Some q-Dixon-like summation formulas

Victor J. W. Guo[1] and Ya-Zhen Wang[2]

[1]School of Mathematical Sciences, Huaiyin Normal University, Huai'an, Jiangsu 223300, People's Republic of China

[2]Department of Mathematics, East China Normal University, Shanghai 200241, People's Republic of China

jwguo@hytc.edu.cn, wang0ya0zhen@126.com

Abstract. We give a q-analogue of some Dixon-like summation formulas obtained by Gould and Quaintance [Fibonacci Quart. 48 (2010), 56–61] and Chu [Integral Transforms Spec. Funct. 23 (2012), 251–261], respectively. For example, we prove that

$$\sum_{k=0}^{2m}(-1)^{m-k}q^{\binom{m-k}{2}}\begin{bmatrix}2m\\k\end{bmatrix}\begin{bmatrix}x+k\\2m+r\end{bmatrix}\begin{bmatrix}x+2m-k\\2m+r\end{bmatrix}$$
$$=\frac{q^{m(x-m-r)}\begin{bmatrix}2m\\m\end{bmatrix}}{\begin{bmatrix}2m+r\\m\end{bmatrix}}\begin{bmatrix}x\\m+r\end{bmatrix}\begin{bmatrix}x+m\\m+r\end{bmatrix},$$

where $\begin{bmatrix}x\\k\end{bmatrix}$ denotes the q-binomial coefficient.

1 Introduction

Gould and Quaintance [4] established the following identity:

$$\sum_{k=0}^{2m}(-1)^{m-k}\binom{2m}{k}\binom{x+k}{2m+r}\binom{x+2m-k}{2m+r}=\binom{2m}{m}\binom{x+m}{2m+r}\frac{\binom{x+m}{m+r}}{\binom{x+m}{m}}, \tag{1.1}$$

which is a generalization of Vosmansky's identity [6]. Recently, by employing the finite difference method, Chu [3] further established some alternating binomial coefficient identities, such as (see [3, Theorem 2]):

$$\sum_{k=0}^{2m}(-1)^{m-k}\binom{2m}{k}\binom{x+k}{2m+r}\binom{x+2m-k-1}{2m+r}$$
$$=\frac{\binom{2m}{m}}{\binom{2m+r}{m}}\binom{x-1}{m+r}\binom{x+m}{m+r}, \tag{1.2}$$

where we replaced $\binom{k-x+r}{2m+r}$ in [3] by its equivalent form $(-1)^r\binom{x+2m-k-1}{2m+r}$.

It is well known that binomial coefficient identities usually have nice q-analogues. For example, the Dixon identity can be generalized to the q-Dixon identity (see [5]). In this paper, we shall give q-analogues of (1.1) and almost all of the identities including (1.2) obtained by Chu [3].

Recall that the q-binomial coefficients $\begin{bmatrix} x \\ k \end{bmatrix}$ are defined by

$$\begin{bmatrix} x \\ k \end{bmatrix} = \begin{bmatrix} x \\ k \end{bmatrix}_q = \begin{cases} \prod_{i=1}^{k} \dfrac{1-q^{x-i+1}}{1-q^i}, & \text{if } k \geqslant 0, \\ 0, & \text{if } k < 0. \end{cases}$$

Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{Z}$ the set of integers. Three of our main results are as follows:

Theorem 1.1 *For $m \in \mathbb{N}$ and $r \in \mathbb{Z}$, there holds*

$$\sum_{k=0}^{2m} (-1)^{m-k} q^{\binom{m-k}{2}} \begin{bmatrix} 2m \\ k \end{bmatrix} \begin{bmatrix} x+k \\ 2m+r \end{bmatrix} \begin{bmatrix} x+2m-k \\ 2m+r \end{bmatrix}$$
$$= q^{m(x-m-r)} \begin{bmatrix} 2m \\ m \end{bmatrix} \begin{bmatrix} x+m \\ 2m+r \end{bmatrix} \frac{\begin{bmatrix} x+m \\ m+r \end{bmatrix}}{\begin{bmatrix} x+m \\ m \end{bmatrix}}. \tag{1.3}$$

Theorem 1.2 *For $m \in \mathbb{N}$ and $r \in \mathbb{Z}$, there holds*

$$\sum_{k=0}^{2m} (-1)^{m-k} q^{\binom{m-k}{2}} \begin{bmatrix} 2m \\ k \end{bmatrix} \begin{bmatrix} x+k \\ 2m+r \end{bmatrix} \begin{bmatrix} x+2m-k-1 \\ 2m+r \end{bmatrix}$$
$$= \frac{q^{m(x-m-r)} \begin{bmatrix} 2m \\ m \end{bmatrix}}{\begin{bmatrix} 2m+r \\ m \end{bmatrix}} \begin{bmatrix} x-1 \\ m+r \end{bmatrix} \begin{bmatrix} x+m \\ m+r \end{bmatrix}. \tag{1.4}$$

Theorem 1.3 *For $m \in \mathbb{N}$ and $r \in \mathbb{Z}$, there holds*

$$\sum_{k=0}^{2m} (-1)^{m-k} q^{\binom{m-k}{2}} \begin{bmatrix} 2m \\ k \end{bmatrix} \begin{bmatrix} x+k \\ 2m+1 \end{bmatrix} \begin{bmatrix} x+2m-k-3 \\ 2m+1 \end{bmatrix}$$
$$= q^{mx-m^2-m} \begin{bmatrix} x-3 \\ m \end{bmatrix} \begin{bmatrix} x+m-1 \\ m \end{bmatrix}$$
$$\times \frac{(1+q^{x-m-1}+q^{x-m-2}+q^{2x-3}-q^{x+m}-q^{x-1}-q^{x-2}-q^{x-3m-3})}{(1-q^{m+1})(1-q^{2m+1})}. \tag{1.5}$$

It is easy to see that (1.3) and (1.4) are q-analogues of (1.1) and (1.2) respectively, and Theorem 1.3 is a q-analogue of [3, Theorem 4].

We should concede that it is sometimes quite a routine matter to write down q-analogues of binomial coefficient identities. However, this is not always the case for the Dixon-like identities in Chu [3]. For example, the identity (1.5) is a little different from classical q-binomial coefficient identities, for the right-hand side of (1.5) has a strange big factor. Moreover, it is rather difficult to find q-analogues of the following identities in [3]:

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}\binom{k+x}{2m-1}\binom{k-x+1}{2m-1}$$
$$=\frac{m^2(2m-x)(2m+x-1)}{12\binom{x+1}{4}}\binom{x-1}{m}\binom{-x}{m},$$

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}\binom{1-x}{k-1}\binom{-x-2}{2m-k-1}$$
$$=\frac{m^2(x-2m)(x+2m-1)}{12\binom{x+1}{4}}\binom{x-1}{m}\binom{-x}{m}.$$

The paper is organized as follows. In the next section, we shall give a detailed proof of Theorem 1.1 by applying the fundamental theorem of algebra and the q-binomial theorem. In Section 3, we shall give proofs of the other five theorems including Theorems 1.2 and 1.3 in a similar way. In Section 4, another kind of alternating q-binomial coefficient identities will be proved. In the last section, we shall point out that eight couples of the identities in [3] are in fact equivalent to each other.

2 Proof of Theorem 1.1

Note that, by the q-binomial theorem

$$\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}n\\k\end{bmatrix}z^k=\prod_{i=0}^{n-1}(1-zq^i)$$

(see, for example, [1, p. 36]), we have

$$\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix}n\\k\end{bmatrix}q^{\binom{n-k}{2}+ik}=0 \quad \text{for} \quad 0\leqslant i\leqslant n-1. \tag{2.1}$$

Define the polynomial by the q-binomial sum

$$F_q(x)=\sum_{k=0}^{2m}(-1)^{m-k}q^{\binom{m-k}{2}}\begin{bmatrix}2m\\k\end{bmatrix}\begin{bmatrix}x+k\\2m+r\end{bmatrix}\begin{bmatrix}x+2m-k\\2m+r\end{bmatrix}$$
$$=\sum_{k=0}^{2m}(-1)^{m-k}q^{\binom{2m-k}{2}-\frac{3m^2-m}{2}+mk}\begin{bmatrix}2m\\k\end{bmatrix}\begin{bmatrix}x+k\\2m+r\end{bmatrix}\begin{bmatrix}x+2m-k\\2m+r\end{bmatrix}.$$

It is easy to see that the coefficient of q^{ax} in $\begin{bmatrix} x+k \\ 2m+r \end{bmatrix}\begin{bmatrix} x+2m-k \\ 2m+r \end{bmatrix}$ is a Laurent polynomial in q^k consisting of terms of degree between $-a$ and a if $a \leqslant 2m+r$, and between $a-4m-2r$ and $4m+2r-a$ if $a > 2m+r$. This means that the coefficient of q^{ax} in $q^{mk}\begin{bmatrix} x+k \\ 2m+r \end{bmatrix}\begin{bmatrix} x+2m-k \\ 2m+r \end{bmatrix}$ is a polynomial in q^k of degree less than $2m$ if $a < m$ or $a > 3m+2r$. By (2.1), one sees that $F_q(x)$ is a polynomial in q^x consisting of terms of degree between m and $3m+2r$.

When $r \leqslant 0$, we have $F_q(x) = 0$ for $x = -m, \ldots, r-1, 0, \ldots, m+r-1$ by the following facts:

- If $-m \leqslant x \leqslant r-1$, then $\begin{bmatrix} x+k \\ 2m+r \end{bmatrix} = 0$ for $m \leqslant k \leqslant 2m$ and $\begin{bmatrix} x+2m-k \\ 2m+r \end{bmatrix} = 0$ for $0 \leqslant k \leqslant m-1$.
- If $0 \leqslant x \leqslant m+r-1$, then $\begin{bmatrix} x+k \\ 2m+r \end{bmatrix} = 0$ for $0 \leqslant k \leqslant m-1$ and $\begin{bmatrix} x+2m-k \\ 2m+r \end{bmatrix} = 0$ for $m \leqslant k \leqslant 2m$.

This implies that $F_q(x)$ have the same $2m+2r$ zeros as

$$G_q(x) := q^{m(x-m-r)}\begin{bmatrix} 2m \\ m \end{bmatrix}\begin{bmatrix} x+m \\ 2m+r \end{bmatrix}\frac{\begin{bmatrix} x+m \\ m+r \end{bmatrix}}{\begin{bmatrix} x+m \\ m \end{bmatrix}}.$$

When $r > 0$, we have the following facts for $F_q(x)$:

- If $-m \leqslant x \leqslant -1$, then $\begin{bmatrix} x+k \\ 2m+r \end{bmatrix} = 0$ for $m \leqslant k \leqslant 2m$ and $\begin{bmatrix} x+2m-k \\ 2m+r \end{bmatrix} = 0$ for $0 \leqslant k \leqslant m-1$.
- If $0 \leqslant x \leqslant r-1$, then $\begin{bmatrix} x+k \\ 2m+r \end{bmatrix} = \begin{bmatrix} x+2m-k \\ 2m+r \end{bmatrix} = 0$ for $0 \leqslant k \leqslant 2m$. That is to say, as a polynomial in q^x, $\begin{bmatrix} x+k \\ 2m+r \end{bmatrix}\begin{bmatrix} x+2m-k \\ 2m+r \end{bmatrix}$ is divisible by $(1-q^x)^2(1-q^{x-1})^2\cdots(1-q^{x-r+1})^2$, which implies that $0, 1, \ldots, r-1$ are double roots of $F_q(x)$.
- If $r \leqslant x \leqslant m+r-1$, then $\begin{bmatrix} x+k \\ 2m+r \end{bmatrix} = 0$ for $0 \leqslant k \leqslant m-1$ and $\begin{bmatrix} x+2m-k \\ 2m+r \end{bmatrix} = 0$ for $m \leqslant k \leqslant 2m$.

This again implies that $F_q(x)$ have the same $2m+2r$ zeros as $G_q(x)$.

Moreover, $F_q(m+r)$ has only one nonzero term $\begin{bmatrix} 2m \\ m \end{bmatrix}$, which is equal to $G_q(m+r)$. Since both $q^{-mx}F_q(x)$ and $q^{-mx}G_q(x)$ are polynomials in q^x of degree at most $2m+2r$, they must be identical. This completes the proof.

3 Proof of four Theorems

Proof of Theorem 1.2. Let

$$A_q(x) = \sum_{k=0}^{2m}(-1)^{m-k}q^{\binom{m-k}{2}}\begin{bmatrix} 2m \\ k \end{bmatrix}\begin{bmatrix} x+k \\ 2m+r \end{bmatrix}\begin{bmatrix} x+2m-k-1 \\ 2m+r \end{bmatrix}.$$

When $r \leqslant 0$, we can prove this theorem by verifying the following statements:

- $A_q(x)$ is a polynomial in q^x consisting of terms of degree between m and $3m+2r$.
- All the zeros of $q^{-mx}A_q(x)$ are $\{i, r-i\colon 1 \leqslant i \leqslant m+r\}$.
- For $x = m+r+1$, both sides of (1.4) are equal to $q^m \left[{2m \atop m}\right] \frac{1-q^{2m+r+1}}{1-q^{m+1}}$.

When $r > 0$, let $\widehat{A}_q(x) = A_q(x)\left[{x-1 \atop r-1}\right]^{-2}$. Then we can confirm this theorem by checking the following statements:

- $\widehat{A}_q(x)$ is a polynomial q^x consisting of terms of degree between m and $3m+2$.
- All the zeros of $q^{-mx}A_q(x)$ are $\{i, r-i : r \leqslant i \leqslant m+r\}$.
- For $x = m+r+1$, the two sides of (1.4) are equal. □

Proof of Theorem 1.3. Let

$$B_q(x) = \sum_{k=0}^{2m}(-1)^{m-k}q^{\binom{m-k}{2}}\begin{bmatrix}2m\\ k\end{bmatrix}\begin{bmatrix}x+k\\ 2m+1\end{bmatrix}\begin{bmatrix}x+2m-k-3\\ 2m+1\end{bmatrix}.$$

Then we can prove this theorem by verifying the following statements:

- $B_q(x)$ is a polynomial in q^x consisting of terms of degree between m and $3m+2$.
- All the zeros of $q^{-mx}B_q(x)$ are $\{i, 3-i : 3 \leqslant i \leqslant m+2\}$.
- The identity (1.5) holds for $x = 1, 2, m+3$ by noticing that the left-hand side of (1.5) has only one or two non-zero terms for such x's. When $x = 1$, since $\left[{1+k \atop 2m+1}\right] = 0$ for $0 \leqslant k \leqslant 2m-1$, we have

$$\begin{aligned} B_q(1) &= (-1)^m q^{\binom{-m}{2}}\begin{bmatrix}-2\\ 2m+1\end{bmatrix} \\ &= (-1)^{m-1}q^{\frac{-3m^2-9m-4}{2}}\frac{(1-q^{2m+2})}{(1-q)}, \end{aligned}$$

  which is equal to the right-hand side of (1.5) with $x = 1$. Similarly, we have

$$B_q(2) = (-1)^m q^{\binom{-m}{2}}\begin{bmatrix}2m+2\\ 2m+1\end{bmatrix}\begin{bmatrix}-1\\ 2m+1\end{bmatrix},$$

$$B_q(m+3) = q\begin{bmatrix}2m\\ m-2\end{bmatrix}\begin{bmatrix}2m+2\\ 2m+1\end{bmatrix} - \begin{bmatrix}2m\\ m-1\end{bmatrix}\begin{bmatrix}2m+2\\ 2m+1\end{bmatrix}.$$

The following theorem is a q-analogue of [3, Theorem 7].

Theorem 3.1 *For $m \in \mathbb{N}$ and $r \in \mathbb{Z}$, there holds*

$$\sum_{k=0}^{2m+1} (-1)^{m-k} q^{\binom{m-k+1}{2}} \begin{bmatrix} 2m+1 \\ k \end{bmatrix} \begin{bmatrix} x+k \\ 2m+r \end{bmatrix} \begin{bmatrix} x+2m-k-1 \\ 2m+r \end{bmatrix}$$
$$= q^{(m+1)(x-m-r-1)} \frac{\begin{bmatrix} 2m+2 \\ m+1 \end{bmatrix}}{\begin{bmatrix} 2m+r \\ m+1 \end{bmatrix}} \begin{bmatrix} x-2 \\ m+r-1 \end{bmatrix} \begin{bmatrix} x+m \\ m+r-1 \end{bmatrix}. \qquad (3.1)$$

Proof. Let

$$C_q(x) = \sum_{k=0}^{2m+1} (-1)^{m-k} q^{\binom{m-k+1}{2}} \begin{bmatrix} 2m+1 \\ k \end{bmatrix} \begin{bmatrix} x+k \\ 2m+r \end{bmatrix} \begin{bmatrix} x+2m-k-1 \\ 2m+r \end{bmatrix}.$$

When $r \leqslant 0$, we can prove this theorem by checking the following statements:

- $C_q(x)$ is a polynomial in q^x consisting of terms of degree between $m+1$ and $3m+2r-1$.
- All the zeros of $q^{-(m+1)x}C_q(x)$ are $\{i, r-i\colon 2 \leqslant i \leqslant m+r\}$.
- For $x = m+r+1$, both sides of (3.1) are equal to $\begin{bmatrix} 2m+2 \\ m+1 \end{bmatrix} \frac{1-q^{2m+r+1}}{1-q^{m+2}}$.

When $r > 0$, we can prove this theorem by defining

$$\widehat{C}_q(x) = C_q(x) \begin{bmatrix} x-2 \\ r-1 \end{bmatrix}^{-1} \begin{bmatrix} x \\ r-1 \end{bmatrix}^{-1}$$

and then verifying the following statements:

- $\widehat{C}_q(x)$ is a polynomial in q^x consisting of terms of degree between $m+1$ and $3m+1$.
- All the zeros of $q^{-(m+1)x}\widehat{C}_q(x)$ are $\{i, r-i\colon r+1 \leqslant i \leqslant m+r\}$.
- For $x = m+r+1$, both sides of (3.1) are equal. □

The following theorem is a q-analogue of [3, Theorem 8].

Theorem 3.2 *For $m \in \mathbb{N}_0$ and $r \in \mathbb{Z}$, there holds*

$$\sum_{k=0}^{2m+1} (-1)^{m-k} q^{\binom{m-k+1}{2}} \begin{bmatrix} 2m+1 \\ k \end{bmatrix} \begin{bmatrix} x+k \\ 2m+r+1 \end{bmatrix} \begin{bmatrix} x+2m-k \\ 2m+r+1 \end{bmatrix}$$
$$= q^{(m+1)(x-m-r-1)} \frac{\begin{bmatrix} 2m+1 \\ m+1 \end{bmatrix}}{\begin{bmatrix} 2m+r+1 \\ m+1 \end{bmatrix}} \begin{bmatrix} x-1 \\ m+r \end{bmatrix} \begin{bmatrix} x+m \\ m+r \end{bmatrix}. \qquad (3.2)$$

Proof. Let

$$D_q(x) = \sum_{k=0}^{2m+1} (-1)^{m-k} q^{\binom{m-k+1}{2}} \begin{bmatrix} 2m+1 \\ k \end{bmatrix} \begin{bmatrix} x+k \\ 2m+r+1 \end{bmatrix} \begin{bmatrix} x+2m-k \\ 2m+r+1 \end{bmatrix}.$$

When $r \leqslant 0$, we can prove this theorem by showing that

- $D_q(x)$ is a polynomial in q^x consisting of terms of degree between $m+1$ and $3m+2r+1$.
- All the zeros of $q^{-(m+1)x} D_q(x)$ are $\{i, r-i \colon 1 \leqslant i \leqslant m+r\}$.
- For $x = m+r+1$, both sides of (3.2) are equal to $\begin{bmatrix} 2m+1 \\ m+1 \end{bmatrix}$.

When $r > 0$, we can prove this theorem by defining

$$\widehat{D}_q(x) = C_q(x) \begin{bmatrix} x-1 \\ r \end{bmatrix}^{-1} \begin{bmatrix} x \\ r \end{bmatrix}^{-1}$$

and then verifying the following statements:

- $\widehat{D}_q(x)$ is a polynomial in q^x consisting of terms of degree between $m+1$ and $3m+1$.
- All the zeros of $q^{-(m+1)x} \widehat{D}_q(x)$ are $\{i, r-i \colon r+1 \leqslant i \leqslant m+r\}$.
- For $x = m+r+1$, both sides of (3.2) are equal. □

We end this section with the following q-analogue of [3, Theorem 9].

Theorem 3.3 *For $m \in \mathbb{N}_0$ and $r \in \mathbb{Z}$, there holds*

$$\sum_{k=0}^{2m+1} (-1)^k q^{\binom{m-k+1}{2}} \begin{bmatrix} 2m+1 \\ k \end{bmatrix} \begin{bmatrix} x+k \\ 2m+r+2 \end{bmatrix} \begin{bmatrix} x+2m-k+1 \\ 2m+r+2 \end{bmatrix} = 0.$$

Proof. Let

$$E_q(x) = \sum_{k=0}^{2m+1} (-1)^k q^{\binom{m-k+1}{2}} \begin{bmatrix} 2m+1 \\ k \end{bmatrix} \begin{bmatrix} x+k \\ 2m+r+2 \end{bmatrix} \begin{bmatrix} x+2m-k+1 \\ 2m+r+2 \end{bmatrix}.$$

When $r < 0$, we can confirm this theorem by checking the following statements:

- $E_q(x)$ is a polynomial in q^x consisting of terms of degree between $m+1$ and $3m+2r+3$.

- $q^{-(m+1)x}E_q(x)$ has zeros $\{i, r-i\colon 0 \leqslant i \leqslant m+r+1\}$, whose cardinality is $2m+2r+4$, greater than $2m+2r+2$, which means that $E_q(x) \equiv 0$.

When $r \geqslant 0$, we can confirm this theorem by defining $\widehat{E}_q(x) = E_q(x)\left[{x \atop r+1}\right]^{-2}$ and then checking the following statements:

- $\widehat{E}_q(x)$ is a polynomial in q^x consisting of terms of degree between $m+1$ and $3m+1$.
- $q^{-(m+1)x}\widehat{E}_q(x)$ has zeros $\{i, r-i\colon r+1 \leqslant i \leqslant m+r+1\}$, whose cardinality is $2m+2$, greater than $2m$, which leads to $\widehat{E}_q(x) \equiv 0$. □

4 Another kind of q-series identities

Applying the Leibniz rule for the product of two functions, Chu [3, Corollary 13] establishes the following transformation on alternating binomial sums:

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{k+x}{n+\varepsilon}\binom{k-x+r}{n+\varepsilon}\\ =\sum_{k=0}^{n}(-1)^{k+\varepsilon}\binom{n}{k}\binom{r-x}{k+\varepsilon}\binom{\varepsilon-x-1}{n+\varepsilon-k},\end{aligned} \tag{4.1}$$

which enables him to deduce some other closed formulas including Dixon's identity.

Although we cannot find a q-analogue of (4.1), we may give a q-analogue of most of the binomial coefficient identities in [3, Section 4]. The following theorem is a q-analogue of [3, $V_{2m}(r, r|x)$]. Its proof is a little different from those in the previous section, but much similar to that in [5].

Theorem 4.1 *For $m \in \mathbb{N}$ and $r \in \mathbb{Z}$, there holds*

$$\begin{aligned}\sum_{k=0}^{2m}(-1)^{m-k}q^{\frac{3(m-k)^2+m-k}{2}}\begin{bmatrix}2m\\k\end{bmatrix}\begin{bmatrix}x\\k+r\end{bmatrix}\begin{bmatrix}x-1\\2m+r-k\end{bmatrix}\\ =\frac{\left[{2m \atop m}\right]}{\left[{2m+r \atop m}\right]}\begin{bmatrix}x+m\\m+r\end{bmatrix}\begin{bmatrix}x-1\\m+r\end{bmatrix}.\end{aligned} \tag{4.2}$$

Proof. Let

$$H_q(x) = \sum_{k=0}^{2m}(-1)^{m-k}q^{\frac{3(m-k)^2+m-k}{2}}\begin{bmatrix}2m\\k\end{bmatrix}\begin{bmatrix}x\\k+r\end{bmatrix}\begin{bmatrix}x-1\\2m+r-k\end{bmatrix}.$$

Then $H_q(x)$ is a polynomial q^x of degree less than or equal to $2m+2r$. We first consider the $r \leqslant 0$ case. It suffices to verify (4.2) for $2m+2r+1$ distinct values of x. For $x = 1, \ldots, m+r$, we have $H_q(x) = 0$ since $\begin{bmatrix} x \\ k+r \end{bmatrix} = 0$ or $\begin{bmatrix} x-1 \\ 2m+r-k \end{bmatrix} = 0$. For $x = m+r+1$, both sides of (4.2) are equal to $\begin{bmatrix} 2m \\ m \end{bmatrix} \frac{1-q^{2m+r+1}}{1-q^{m+1}}$. For $x = -p$ $(1-r \leqslant p \leqslant m)$, noticing that

$$\begin{bmatrix} -n \\ k \end{bmatrix} = (-1)^k q^{-nk-\binom{k}{2}} \begin{bmatrix} n+k-1 \\ k \end{bmatrix},$$

we have

$$\begin{aligned} H_q(-p) &= \sum_{k=0}^{2m} (-1)^{m-k} q^{\frac{3(m-k)^2+m-k}{2}} \begin{bmatrix} 2m \\ k \end{bmatrix} \begin{bmatrix} -p \\ k+r \end{bmatrix} \begin{bmatrix} -p-1 \\ 2m+r-k \end{bmatrix} \\ &= \sum_{k=0}^{2m} (-1)^{m-k} q^{\binom{2m-k}{2}+U} \begin{bmatrix} 2m \\ k \end{bmatrix} \begin{bmatrix} k+r+p-1 \\ p-1 \end{bmatrix} \begin{bmatrix} 2m+r+p-k \\ p \end{bmatrix} \\ &= \sum_{k=0}^{2m} (-1)^{m-k+p} q^{\binom{2m-k}{2}+V} \begin{bmatrix} 2m \\ k \end{bmatrix} \begin{bmatrix} k+r+p-1 \\ p-1 \end{bmatrix} \begin{bmatrix} k-2m-r-1 \\ p \end{bmatrix}, \end{aligned} \tag{4.3}$$

where

$$U = mk + (m-5m^2)/2 - 2p(m+r) - 2mr - r^2,$$
$$V = U + p(2m+r+p-k) - \binom{p}{2}.$$

Since $q^V \begin{bmatrix} k+r+p-1 \\ p-1 \end{bmatrix} \begin{bmatrix} k-2m-r-1 \\ p \end{bmatrix}$ is a polynomial in q^k of degree

$$m - p + 2p - 1 = m + p - 1 \leqslant 2m - 1,$$

and by (2.1) we have

$$\sum_{k=0}^{2m} (-1)^k q^{\binom{2m-k}{2}} \begin{bmatrix} 2m \\ k \end{bmatrix} q^{ik} = 0, \quad \text{for} \quad 0 \leqslant i \leqslant 2m-1.$$

Namely, the right-hand side of (4.3) vanishes for $x = -p$ $(1-r \leqslant p \leqslant m)$. This proves the $r \leqslant 0$ case.

For the $r > 0$ case, let $\widehat{H}_q(x) = H_q(x) \begin{bmatrix} x-1 \\ r \end{bmatrix}^{-1} \begin{bmatrix} x \\ r \end{bmatrix}^{-1}$. Then $\widehat{H}_q(x)$ is a polynomial in q^x of degree no more than $2m$. Similarly to the $r \leqslant 0$ case, we can show that all the zeros of $\widehat{H}_q(x)$ are $\{i, r-i \colon r+1 \leqslant i \leqslant m+r\}$, and for $x = m+r+1$, the identity (4.2) holds. □

We now give a q-analogue of [3, $V_{2m}(3,1|x)$].

Theorem 4.2 *For $m \in \mathbb{N}$, there holds*

$$\sum_{k=0}^{2m}(-1)^{m-k}q^{\frac{3(m-k)^2+m-3k}{2}}\begin{bmatrix}2m\\k\end{bmatrix}\begin{bmatrix}x\\k+1\end{bmatrix}\begin{bmatrix}x-3\\2m-k+1\end{bmatrix}$$
$$=\frac{(1-q^{x-2m-3})(1+q-q^{m+1}-q^x-q^{x-1}+q^{x-m-1})}{(1-q^{m+1})(1-q^{2m+1})}$$
$$\times\begin{bmatrix}x+m-1\\m\end{bmatrix}\begin{bmatrix}x-3\\m\end{bmatrix}. \qquad (4.4)$$

Sketch of Proof. Both sides of (4.4) are polynomials in q^x of degree less than or equal to $2m+2$, and have zeros $\{i, 3-i\colon 3 \leqslant i \leqslant m+2\} \cup \{2m+3\}$. For $x = 1$, both sides of (4.4) are equal to $(-1)^{m-1}q^{-(m^2+9m+4)/2}(1-q^{2m+2})/(1-q)$, and for $x = 2$, both sides of (4.4) are equal to $(-1)^{m-1}q^{-(m^2+7m)/2}(1+q^{m-1})(1-q^{m+1})/(1-q)$.

Similarly, applying another special case of the binomial theorem

$$\sum_{k=0}^{2m+1}(-1)^k q^{\binom{2m+1-k}{2}}\begin{bmatrix}2m+1\\k\end{bmatrix}q^{ik}=0, \quad \text{for} \quad 0 \leqslant i \leqslant 2m,$$

we can prove the following result, which is a q-analogue of [3, $V_{2m+1}(r, r|x)$].

Theorem 4.3 *For $m \in \mathbb{N}$ and $r \in \mathbb{Z}$, there holds*

$$\sum_{k=0}^{2m+1}(-1)^{m-k-1}q^{\frac{3(m-k)^2+m-5k}{2}}\begin{bmatrix}2m+1\\k\end{bmatrix}\begin{bmatrix}x\\k+r\end{bmatrix}\begin{bmatrix}x-1\\2m+r-k+1\end{bmatrix}$$
$$=\frac{q^{-2m-1}\begin{bmatrix}2m+1\\m\end{bmatrix}}{\begin{bmatrix}2m+r+1\\m+1\end{bmatrix}}\begin{bmatrix}x+m\\m+r\end{bmatrix}\begin{bmatrix}x-1\\m+r\end{bmatrix}. \qquad (4.5)$$

Sketch of Proof. When $r \leqslant 0$, both sides of (4.5) are polynomials in q^x of degree less than or equal to $2m+2r+1$ and have zeros $\{i, r-i\colon 1 \leqslant i \leqslant m+r\}$. For $x = m+r+1$, both sides of (4.5) are equal to $q^{-2m-1}\begin{bmatrix}2m+1\\m\end{bmatrix}$. For $x = m+r+2$, both sides of (4.5) are equal to $q^{-2m-1}\begin{bmatrix}2m+1\\m\end{bmatrix}\frac{(1-q^{m+r+1})(1-q^{2m+r+2})}{(1-q)(1-q^{m+2})}$.

When $r > 0$, divide both sides of (4.5) by $\begin{bmatrix}x-1\\r\end{bmatrix}\begin{bmatrix}x\\r\end{bmatrix}$ and then we can show that both sides are polynomials in q^x of degree no more than $2m$ and have zeros $\{i, r-i\colon r+1 \leqslant i \leqslant m+r\}$. For $x = m+r+1$, the identity holds. □

Replacing q by q^{-1} in (4.4) and noticing that $\begin{bmatrix}n\\k\end{bmatrix}_{q^{-1}} = q^{k(k-n)}\begin{bmatrix}n\\k\end{bmatrix}$, we have

Corollary 4.4 *For $m \in \mathbb{N}$ and $r \in \mathbb{Z}$, there holds*

$$\sum_{k=0}^{2m+1} (-1)^{m-k-1} q^{\frac{3(m-k)^2+m-3k}{2}} \begin{bmatrix} 2m+1 \\ k \end{bmatrix} \begin{bmatrix} x \\ k+r \end{bmatrix} \begin{bmatrix} x-1 \\ 2m+r-k+1 \end{bmatrix}$$
$$= \frac{q^{x-2m-r-1} \begin{bmatrix} 2m+1 \\ m \end{bmatrix}}{\begin{bmatrix} 2m+r+1 \\ m+1 \end{bmatrix}} \begin{bmatrix} x+m \\ m+r \end{bmatrix} \begin{bmatrix} x-1 \\ m+r \end{bmatrix}.$$

Similarly, we can prove the following q-analogue of [3, $V_{2m+1}(r, r-1|x)$].

Theorem 4.5 *For $m \in \mathbb{N}$ and $r \in \mathbb{Z}$, there holds*

$$\sum_{k=0}^{2m+1} (-1)^{m-k-1} q^{\frac{3(m-k)^2+m-5k}{2}} \begin{bmatrix} 2m+1 \\ k \end{bmatrix} \begin{bmatrix} x \\ k+r-1 \end{bmatrix} \begin{bmatrix} x-2 \\ 2m+r-k \end{bmatrix}$$
$$= \frac{(q^{x-3m-r-1} + q^{-2m-1}) \begin{bmatrix} 2m+1 \\ m \end{bmatrix}}{\begin{bmatrix} 2m+r \\ m+1 \end{bmatrix}} \begin{bmatrix} x+m \\ m+r-1 \end{bmatrix} \begin{bmatrix} x-2 \\ m+r-1 \end{bmatrix}. \tag{4.6}$$

Letting $r = 2$ in (4.6), we obtain the following result.

Corollary 4.6 *For $m \in \mathbb{N}$, there holds*

$$\sum_{k=0}^{2m+1} (-1)^{m-k-1} q^{\frac{(3m-3k+2)(m-k+1)}{2}} \begin{bmatrix} 2m+1 \\ k \end{bmatrix} \begin{bmatrix} x \\ k+1 \end{bmatrix} \begin{bmatrix} x-2 \\ 2m-k+2 \end{bmatrix}$$
$$= \frac{1+q^{x-m-2}}{1+q^{m+1}} \begin{bmatrix} x+m \\ m+1 \end{bmatrix} \begin{bmatrix} x-2 \\ m+1 \end{bmatrix}. \tag{4.7}$$

5 Concluding Remarks

Let

$$U_n(r, \varepsilon|x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{k+x}{n+\varepsilon} \binom{k-x+r}{n+\varepsilon},$$

$$V_n(r, \varepsilon|x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{r-x}{k+\varepsilon} \binom{\varepsilon-x-1}{n+\varepsilon-k}.$$

Chu [3] gave more theorems on $U_n(r, \varepsilon|x)$ and $V_n(r, \varepsilon|x)$ than those q-analogues we give in the previous sections. Here we want to point out that the following four couples of the identities on $U_n(r, \varepsilon|x)$ in [3] are equivalent to each other:

Theorems 2 and 3; Theorems 4 and 5; Theorems 7 and 11; Theorems 8 and 10.

For example, Theorems 4 and 5 in [3] can be respectively written as

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}\binom{k+x}{2m+1}\binom{k-x+3}{2m+1}$$
$$=\frac{(2m+x)(2m-x+3)}{(m+1)(2m+1)}\binom{x-3}{m}\binom{-x}{m}, \tag{5.1}$$

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}\binom{k+x}{2m+1}\binom{k-x-3}{2m+1}$$
$$=\frac{(2m-x)(2m+x+3)}{(m+1)(2m+1)}\binom{x}{m}\binom{-x-3}{m}. \tag{5.2}$$

Replacing k by $2m-k$ and x by $x+3$ in (5.1), we have

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}\binom{x+2m-k+3}{2m+1}\binom{2m-k-x}{2m+1}$$
$$=\frac{(2m-x)(2m+x+3)}{(m+1)(2m+1)}\binom{x}{m}\binom{-x-3}{m},$$

which is equivalent to (5.2) since $\binom{x+2m-k+3}{2m+1}=-\binom{k-x-3}{2m+1}$ and $\binom{2m-k-x}{2m+1}=-\binom{k+x}{2m+1}$.

Similarly, substituting $\lambda\to\lambda+2$, $y\to y+1$, and $k\to 2m-k$ in [3, Theorem 2], we obtain [3, Thoerem 3]; substituting $\lambda\to\lambda+4$, $y\to y+2$, and $k\to 2m+1-k$ in [3, Theorem 7], we get [3, Thoerem 11]; substituting $\lambda\to\lambda+2$, $y\to y+1$, and $k\to 2m-k$ in [3, Theorem 8], we are led to [3, Thoerem 10]. Finally, by (4.1), the corresponding four couples of the identities on $V_n(r,\varepsilon|x)$ are equivalent to one another.

References

[1] G.E. Andrews, The Theory of Partitions, Cambridge University Press, Cambridge, 1988.

[2] L. Carlitz, J. Kaucky, and J. Vosmansky, On two combinatorial identities, Math. Slovaca 32 (1982), 297–300.

[3] W. Chu, Finite differences and Dixon-like binomial sums, Integral Transforms Spec. Funct. 23 (2012), 251-261.

[4] H.W. Gould and J. Quaintance, Generalizations of Vosmansky's identity, Fibonacci Quart. 48 (2010), 56–61.

[5] V.J.W. Guo and J. Zeng, A short proof of the q-Dixon identity, Discrete Math. 296 (2005), 259–261.

[6] J. Vosmansky, Monotonic properties of zeros and extremants of the differential equation $y''+q(t)y=0$, Arch. Math. 6 (1970), 37–74.