On edge irregularity strength of products of certain families of graphs with path P_2

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Abstract

For a simple graph $G = (V, E)$, a vertex labeling $\phi : V \to \{1, 2, \ldots k\}$ is called k-labeling. The weight of an edge xy in G, denoted by $w_\phi(xy)$, is the sum of the labels of end vertices x and y, i.e $w_\phi(xy) = \phi(x) + \phi(y)$. A vertex k-labeling is defined to be an edge irregular k labeling of the graph G if for every two different edges e and f there is $w_\phi(e) \neq w_\phi(f)$. The minimum k for which the graph G has an edge irregular k-labeling is called the edge irregularity strength of G, denoted by $es(G)$.

In this paper, determine the exact value for certain families of graphs with path P_2.

Keywords : *irregular assignment, irregularity strength, edge irregularity strength .*

MR (2000) Subject Classification : 05C78.

1 Introduction

Labeled graphs are becoming an increasingly useful family of Mathematical Models for a wide range of applications. While the qualitative labelings of graph elements have inspired research in diverse fields of human enquiry such as conflict resolution in social psychology, electrical circuit theory and energy crisis, these labelings have led to quite intricate fields of application such as coding theory problems, including the design of good radar location codes, synch-set codes; missile guidance codes and convolution codes with optimal autocorrelation properties. Labeled graphs have also been applied in determining ambiguities in X-Ray crystallographic analysis, to design communication network addressing systems, in determining optimal circuit layouts, radio-Astronomy., etc. More detailed discussions about applications of graph labelings can be found in [5, 6].

Let $G = (V, E)$ be the connected, simple and undirected graph with vertex set V and edge set E. By a *labeling* we mean any mapping that carries a set of graph elements to a set of numbers (usually positive integers), called *labels*. If the domain is the vertex-set or the edge-set, the labelings are called respectively *vertex labelings* or *edge labelings*. If the domain is $V \cup E$ then we call the labeling *total labeling*. Thus, for an edge k-labeling $\delta : E(G) \to \{1, 2, \ldots, k\}$ the associated weight of a vertex $x \in V(G)$ is

$$w_\delta(x) = \sum \delta(xy),$$

where the sum is over all vertices y adjacent to x, and for a total k-labeling $\varphi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ the associated weight of edge $xy \in E(G)$ is

$$wt_\varphi(xy) = \varphi(x) + \varphi(xy) + \varphi(y).$$

Chartrand *et al.* in [11] introduced edge k-labeling δ of a graph G such that $w_\delta(x) \neq w_\delta(y)$ for all vertices $x, y \in V(G)$ with $x \neq y$. Such labelings were called *irregular assignments* and the *irregularity strength* $s(G)$ of a graph G is known as the minimum k for which G has an irregular assignment using labels at most k. This parameter has attracted much attention [7, 8, 10, 12].

Motivated by these papers, Bača *et al.* in [9] started to investigate two modifications of the irregularity strength of graphs, namely a *total edge irregularity strength*, denoted by $tes(G)$, and a *total vertex irregularity strength*, denoted by $tvs(G)$. Some results on total edge irregularity strength and total vertex irregularity strength can be found in [1, 2, 4, 15, 16, 18, 19, 20].

Combining both previous modifications of the irregularity strength, Marzuki, Salman and Miller [17] introduced a new irregular total k-labeling of a graph G called *totally irregular total* k-labeling, which is required to be at the same time vertex irregular total and also edge irregular total. They have given an upper bond and a lower bond of the totally irregular total k-labeling, denoted by $ts(G)$. The most complete recent survey of graph labelings is [13].

A vertex k labeling $\phi : V \to \{1, 2, \ldots, k\}$ is defined to be an *edge irregular k-labeling* of the graph G if for every two different edges e and f there is $w_\phi(e) \neq w_\phi(f)$, where the weight of an edge $e = xy \in E(G)$ is $w_\phi(xy) =$

$\phi(x) + \phi(y)$. The minimum k for which the graph G has an edge irregular k-labeling is called the *edge irregularity strength* of G, denoted by $es(G)$.

The following theorem proved in [3], establishes lower bound for the edge irregularity strength of a graph G.

Theorem 1. *[3] Let $G = (V, E)$ be a simple graph with maximum degree $\Delta = \Delta(G)$. Then*

$$es(G) \geq \max\left\{\left\lceil \frac{|E(G)|+1}{2} \right\rceil, \Delta(G)\right\}.$$

In this paper, determine the exact values of the edge irregularity strength of Cartesian product of star, cycle with path P_2, and strong product of path P_n with P_2.

2 The Cartesian Product

The Cartesian product $G \square H$ of graphs G and H is a graph such that the vertex set of $G \square H$ is the Cartesian product $V(G) \square V(H)$ and any two vertices (x, x') and (y, y') are adjacent in $G \square H$ if and only if either $x = y$ and x' is adjacent with y' in H, or $x' = y'$ and x is adjacent with y in G.

Let $K_{1,n}$ be a star with order $n+1$ and P_2 be a path of order 2. The Cartesian product $K_{1,n} \square P_2$ is a graph with the vertex set $V(K_{1,n} \square P_2) = \{x_i, y_i : 1 \leq i \leq n\} \cup \{x, y\}$ and the edge set $E(K_{1,n} \square P_2) = \{xx_i, yy_i, x_iy_i : 1 \leq i \leq n\} \cup \{xy\}$. The Cartesian product of $C_n \square P_2$ is a graph with the vertex set $V(C_n \square P_2) = \{x_i, y_i : 1 \leq i \leq n\}$ and the edge set $E(C_n \square P_2) = \{x_ix_{i+1}, y_iy_{i+1}, x_iy_i : 1 \leq i \leq n\}$. From the next theorems it follows that the lower bound in Theorem 1 is tight.

Theorem 2. $es(K_{1,n} \square P_2) = \lceil \frac{3n+2}{2} \rceil$ *for $n \geq 2$.*

Proof. $K_{1,n} \square P_2$ has $2n+2$ vertices and $3n+1$ edges. The minimum degree of $K_{1,n} \square P_2$ is $\delta(K_{1,n} \square P_2) = 2$ and the maximum degree is $\Delta(K_{1,n} \square P_2) = n+1$. According to Theorem 1 we have that $es(K_{1,n} \square P_2) \geq \max\{\lceil \frac{3n+2}{2} \rceil, n+1\} = \lceil \frac{3n+2}{2} \rceil$. To prove the equality, it suffices to prove the existence of an optimal edge irregular $\lceil \frac{3n+2}{2} \rceil$-labeling.

Let $\phi_1 : V(K_{1,n} \square P_2) \to \{1, 2, \ldots, \lceil \frac{3n+2}{2} \rceil\}$ be the vertex labeling such that

$$\phi_1(x) = 1, \phi_1(y) = \left\lceil \frac{3n+2}{2} \right\rceil, \phi_1(x_i) = i, \quad \text{for} \quad 1 \leq i \leq n.$$

$$\phi_1(y_i) = \begin{cases} n+1-\lceil \frac{i-1}{2} \rceil, & \text{for } 1 \le i \le n \text{ and } i \text{ odd} \\ \lceil \frac{3n+2}{2} \rceil + 1 - \lceil \frac{i}{2} \rceil, & \text{for } 1 \le i \le n \text{ and } i \text{ even.} \end{cases}$$

Since $w_{\phi_1}(xy) = \lceil \frac{3n+2}{2} \rceil + 1$, $w_{\phi_1}(xx_i) = i+1$, for $1 \le i \le n$,

$$w_{\phi_1}(x_i y_i) = \begin{cases} n+1+i-\lceil \frac{i-1}{2} \rceil, & \text{for } 1 \le i \le n \text{ and } i \text{ odd} \\ \lceil \frac{3n+2}{2} \rceil + 1 + i - \lceil \frac{i}{2} \rceil, & \text{for } 1 \le i \le n \text{ and } i \text{ even.} \end{cases}$$

$$w_{\phi_1}(yy_i) = \begin{cases} n+1+\lceil \frac{3n+2}{2} \rceil - \lceil \frac{i-1}{2} \rceil, & \text{for } 1 \le i \le n \text{ and } i \text{ odd} \\ 2\lceil \frac{3n+2}{2} \rceil + 1 - \lceil \frac{i}{2} \rceil, & \text{for } 1 \le i \le n \text{ and } i \text{ even.} \end{cases}$$

So the edge weights form an arithmetic sequence $2, 3, 4, \ldots, 2\lceil \frac{3n+2}{2} \rceil$. Now clearly the edge weights are distinct for all pairs of distinct edges. Thus, the vertex labeling ϕ_1 is an optimal edge irregular $\lceil \frac{3n+2}{2} \rceil$-labeling. This completes the proof. □

In the above Theorem is determined the exact value of the edge irregularity strength of $K_{1,n}\Box P_2$ for $n > 2, m = 2$. I have try to find an edge irregularity strength $K_{1,n}\Box P_m$ for $n, m > 3$ but so far without success. So I conclude the following Conjecture.

Conjecture 1. *For the Cartesian product $K_{1,n}\Box P_m$, $n, m \ge 3$,*

$$es(K_{1,n}\Box P_m) = \left\lceil \frac{2mn - n + m}{2} \right\rceil.$$

The following theorem gives the exact value of the edge irregularity strength for Cartesian product $C_n\Box P_2$.

Theorem 3. *$es(C_n\Box P_2) = \lceil \frac{3n+1}{2} \rceil$ for $n > 4$.*

Proof. $C_n\Box P_2$ has $2n$ vertices and $3n$ edges. The minimum degree of $C_n\Box P_2$ is $\delta(C_n\Box P_2) = 3$ and the maximum degree is $\Delta(C_n\Box P_2) = 3$. According to Theorem 1 we have that $es(C_n\Box P_2) \ge \max\{\lceil \frac{3n+1}{2} \rceil, 3\} = \lceil \frac{3n+1}{2} \rceil$. To prove the equality, it suffices to prove the existence of an optimal edge irregular $\lceil \frac{3n+1}{2} \rceil$-labeling. Let us distinguish four cases:

Case 1. $n \equiv 0 \pmod 4$

We construct the vertex labeling ϕ_2 in the following way

$$\phi_2(x_i) = \begin{cases} \frac{3i-1}{2}, & \text{if } 1 < i \le \frac{n}{2} \text{ and } i \text{ odd} \\ \frac{3(i+1)}{2}, & \text{if } \frac{n}{2}+1 < i \le n \text{ and } i \text{ odd} \\ \frac{3i}{2}, & \text{if } 1 < i < n \text{ and } i \text{ even.} \end{cases}$$

$$\phi_2(y_i) = \begin{cases} \frac{3i-1}{2}, & \text{if } 1 \le i \le n \text{ and } i \text{ odd} \\ \frac{3i}{2} - 1, & \text{if } 1 \le i \le \frac{n}{2} \text{ and } i \text{ even} \\ \frac{3i}{2} + 1, & \text{if } \frac{n}{2} + 1 \le i \le n \text{ and } i \text{ even.} \end{cases}$$

The weights of edges are as follows:

$$w_{\phi_2}(x_iy_i) = \begin{cases} 3i - 1, & \text{if } 1 \le i \le \frac{n}{2} \\ 3i + 1, & \text{if } \frac{n}{2} + 1 \le i \le n \end{cases}$$

$$w_{\phi_2}(x_ix_{i+1}) = \begin{cases} 3i + 1, & \text{if } 1 \le i \le \frac{n}{2} - 1 \\ 3(i + 1), & \text{if } \frac{n}{2} \le i \le n - 1 \\ \lceil \frac{3n+1}{2} \rceil, & \text{if } i = n \end{cases}$$

$$w_{\phi_2}(y_iy_{i+1}) = \begin{cases} 3i, & \text{if } 1 \le i \le \frac{n}{2} \\ 3i + 2, & \text{if } \frac{n}{2} + 1 \le i \le n - 1 \\ \lceil \frac{3n+1}{2} \rceil + 1, & \text{if } i = n \end{cases}$$

We can see that all vertex labels are at most $\lceil \frac{3n+1}{2} \rceil$ and the edge weights successively attain values $2, 3, \ldots, 3n + 1$. Thus the labeling ϕ_2 provides the upper bound on $es(C_n \square P_2)$.

Case 2. $n \equiv 1 \pmod 4$

For $n = 5, 9$ define a suitable an optimal edge irregular labeling is defined in Figure 1. For $n > 10$ construct the vertex labeling ϕ_3 in the following

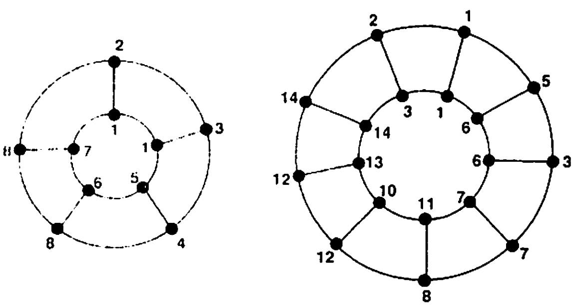

Figure 1: The edge irregular labelings for $C_5 \square P_2$ and $C_9 \square P_2$.

way:

$\phi_3(x_1) = 1, \phi_3(x_2) = 6, \phi_3(x_4) = 7, \phi_3(x_n) = 3, \phi_3(y_1) = 1, \phi_3(y_2) = 5, \phi_3(y_3) = 3, \phi_3(y_4) = 7.$

$$\phi_3(x_i) = \begin{cases} \frac{3(i+1)}{2}, & \text{if } 3 \le i \le n-1 \text{ and } i \text{ odd} \\ \frac{3i}{2}, & \text{if } 6 \le i < \frac{n-1}{2} \text{ and } i \text{ even} \\ \frac{3i}{2}+2, & \text{if } \frac{n-1}{2}+1 < i < n-1 \text{ and } i \text{ even} \end{cases}$$

$$\phi_3(y_i) = \begin{cases} \frac{3(i+1)}{2}-1, & \text{if } 5 < i < \frac{n-1}{2}-2 \text{ and } i \text{ odd} \\ \frac{3(i+1)}{2}+1, & \text{if } \frac{n-1}{2}-1 < i \le n-2 \text{ and } i \text{ odd} \\ \frac{3i}{2}+2, & \text{if } 6 < i \le n-1 \text{ and } i \text{ even} \end{cases}$$

The weights of edges are as follows:

$w_{\phi_3}(x_1y_1) = 2, w_{\phi_3}(x_2y_2) = 11, w_{\phi_3}(x_3y_3) = 9, w_{\phi_3}(x_ny_n) = 5, w_{\phi_3}(x_1x_2) = 7, w_{\phi_3}(x_2x_3) = 12, w_{\phi_3}(x_{n-1}x_n) = \lceil\frac{3n+1}{2}\rceil + 3, w_{\phi_3}(x_nx_1) = 4, w_{\phi_3}(y_4y_5) = 15, w_{\phi_3}(y_{n-1}y_n) = \lceil\frac{3n+1}{2}\rceil + 2, w_{\phi_3}(y_ny_1) = 3.$

$$w_{\phi_3}(x_iy_i) = \begin{cases} 3i+2, & \text{if } 4 \le i \le \frac{n-1}{2} \\ 3i+4, & \text{if } \frac{n-1}{2}+1 \le i \le n-1 \end{cases}$$

$$w_{\phi_3}(x_ix_{i+1}) = \begin{cases} 3(i+1)+1, & \text{if } i = 3,4 \\ 3(i+1), & \text{if } 5 \le i \le \frac{n-1}{2} \\ 3(i+1)+2, & \text{if } \frac{n-1}{2}+1 \le i \le n-2 \end{cases}$$

$$w_{\phi_3}(y_iy_{i+1}) = \begin{cases} 2(i+2), & \text{if } 1 < i < 3 \\ 3(i+1)+1, & \text{if } 5 < i < \frac{n-1}{2}-1 \\ 3(i+2), & \text{if } \frac{n-1}{2} \le i \le n-2 \end{cases}$$

We can see that all vertex labels are at most $\lceil\frac{3n+1}{2}\rceil$ and the edge weights successively attain values $2, 3, \ldots, 3n+1$. Thus the labeling ϕ_3 provides the upper bound on $es(C_n \square P_2)$.

Case 3. $n \equiv 2 \pmod 4$

We construct the vertex labeling ϕ_4 in the following way:

$\phi_4(x_1) = 1, \phi_4(x_n) = \lceil\frac{3n+1}{2}\rceil - 1, \phi_4(y_1) = 1, \phi_4(y_n) = \lceil\frac{3n+1}{2}\rceil,$

$$\phi_4(x_i) = \begin{cases} \frac{3i-2}{2}, & \text{if } 2 \le i \le \frac{n}{2} \text{ and } i \text{ even} \\ \frac{3i+1}{2}, & \text{if } 3 \le i < n \text{ and } i \text{ odd} \\ \frac{3i+2}{2}, & \text{if } \frac{n}{2}+1 \le i \le n-1 \text{ and } i \text{ even} \end{cases}$$

$$\phi_4(y_i) = \begin{cases} \frac{3i}{2}, & \text{if } 2 < i < n-1 \text{ and } i \text{ even} \\ \frac{3(i-1)}{2}, & \text{if } 3 < i < \frac{n}{2} \text{ and } i \text{ odd} \\ \frac{3i+1}{2}, & \text{if } \frac{n}{2}+1 \le i < n \text{ and } i \text{ odd} \end{cases}$$

The weights of edges are as follows:

$w_{\phi_4}(x_1x_2) = 3, w_{\phi_4}(x_{n-1}x_n) = 2\lceil\frac{3n+1}{2}\rceil - 3, w_{\phi_4}(x_nx_1) = \lceil\frac{3n+1}{2}\rceil, w_{\phi_4}(y_1y_2) = 4, w_{\phi_4}(y_{n-1}y_n) = 2\lceil\frac{3n+1}{2}\rceil - 2, w_{\phi_4}(y_ny_1) = \lceil\frac{3n+1}{2}\rceil$.

$$w_{\phi_4}(x_iy_i) = \begin{cases} 3i - 1, & \text{if } 1 \le i \le \frac{n}{2} \\ 3i + 1, & \text{if } \frac{n}{2} + 1 \le i \le n \end{cases}$$

$$w_{\phi_4}(x_ix_{i+1}) = \begin{cases} 3i + 1, & \text{if } 2 \le i \le \frac{n}{2} - 1 \\ 3(i + 1), & \text{if } \frac{n}{2} \le i \le n - 2 \end{cases}$$

$$w_{\phi_4}(y_iy_{i+1}) = \begin{cases} 3i, & \text{if } 2 \le i \le \frac{n}{2} \\ 3i + 2, & \text{if } \frac{n}{2} + 1 \le i \le n - 2 \end{cases}$$

We can see that all vertex labels are at most $\lceil\frac{3n+1}{2}\rceil$ and the edge weights successively attain values $2, 3, \ldots, 3n + 1$. Thus the labeling ϕ_4 provides the upper bound on $es(C_n \square P_2)$.

Case 4. $n \equiv 3 \pmod 4$

We construct the vertex labeling ϕ_5 in the following way:

$$\phi_5(x_i) = \begin{cases} \frac{3i+1}{2} + 1, & \text{if } 4 \le i \le n - 2 \text{ and } i \text{ even} \\ \frac{3(i-1)}{2} + 2, & \text{if } 4 \le i \le \frac{n-1}{2} \text{ and } i \text{ odd} \\ \frac{3i+5}{2}, & \text{if } \frac{n-1}{2} + 1 \le i \le n - 1 \text{ and } i \text{ odd} \end{cases}$$

$$\phi_5(y_i) = \begin{cases} \frac{3i+1}{2} + 1, & \text{if } 4 \le i \le \frac{n-1}{2} \text{ and } i \text{ even} \\ \frac{3i+1}{2} + 3, & \text{if } \frac{n-1}{2} + 1 \le i \le n - 2 \text{ and } i \text{ even} \\ \frac{3(i-1)}{2} + 3, & \text{if } 4 \le i \le \frac{n-1}{2} \text{ and } i \text{ odd} \\ \frac{3i+5}{2} - 1, & \text{if } \frac{n-1}{2} + 1 \le i \le n - 1 \text{ and } i \text{ odd} \end{cases}$$

The weights of edges are as follows:

$w_{\phi_5}(x_1y_1) = 2, w_{\phi_5}(x_2y_2) = 11, w_{\phi_5}(x_3y_3) = 9, w_{\phi_5}(x_ny_n) = 5, w_{\phi_5}(x_1x_2) = 7, w_{\phi_5}(x_2x_3) = 12, w_{\phi_5}(x_{n-1}x_n) = \lceil\frac{3n+1}{2}\rceil + 3, w_{\phi_5}(x_nx_1) = 4, w_{\phi_5}(y_4y_5) = 15, w_{\phi_5}(y_{n-1}y_n) = \lceil\frac{3n+1}{2}\rceil + 2, w_{\phi_5}(y_ny_1) = 3$,

$$w_{\phi_5}(x_iy_i) = \begin{cases} 3i + 2, & \text{if } 4 \le i \le \frac{n-1}{2} \\ 3i + 1, & \text{if } \frac{n-1}{2} + 1 \le i \le n - 1 \end{cases}$$

$$w_{\phi_5}(x_ix_{i+1}) = \begin{cases} 3(i + 1) + 1, & \text{if } i = 3, 4 \\ 3(i + 1), & \text{if } 5 \le i \le \frac{n-1}{2} \\ 3(i + 1) + 2, & \text{if } \frac{n-1}{2} + 1 \le i \le n - 2 \end{cases}$$

$$w_{\phi_5}(y_iy_{i+1}) = \begin{cases} 2(i+2), & \text{if } 1 \le i \le 3 \\ 3(i+1)+1, & \text{if } 5 \le i \le \frac{n-1}{2}-1 \\ 3(i+2), & \text{if } \frac{n-1}{2} \le i \le n-2 \end{cases}$$

We can see that all vertex labels are at most $\lceil \frac{3n+1}{2} \rceil$ and the edge weights successively attain values $2, 3, \ldots, 3n+1$. Thus the labeling ϕ_5 provides the upper bound on $es(C_n \square P_2)$. This completes the proof. $\square$

Although I have not found the result for edge irregularity strength of $C_n \square P_m$, $m \ge 3$, the lower bound in Theorem 1 is considered to be an exact value of the parameter es, the result from Theorem 3 leads me to suggest the following.

Conjecture 2. *Let $C_n \square P_m$ be the Cartesian product of cycle C_n and path P_m, $n, m \ge 3$. Then*

$$es(C_n \square P_m) = \left\lceil \frac{n(2m-1)+1}{2} \right\rceil.$$

3 The strong product

The strong product $G_1 \boxtimes G_2$ of graphs G_1 and G_2 has as vertices the pairs (x, y) where $x \in V(G_1)$ and $y \in V(G_2)$. Vertices (x_1, y_1) and (x_2, y_2) are adjacent if either x_1x_2 is an edge of G_1 and $y_1 = y_2$ or if $x_1 = x_2$ and y_1y_2 is an edge of G_2 or if x_1x_2 is an edge of G_1 and y_1y_2 is an edge of G_2. Note that the edge set of the strong product $G_1 \boxtimes G_2$ is the union of the edge sets of the Cartesian product $G_1 \square G_2$ and categorical product $G_1 \times G_2$, see e.g. [14].

For integers a and b let $[a, b]$ be an interval of integers c, $a \le c \le b$. If we consider graph G_1 as the path P_n with $V(P_n) = \{x_1, x_2, \ldots, x_n\}$, $E(P_n) = \{x_ix_{i+1} : i \in [1, n-1]\}$ and graph G_2 as the path P_m with $V(P_m) = \{y_1, y_2, \ldots, y_m\}$, $E(P_m) = \{y_jy_{j+1} : j \in [1, m-1]\}$ then

$V(P_n \boxtimes P_m) = \{(x_i, y_j) : i \in [1, n], \ j \in [1, m]\}$ is the vertex set and

$E(P_n \boxtimes P_m) = \{(x_i, y_j)(x_{i+1}, y_j) : i \in [1, n-1], \ j \in [1, m]\}$

$\cup \{(x_{i+1}, y_j)(x_i, y_{j+1}), (x_i, y_j)(x_{i+1}, y_{j+1}) : i \in [1, n-1], \ j \in [1, m-1]\}$

$\cup \{(x_i, y_j)(x_i, y_{j+1}) : i \in [1, n], \ j \in [1, m-1]\}$ is the edge set of $P_n \boxtimes P_m$.

In the next theorem, determine the exact value of the edge irregularity strength of $P_n \boxtimes P_2$.

Theorem 4. *Let $P_n \boxtimes P_2$ be a strong product of path P_n and P_2 with $n \geq 2$. Then*

$$es(P_n \boxtimes P_2) = \begin{cases} \lceil \frac{5n}{2} \rceil - 1, & \text{if } n \equiv 1 \ (mod\ 2) \\ \lceil \frac{5n}{2} \rceil. & \text{if } n \equiv 0 \ (mod\ 2). \end{cases}$$

Proof. The graph $P_n \boxtimes P_2$ has $2n$ vertices, $5n - 4$ edges. As every vertex of $P_n \boxtimes P_2$ is also a vertex of at least one complete graph K_4. Therefore no two adjacent vertices have the same label. This implies that the weights of the edges must be greater than 2. To prove the lower bound consider the weight of the edges. The smallest weight among all edges is at least 3, so the largest value among the weights of the edges is at least $5n - 2$. Since the weight of an edge is the sum of two positive integers, so at least one label is at least $\lceil \frac{5n-2}{2} \rceil = \lceil \frac{5n}{2} \rceil - 1$.

Define the vertex labeling ϕ_6 as follows:

$$\phi_6((x_i, y_j)) = \begin{cases} \frac{5(i-1)}{2} + j. & \text{if } 1 \leq i \leq n \text{ and } i \text{ odd} \ \& \ 1 \leq j \leq 2 \\ \frac{5i}{2} + 2(j - 2). & \text{if } 1 < i \leq n \text{ and } i \text{ even} \ \& \ 1 \leq j \leq 2 \end{cases}$$

For edge weights we have:

$$w_{\phi_6}((x_i, y_1)(x_{i+1}, y_2)) = \begin{cases} 5i + 1, & \text{if } 1 \leq i \leq n - 1 \text{ and } i \text{ odd} \\ 5i, & \text{if } 1 \leq i \leq n - 1 \text{ and } i \text{ even} \end{cases}$$

$$w_{\phi_6}((x_i, y_2)(x_{i+1}, y_1)) = \begin{cases} 5i, & \text{if } 1 \leq i \leq n - 1 \text{ and } i \text{ odd} \\ 5i + 1, & \text{if } 1 \leq i \leq n - 1 \text{ and } i \text{ even} \end{cases}$$

$$w_{\phi_6}((x_i, y_j)(x_{i+1}, y_j)) = 5i - 1 + 3(j - 1) \text{ for } 1 \leq i \leq n - 1, 1 \leq j \leq 2,$$

$$w_{\phi_6}((x_i, y_1)(x_i, y_2)) = 5i - 2 \text{ for } 1 \leq i \leq n.$$

For $n \equiv 1$ $(mod\ 2)$ we can see that all vertex labels are at most $\lceil \frac{5n}{2} \rceil - 1$ and the edge weights successively attain values $3, 4, \ldots, 5n-2$, i.e. the edge weights are distinct for all pairs of distinct edges. For $n \equiv 0$ $(mod\ 2)$, it is easy to see that the that the vertex label $\phi_6((x_n, y_2)) = \lceil \frac{5n}{2} \rceil$, if we replace by the vertex label $\lceil \frac{5n}{2} \rceil - 1$ then weights of the edges $(x_{n-1}, y_1)(x_n, y_2)$ and $(x_{n-1}, y_2)(x_n, y_1)$ are the same. Therefore the labeling ϕ_6 is an optimal labeling and gives the upper bound on the edge irregularity strength with the maximum vertex label $\lceil \frac{5n}{2} \rceil$, for $n \equiv 0$ $(mod\ 2)$. This completes the proof. □

Theorem 4 is determined the exact value of the edge irregularity strength of $P_n \boxtimes P_m$ for $n \geq 2, m = 2$. I have try to find an edge irregularity strength

$P_n \boxtimes P_m$ for $n, m > 3$ but so far without success. So I conclude the following open problem.

Open Problem 1. *For the strong product $P_n \boxtimes P_m$, $n, m \geq 3$, determine the exact value of edge irregularity strength.*

4 Conclusion

This paper introduced a new graph characteristic, the edge irregularity strength, as a modification of the well-known irregularity strength, total edge irregularity strength and total vertex irregularity strength. In this paper determined the precise values for Cartesian product of star $K_{1,n}$, cycle C_n with P_2 and strong product of path P_n with P_2. To encourage the researchers for further research, two conjectures and one open problem is proposed. In fact, it seems to be a very challenging problem to find the exact value for the edge irregularity strength of families of graphs.

Acknowledgement
The authors wish to thank the anonymous referee for his/her valuable comments.

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