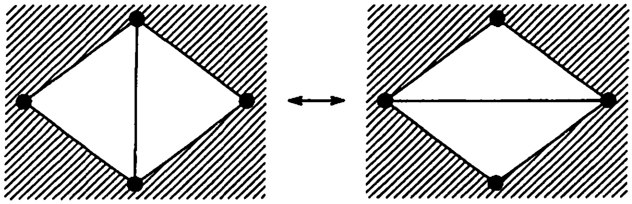

Figure 1: A diagonal flip

In quadrangulations on the sphere, sliding an edge and rotating a path of length 2 are two kinds of diagonal transformations, and they are called a *diagonal slide* and a *diagonal rotation*, respectively (see Figures 2 and 3). Note that any quadrangulation on the sphere is bipartite, and any diagonal slide preserves the bipartition of quadrangulations on the sphere but any diagonal rotation does not (the *bipartition* means the number of black and white vertices in the graph).

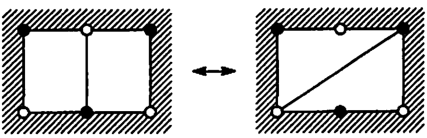

Figure 2: A diagonal slide

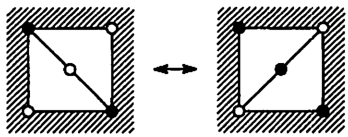

Figure 3: A diagonal rotation

For quadrangulations on the sphere, Nakamoto [3] proved that any two quadrangulations on the sphere with the same number of vertices can be transformed into each other by diagonal transformations. Moreover, Nakamoto [3] also proved that any two quadrangulations on the sphere with the same sizes of partite sets can be transformed into each other only by diagonal slides. Note that if two given quadrangulations on the sphere do not have the same sizes of partite sets, then we need diagonal rotations. For related topics, see [4].

In this paper, we consider diagonal transformations in pentangulations on the sphere. Hence, we first introduce several facts about pentangulations on the sphere.

Let G be a pentangulation on the sphere. The set of its vertices, edges and faces are denoted by $V(G), E(G)$ and $F(G)$, respectively. By Euler's formula and $2|E(G)| = 5|F(G)|$, we have $3|E(G)| = 5(|V(G)| - 2)$ by an easy calculation. Thus $|V(G)| = 3k + 2$ (for $k = 1, 2, \ldots$) since $|V(G)| \equiv 2 \pmod 3$. Note that the smallest pentangulation on the sphere is the 5-cycle.

Now, let us consider two types of *diagonal transformations* in pentangulations on the sphere, called $\mathcal{A}$ and $\mathcal{B}$ shown in Figure 4.

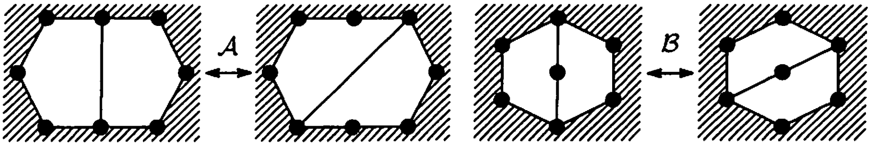

Figure 4: $\mathcal{A}$ and $\mathcal{B}$

A few years ago, the first and third authors (private communication) proved that any two pentangulations on the sphere with the same number of vertices can be transformed into each other by diagonal transformations.[1] On the other hand, the second author recently proved a similar statement for hexangulations [1], and he completely determined the role of three types of diagonal transformations specifically defined for hexangulations by constructing the transition diagram of hexangulations [2]. Therefore, we refocus on diagonal transformations in pentangulations on the sphere, and we establish the following theorem. A *standard form*, shown in Figure 6, is a pentangulation on the sphere with n vertices which consists of two vertices u and v such that $\deg(u) = \deg(v) = \frac{2(n-2)}{3}$ and alternate paths of length 2 and 3 connecting u and v, where the middle vertices have degree exactly 2.

Theorem 1 *Let G and G' be pentangulations on the sphere with the same number of vertices. Then G and G' can be transformed into each other using only $\mathcal{A}$ and $\mathcal{B}$. In particular, if neither G nor G' is isomorphic to the standard form, then we do not need $\mathcal{B}$.*

We note that neither $\mathcal{A}$ nor $\mathcal{B}$ can be omitted from the statement in Theorem 1 since only $\mathcal{A}$ can be applied to the dodecahedron and only $\mathcal{B}$ can be applied to the standard form by the definition of diagonal transformations (see Figures 5 and 6).

2 The structure $\mathcal{X}$ in pentangulations

In this section, we consider the special local structure $\mathcal{X}$ in pentangulations on the sphere, which is a subgraph included in a pentagonal non-facial region. Let $R = v_1v_2v_3v_4v_5$ be the pentagonal non-facial region, where $v_i \neq v_j$ if $i \neq j$ for every $i, j \in \{1,2,3,4,5\}$. The structure $\mathcal{X}$ consists of three vertices a, b and c inside R, three vertices v_1, v_3 and v_4 on the

[1]They showed that any pentangulation on the sphere can be transformed into the standard form (which is defined in this page) by diagonal transformations. However, they did not care about how two transformations $\mathcal{A}$ and $\mathcal{B}$ are used in the procedure of their proof.

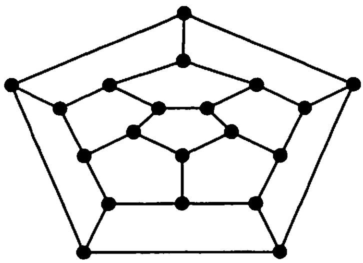

Figure 5: The dodecahedron

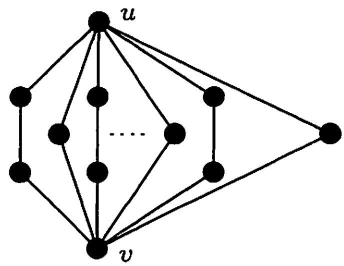

Figure 6: The standard form

boundary of R and edges av_1, ab, ac, bv_3, cv_4, where $\deg(a) = 3, \deg(b) = \deg(c) = 2$ and $\deg(v_k) \geq 3$ for each $k \in \{1,3,4\}$ (see Figure 7). Moreover, we sometimes denote $\mathcal{X}$ by $\mathcal{X}(a,b,c : v_1, v_3, v_4)$, specifying these six vertices in $\mathcal{X}$.

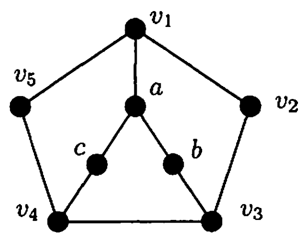

Figure 7: The structure $\mathcal{X}$

Now we consider the following two operations. *Adding* $\mathcal{X}$ means that we add three vertices a, b and c to the interior of a pentagonal face $v_1v_2v_3v_4v_5$ and add edges ab, ac, av_1, bv_3 and cv_4 as shown in Figure 8, and *Removing* $\mathcal{X}$ is the inverse operation of adding $\mathcal{X}$ as shown in Figure 8.

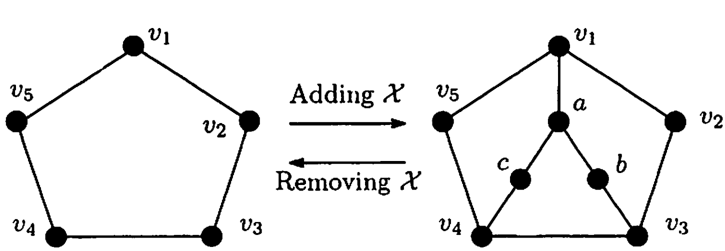

Figure 8: Adding $\mathcal{X}$ and removing $\mathcal{X}$

It is easy to see that adding (or removing) $\mathcal{X}$ preserves that the graph is a pentangulation on the sphere since all v_i's are distinct. Moreover, for adding $\mathcal{X}$, there exist five possibilities for the neighbor v_i of a in the interior of the 5-cycle shown in Figure 8, but Figure 9 shows that all positions can be regarded as the same up to $\mathcal{A}$. In this case, we note that the operation shown in Figure 9 does not break the simpleness and the 2-connectedness. Here, we shall prove the following lemma.

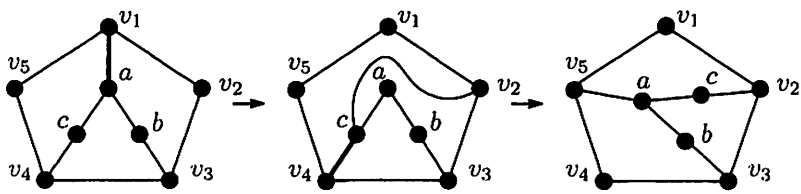

Figure 9: The rotation of $\mathcal{X}$

Lemma 2 *Let G be a pentangulation on the sphere. Any two pentangulations on the sphere obtained from G by adding $\mathcal{X}$ can be transformed into each other by a sequence of only $\mathcal{A}$.*

Proof. It suffices to prove that we can move $\mathcal{X}$ in a face of G into any other face of G only by $\mathcal{A}$. If G is the 5-cycle, then two pentangulations obtained from G by adding $\mathcal{X}$ are clearly isomorphic. Hence, we may suppose that G is not the 5-cycle, that is, G has at least three faces. Let $\Gamma = e_1e_2e_3e_4e_5$ be a pentagonal region which has three inner vertices of $\mathcal{X}$ in the interior, where e_i is an edge for each $i \in \{1,2,3,4,5\}$. Let f be a neighboring face of Γ and, without loss of generality, we may suppose that $e_1 \in f \cap \Gamma$ and $e_5 \notin f \cap \Gamma$. In this case, we can move $\mathcal{X}$ to f only by $\mathcal{A}$ as shown in Figure 10. Therefore, by repeating the operations shown in Figures 9 and 10, $\mathcal{X}$ can be moved to any other face of G only by $\mathcal{A}$. ■

3 Proof of Theorem 1

In this section, we shall prove Theorem 1. We first prove the following lemma. Let xy be an edge of a pentangulation on the sphere and we suppose that xy can be flipped by $\mathcal{A}$ to join two vertices a and b. In this case, we denote it by $xy \to ab$.

Lemma 3 *Suppose that xy is an edge of a pentangulation G on the sphere such that* $\deg(x) \geq 3$ *and* $\deg(y) \geq 3$. *Let $xyu_1u_2u_3$ and $xyv_1v_2v_3$ be the faces sharing the edge xy. If the operation $xy \to v_3u_1$ cannot be applied, then one of the following situations occurs:*

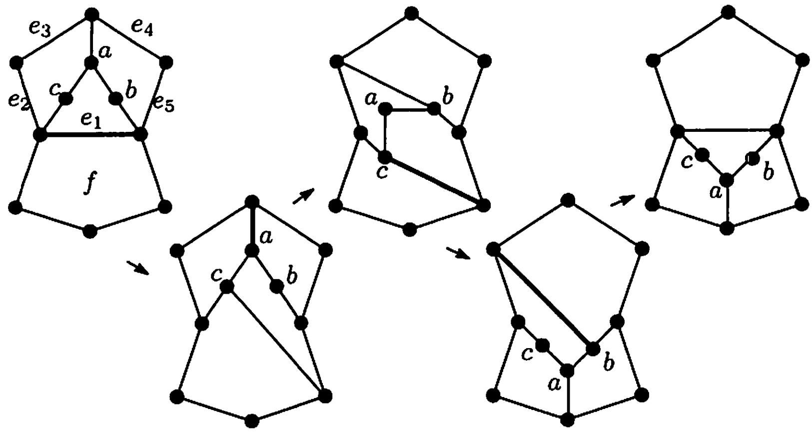

Figure 10: Move $\mathcal{X}$ to f

(1) $\{u_1, u_2, u_3\} \cap \{v_1, v_2, v_3\} = \emptyset$ *and* $u_1v_3 \in E(G)$.

(2) $u_1 = v_2$.

(3) $u_2 = v_3$.

(4) $u_1 = v_3$.

Proof. By the assumption, the operation $xy \to v_3u_1$ yields either a pair of multiple edges or a self-loop. If at least one of the situations (1), (2) and (3) occurs, then the operation yields multiple edges. If the situation (4) occurs, then the operation yields a self-loop. On the other hand, if none of those situations occurs, then it is easy to see that $xy \to v_3u_1$ can be applied preserving the simplicity, which is a contradiction. ■

By the above lemma, we can immediately obtain the following lemma since if G has one of the four situations in Lemma 3, G does not have the similar situation for $xy \to u_3v_1$ by the planarity.

Lemma 4 *Suppose that xy is an edge of a pentangulation on the sphere such that* $\deg(x) \geq 3$ *and* $\deg(y) \geq 3$. *Let $xyu_1u_2u_3$ and $xyv_1v_2v_3$ be the faces sharing the edge xy. Then, one of the operations $xy \to v_3u_1$ or $xy \to u_3v_1$ can be applied.* ■

Next, we show the following two lemmas. The second lemma (Lemma 6) is essential to prove Theorem 1.

Lemma 5 *Let G be a pentangulation on the sphere with $|V(G)| \geq 8$ which is not the standard form. Then we can obtain a face $f = u_1u_2u_3u_4u_5$ such that* $\deg(u_1) \geq 3, \deg(u_2) \geq 3$ *and* $\deg(u_3) = 2$ *by applying $\mathcal{A}$ to G at most once.*

Proof. First, it is clear that each pentagon bounding a face contains at least two vertices of degree at least 3; otherwise, either G is the 5-cycle or G has a cut vertex. Secondly, it is clear that if each pentagon bounding a face contains only two non-adjacent vertices of degree at least 3, then G is a standard form. Hence, there is a pentagon bounding a face which contains at least two adjacent vertices of degree at least 3, and if it does not have the required property, all five vertices have degree at least 3. In this case, since the average degree of G is less than 4, we can obtain a required face by applying $\mathcal{A}$ at most once to make a vertex of degree 2. ■

Lemma 6 *Let G be a pentangulation on the sphere which is not the standard form. Then G can be transformed into a pentangulation on the sphere with at least one $\mathcal{X}$ using only $\mathcal{A}$.*

Proof. If G already has $\mathcal{X}$, then we are done. Hence, we may suppose that G has no $\mathcal{X}$ and $|V(G)| \geq 8$ (otherwise, G is the standard form with five vertices). By Lemma 5, G has a face $f = xyzv_1v_2$ such that $\deg(x) \geq 3$, $\deg(y) \geq 3$ and $\deg(z) = 2$ by applying $\mathcal{A}$ once. Then we consider the following steps for surroundings of f.

Step 1. Make $\deg(y) = 3$

Suppose that $\deg(y) \geq 4$. Let $f' = xyua_1a_2$ be the face sharing xy with f, and let $f'' = yub_1b_2b_3$ be the face sharing yu with f' (see Figure 11). We consider to reduce $\deg(y)$ by $\mathcal{A}$.

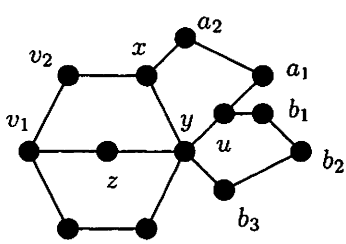

Figure 11: Case 1

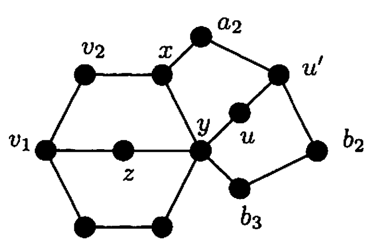

Figure 12: Case 2

Case 1. $\deg(u) \geq 3$ (Figure 11)

We can always reduce $\deg(y)$ by applying $\mathcal{A}$ to yu by Lemma 4.

Case 2. $\deg(u) = 2$ (Figure 12 : This configuration is obtained from Figure 11 by identifying a_1 and b_1.)

After applying $\mathcal{A}$ to xy to make $\deg(u) = 3$, we replace u as x. If this operation is applicable, we can reduce $\deg(y)$. (Note that this operation is also applicable if $a_2 = b_2$.) Otherwise, by Lemma 3, we have $u' = v_2$ since $u \notin \{v_1, v_2\}$. In this case, if $\deg(a_2) \geq 3$, then after applying $\mathcal{A}$ to v_2a_2 to make $\deg(u) = 3$, we reduce $\deg(y)$ similarly to Case 1.

Hence, we suppose that $\deg(a_2) = 2$. Note that we now have $\deg(x) \geq 4$, otherwise, the removal of u' would separate the inside of $u'a_2x$ and the remaining part, which contradicts that G is 2-connected. Let $r \neq v_2, a_2$ be the next vertex of v_2 on the clockwise rotation of x in Figure 13. After $v_2x \to v_1r$, we can reduce $\deg(y)$ by $xy \to ru$. These operations can be applied since the 3-cycle xa_2v_2 separates v_1, u and r. Following this, we replace r and u as v_2 and x, respectively.

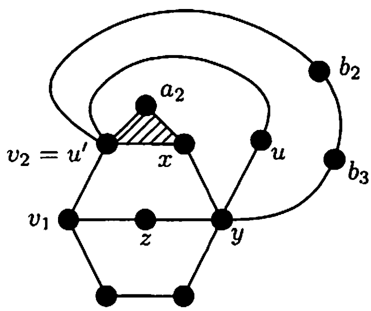

Figure 13: Case 2 ($u' = v_2$)

For the operations in each case, note that we preserve $\deg(x) \geq 3$ and $\deg(z) = 2$. Therefore, since we can reduce $\deg(y)$ preserving $\deg(x) \geq 3$ and $\deg(z) = 2$ in each case, we suppose that $\deg(z) = 2$ and $\deg(y) = 3$, and we consider the next step.

Step 2. Make $\deg(x) = 3$

Let $f' = x'yzv_1v_2'$ be the face sharing a path yzv_1 with f. Also, we let $f_1 = xyx'r_2r_1$ be the face sharing a path xyx' with $f \cup f'$, and let $f_2 = xr_1a_2a_1u$ be the face sharing an edge xr_1 with f_1. Now, if $u = v_2$, then we already have $\deg(x) = 3$. Hence, we may assume that $u \neq v_2$ and $\deg(x) \geq 4$. We consider the following two cases to reduce $\deg(x)$ only by $\mathcal{A}$.

Case 1. $\deg(r_1) \geq 3$

If we can apply $xr_1 \to ur_2$, then we are done. Hence, we suppose that the operation is not applicable, that is, (1) $u = r_2$, (2) $u = x'$, (3) $a_1 = r_2$ or (4) $\{u, a_1, a_2\} \cap \{x', r_2\} = \emptyset$ and an edge ur_2 occurs by Lemma 3. In the case (1) or (4), we can apply $xr_1 \to ya_2 \to x'a_1$ if $a_1 \neq x'$. (If $a_1 = x'$, then we can apply $xr_1 \to ur_2$.) In the case (2) or (3), after $xr_1 \to ya_2$, we apply $yx' \to a_2v_2'$ and replace a_2 with x'. Hence, we can always decrease $\deg(x)$ in this case.

Case 2. $\deg(r_1) = 2$ $(a_2 = r_2)$

After applying $xy \to r_1z$ and $v_1z \to v_2y$, we replace x', r_2, y and r_1 as v_2', x', v_1 and y, respectively. If this operation is applicable, then we are done. Otherwise, we have $v_2 = x'$. In this case, after applying $yx' \to zr_2 \to v_1r_1$ and $xr_1 \to yr_2$, we replace r_1 and r_2 as v_2' and x', respectively. (Note that this operation is now applicable since the 3-cycle $x'yx$ separates v_1 and $a_2(= r_2)$ in the interior and the exterior.)

Therefore, we can reduce $\deg(x)$ only by $\mathcal{A}$, which preserves $\deg(z) = 2$ and $\deg(y) = 3$ in both cases. Hence, we suppose that $\deg(z) = 2$ and $\deg(x) = \deg(y) = 3$ (see Figure 14), and we consider the final step.

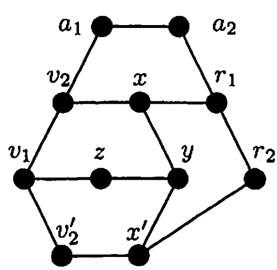

Figure 14: $\deg(z) = 2$ and $\deg(x) = \deg(y) = 3$

Step 3. Make $\mathcal{X}$

If $\deg(x') = 2$, $\deg(v_1) \geq 3$ and $\deg(v_2') \geq 3$, then it is easy to see that we already have $\mathcal{X}(y, x', z : x, v_2', v_1)$. Therefore, we may suppose that $\deg(x') \geq 3$ or $\deg(v_1) = 2$ or $\deg(v_2') = 2$, and hence, it suffices to consider the following cases.

Case 1. $\deg(x') \geq 3$ and $\deg(v_1) \geq 3$

If $\deg(v_2) = \deg(r_1) = 2$ ($a_1 = v_1$ and $a_2 = r_2$), then we already have $\mathcal{X}(x, v_2, r_1 : y, v_1, r_2)$ since r_2 has distinct neighbors v_1, x' and r_1. Hence, we may assume that $\deg(v_2) \geq 3$ or $\deg(r_1) \geq 3$. In this case, we can obtain $\mathcal{X}(y, z, x : x', v_1, r_1)$ (resp., $\mathcal{X}(y, z, x : x', v_1, v_2)$) if $v_2x \to v_1r_1$ (resp., $r_1x \to r_2v_2$) is applicable. If $v_2x \to v_1r_1$ (resp., $r_1x \to r_2v_2$) is not applicable, then $v_1 = r_1$, $v_1 = a_2$ or $v_1r_1 \in E(G)$ and $v_1 \neq \{r_1, a_2\}$ (resp., $r_2 = v_2$, $r_2 = a_1$, $v_2 = x'$ or $v_1r_1 \in E(G)$ and $r_2 \notin \{v_2, a_1\}$) by Lemma 3. However, in each of the three (resp., four) cases, we can apply the other operation by the planarity (for example, if $v_1 = r_1$, then the 4-cycle r_1xyz separates $\{r_2, x'\}$ and $\{v_2, a_1\}$ in the interior and the exterior).

Case 2. $\deg(x') \geq 3$ and $\deg(v_1) = 2$

We now have $v_2 = v_2'$ and $\deg(v_2') \geq 3$ (note that if $\deg(v_2') = 2$, then we have $x = x'$, which contradicts to the simplicity of G). Hence, we can obtain $\mathcal{X}(z, y, v_1 : r_1, x', v_2')$ by $xy \to r_1z$.

Case 3. $\deg(x') = 2$ and $\deg(v_1) = 2$

Now, since $v_2' = r_2 = v_2$ and x, v_1, x', a_1 and r_1 are clearly distinct, we have $\deg(v_2') \geq 5$. In this case, we can obtain $\mathcal{X}(z, y, v_1 : r_1, x', v_2')$ by flipping v_2r_1 to make $\deg(x') = 3$ and $xy \to r_1z$.

Case 4. $\deg(x') = 2, \deg(v_1) \geq 3$ and $\deg(v_2') = 2$ ($v_2' = r_2$ and $v_1 = r_1$)

We now have $\deg(v_1) \geq 4$ and $\deg(v_2) \geq 3$ since $\deg(x) = 3$. Hence we can obtain $\mathcal{X}(x', v_2', y : v_2, v_1, z)$ by flipping v_1v_2 to make $\deg(z) = 3$ and $xy \to v_2x'$.

Hence, since we can make $\mathcal{X}$ only by $\mathcal{A}$ in all cases of Step 3, the lemma holds. ∎

We have prepared to prove Theorem 1.

Proof of Theorem 1. Let G and G' be pentangulations on the sphere with the same number of vertices and let G and G' be not isomorphic to the standard form. Since any pentangulation on the sphere with five vertices is isomorphic to the 5-cycle, we may suppose that $|V(G)| \geq 8$. By induction on $|V(G)|$, we shall prove that G and G' can be transformed into each other only by $\mathcal{A}$.

By Lemma 6, G and G' can be transformed into pentangulations on the sphere with at least one $\mathcal{X}$. Let $\mathcal{X}$ and $\mathcal{X}'$ be two $\mathcal{X}$'s in G and G', respectively, and let H (resp., H') be a pentangulation on the sphere obtained from G (resp., G') by removing $\mathcal{X}$ (resp., $\mathcal{X}'$). If H and H' are not isomorphic to the standard form, then they can be transformed into each other only by $\mathcal{A}$ by the inductive hypothesis, but such a deformation might touch the face to which $\mathcal{X}$ (resp., $\mathcal{X}'$) is added to obtain G (resp., G'). However, we can move $\mathcal{X}$ (resp., $\mathcal{X}'$) to another face by Lemma 2, so that the deformation does not touch $\mathcal{X}$ (resp., $\mathcal{X}'$). Finally, by moving $\mathcal{X}$ (resp., $\mathcal{X}'$) back to the original position to recover G (resp., G'), G and G' can be transformed into each other only by $\mathcal{A}$.

Now, we may suppose that H is isomorphic to the standard form. In this case, Figure 15 suggests that $\mathcal{B}$ deforms the standard form H into another pentangulation H_1, which is not a standard form and that G can be transformed into H_1 with $\mathcal{X}$ added by a sequence of $\mathcal{A}$. Similarly, G' can be transformed into H_2 with $\mathcal{X}$ added, where H_2 is not a standard form. Therefore, G and G' can be transformed into each other by a sequence of $\mathcal{A}$ as well as in the previous case.

Finally, we describe the role of $\mathcal{B}$. If a pentangulation G on the sphere is isomorphic to a standard form with at least eight vertices, then we can transform G into another pentangulation which is not a standard form by applying $\mathcal{B}$ once. Therefore, by the above proof, for any two pentangulations on the sphere with the same number of vertices, they can be transformed into each other by $\mathcal{A}$ and $\mathcal{B}$, where the number of application of $\mathcal{B}$ is at most once. ■

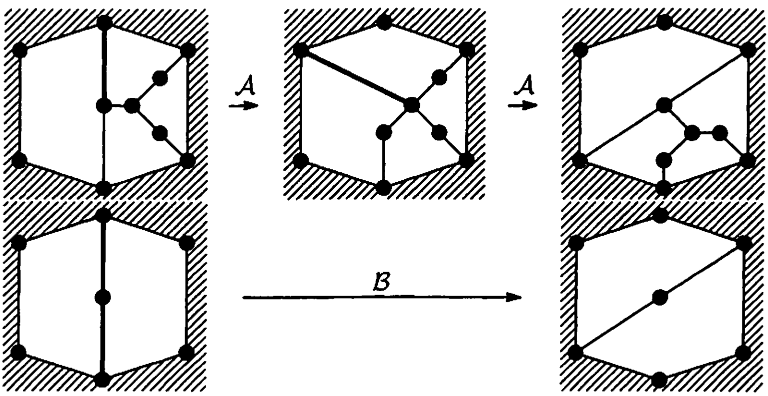

Figure 15: Applying $\mathcal{B}$ and $\mathcal{A}$ twice

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