THE RAINBOW VERTEX CONNECTIVITIES OF SMALL CUBIC GRAPHS

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ABSTRACT. A vertex colored path is *vertex-rainbow* if its internal vertices have distinct colors. For a connected graph G with connectivity $\kappa(G)$ and an integer k with $1 \leq k \leq \kappa(G)$, the *rainbow vertex k-connectivity* of G is the minimum number of colors required to color the vertices of G such that any two vertices of G are connected by k internally vertex disjoint vertex-rainbow paths. In this paper, we determine the rainbow vertex k-connectivities of all small cubic graphs of order 8 or less.

KEYWORDS. Vertex-coloring, vertex-rainbow path, rainbow vertex k-connectivity.

1. INTRODUCTION

All graphs considered in this paper are simple, finite and undirected. We follow the notation and terminology of [1] for those not described here. Recall that the *connectivity* of a connected graph G is $\kappa(G)$=max$\{k$: G is k-connected$\}$. Let G be a connected graph with connectivity $\kappa(G)$. Throughout the paper, let k be an integer satisfying $1 \leq k \leq \kappa(G)$. For convenience, a set of internally vertex disjoint paths will be called *disjoint*.

For a graph G, we denote by $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively. An *edge-coloring* of a graph G is a mapping from $E(G)$ to some finite set of colors. A path in an edge colored graph is said to be a *rainbow path* if no two edges on the path share the same color. *The rainbow k-connectivity* of a connected graph G, denoted by $rc_k(G)$, is the minimum number of colors needed in an edge-coloring of G such that any two distinct vertices of G are connected by k disjoint rainbow paths. The function $rc_k(G)$ was introduced by Chartrand et al.(see [2] for $k = 1$, and [3] for general k). Since then, a considerable amount of research has been carried out towards the study of $rc_k(G)$, see [8] for a survey on this topic.

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Similar to the concept of rainbow k-connectivity, Krivelevich and Yuster[6] (2009), Liu et al.[9](2013) proposed the concept of rainbow vertex k-connectivity. A *vertex-coloring* of a graph G is a mapping from $V(G)$ to some finite set of colors. A vertex colored path is *vertex-rainbow* if its internal vertices have distinct colors. A vertex-coloring of a connected graph G, not necessarily proper, is *rainbow vertex k-connected* if any two vertices of G are connected by k disjoint vertex-rainbow paths. *The rainbow vertex k-connectivity* of G, denoted by $rvc_k(G)$, is the minimum integer t so that there exists a rainbow vertex k-connected coloring of G, using t colors. For convenience, we write $rvc(G)$ for $rvc_1(G)$. By Menger's theorem[10], $rc_k(G)$ and $rvc_k(G)$ are well defined if and only if G is a connected graph satisfying $1 \leq k \leq \kappa(G)$.

Let G be a connected graph. Note that $rvc(G) = 0$ if and only if G is a complete graph. Let $diam(G)$ denote the diameter of G. Then $rvc(G) \geq diam(G) - 1$ with equality if $k = 1$ and $diam(G) = 1$ or 2. For $u, v \in V(G)$, let $d_k(u, v)$ be the minimum possible length of the longest path in a set of k disjoint $u - v$ paths. The *k-diameter* of G is $diam_k(G) = \max_{u,v \in V(G)} d_k(u, v)$. Hence $diam_1(G) = diam(G)$. An easy observation is that $rvc_k(G) \geq diam_k(G) - 1$. If $k \geq 2$, then $rvc_k(G) \geq 1$, and equality holds if G is a complete graph with at least three vertices.

Krivelevich and Yuster [6] proved that if G is a connected graph with n vertices and minimum degree δ, then $rvc(G) < 11n/\delta$. It was shown[4] that the computation of $rvc(G)$ is NP-hard. It was proved in [7] that $rvc(G) = n - 2$ if and only if G is a path of order n. In [9], Liu et al. determined the precise values of $rvc_k(G)$ when G is a cycle, a wheel, and a complete multipartite graph. The foregoing results motivate us to consider the rainbow vertex connectivities of some special graph classes.

In [5], Fujie-Okamoto et al. investigated the rainbow connectivities of all small cubic graphs of order 8 or less. In this paper, we determine the rainbow vertex connectivities of all small cubic graphs of order 8 or less. Suppose that G is a connected cubic graph of order $n \leq 8$. Since $3n = \sum_{v \in V(G)} \deg(v) = 2|E(G)|$ implies that n is even, we have $n = 4, 6, 8$. If $n = 4$, then $G = K_4$. If $n = 6$, then the complement graph $\bar{G}$ is 2-regular, so that $\bar{G} = 2C_3$ or C_6. This gives $G = K_{3,3}$ or $K_3 \Box K_2$, where $\Box$ denotes Cartesian product. If $n = 8$, then we obtain five connected cubic graphs by [11], which are depicted in Figure 1.

It is easy to verify that $rvc(K_4) = 0$, and $rvc_2(K_4) = rvc_3(K_4) = 1$. It was also shown in [9] that $rvc(K_{3,3}) = 1$, and $rvc_2(K_{3,3}) = rvc_3(K_{3,3}) = 2$.

Our main result is stated as follows.

Theorem 1.1. (a) $rvc(K_3 \Box K_2) = 1, rvc_2(K_3 \Box K_2) = 2$, $rvc_3(K_3 \Box K_2) = 3$.

(b) (i) $rvc(Q_3) = rvc_2(Q_3) = 2, rvc_3(Q_3) = 4$.

(ii) $rvc(M_8) = 1, rvc_2(M_8) = 3, rvc_3(M_8) = 4$.
(iii) $rvc(F_1) = 2, rvc_2(F_1) = 3, rvc_3(F_1) = 5$.
(iv) $rvc(F_2) = 2, rvc_2(F_2) = 4$.
(v) $rvc(F_3) = 1, rvc_2(F_3) = 3, rvc_3(F_3) = 4$.

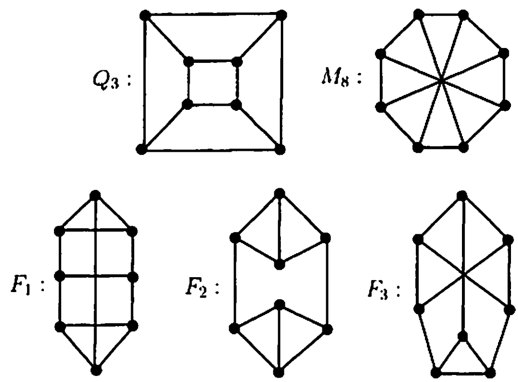


Figure 1: Connected cubic graphs of order 8.

2. Proof of Theorem 1.1

By proving the following lemma, we determine the rainbow vertex connectivities of $K_3 \square K_2$.

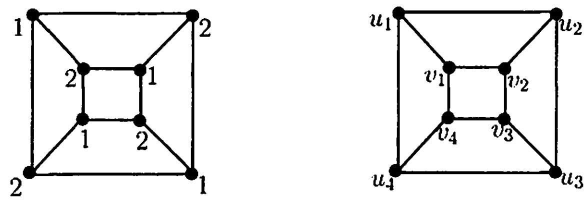
Lemma 2.1. *Let* $G = K_3 \square K_2$. *Then* $rvc(G) = 1, rvc_2(G) = 2$ *and* $rvc_3(G) = 3$.

Proof. Let $V(G) = \{u_1, u_2, u_3\} \cup \{v_1, v_2, v_3\}$ such that $u_iu_j, v_iv_j, u_iv_i \in E(G)$, where $1 \leq i, j \leq 3$ with $i \neq j$. Since $diam(G) = 2$, we have $rvc(G) = 1$. It is not hard to see that $diam_2(G) = 3$. Thus $rvc_2(G) \geq 2$. By giving u_i color 1 and v_i color 2 for $1 \leq i \leq 3$, this is a vertex-coloring of G with $rvc_2(G) \leq 2$.

Suppose $rvc_3(G) = 2$. Assign a rainbow vertex 3-connected coloring c with colors 1 and 2 to G. Since one of the three vertex-rainbow paths between v_1 and v_2 must be $v_1u_1u_2v_2$, this implies $c(u_1) \neq c(u_2)$. By the same argument, we obtain that $c(u_2) \neq c(u_3)$ and $c(u_1) \neq c(u_3)$, a contradiction. Thus $rvc_3(G) \geq 3$. The following coloring c' with colors 1, 2 and 3 induces a vertex-coloring of G with $rvc_3(G) \leq 3$: $c'(u_1) = c'(v_3) = 1, c'(u_2) = c'(v_1) = 2$ and $c'(u_3) = c'(v_2) = 3$. □

We now consider the rainbow vertex connectivities of the five connected cubic graphs as depicted in Figure 1.

Recall that the 3-dimensional cube Q_3 is a cubic graph of diameter 3 and connectivity 3. Hence $rvc_3(Q_3) \geq rvc_2(Q_3) \geq rvc(Q_3) \geq 2$. Assigning a vertex-coloring to Q_3 with colors 1 and 2 as Figure 2(a), we can easily check that any two distinct vertices of Q_3 are connected by two disjoint vertex-rainbow paths. Thus $rvc_2(Q_3) = rvc(Q_3) = 2$. Now we only need to determine $rvc_3(Q_3)$(see Figure 2(b)).



(a) (b)

Figure 2: The rainbow vertex 2-connectivity of Q_3.

Lemma 2.2. $rvc_3(Q_3) = 4$.

Proof. Let c be a rainbow vertex 3-connected coloring of Q_3.

(i) Without loss of generality, consider u_1 and u_2. Since in any set of three disjoint $u_1 - u_2$ paths, one path contains v_1 and v_2, we must have $c(v_1) \neq c(v_2)$. By symmetry, any two adjacent vertices of Q_3 must be colored by distinct colors.

(ii) Since one of the three vertex-rainbow $u_1 - v_2$ paths must be $u_1u_4v_4v_3v_2$ or $u_1u_4u_3v_3v_2$, this implies $c(u_4) \neq c(v_3)$. By symmetry, for any distinct vertices u, v of Q_3 satisfying $d(u, v) = 2$, we obtain $c(u) \neq c(v)$.

Combining (i) and (ii), we conclude that $c(u_1), c(u_2), c(u_3), c(u_4)$ are distinct, so that $rvc_3(Q_3) \geq 4$. Now, define the vertex-coloring c' on Q_3 as follows: $c'(v_1) = c'(u_3) = 1, c'(v_3) = c'(u_1) = 2, c'(v_2) = c'(u_4) = 3$, and $c'(v_4) = c'(u_2) = 4$. It is easy to verify that the vertex-coloring c' is rainbow vertex 3-connected. Therefore, $rvc_3(Q_3) \leq 4$. □

Recall that M_8 is the Möbius ladder of order 8, or the Wagner graph. Since $diam(M_8) = 2$, it follows that $rvc(M_8) = 1$. Observe that $\kappa(M_8) = 3$. This implies that we need to consider $rvc_2(M_8)$ and $rvc_3(M_8)$(see Figure 3(a)).

Lemma 2.3. $rvc_2(M_8) = 3$ *and* $rvc_3(M_8) = 4$.

Proof. First, it is easy to see that $diam_2(M_8) = 3$, so that $rvc_2(M_8) \geq 2$. Suppose $rvc_2(M_8) = 2$. Let c be a rainbow vertex 2-connected coloring with colors 1 and 2. One of the following must occur.

(i) $c(u_{2i-1}) = 1$ and $c(u_{2i}) = 2$, where $1 \leq i \leq 4$. However, there is no set of two disjoint vertex-rainbow $u_1 - u_5$ paths, a contradiction.

(ii) There exist two adjacent vertices, without loss of generality, u_1 and u_2 satisfying $c(u_1) = c(u_2)$. However, there is no set of two disjoint vertex-rainbow $u_5 - u_6$ paths, another contradiction.

By (i) and (ii), we have $rvc_2(M_8) \geq 3$. Since there exists a rainbow vertex 2-connected coloring with three colors shown in Figure 3(b), this implies that $rvc_2(M_8) = 3$.

Next, we show that $rvc_3(M_8) = 4$. Since there exists a rainbow vertex 3-connected coloring with four colors(see Figure 3(c)), we have $3 \leq rvc_3(M_8) \leq 4$. Now we only need to prove that $rvc_3(M_8) \neq 3$. To the contrary, suppose there exists a rainbow vertex 3-connected coloring c of M_8, using colors 1, 2 and 3.

Let $C = u_1u_2 \cdots u_8u_1$ be a Hamiltonian cycle in M_8 and consider two adjacent vertices u and v of C. By symmetry, assume that $u = u_1$ and $v = u_2$. If $c(u_1) = c(u_2)$, then there is no set of three disjoint vertex-rainbow paths between u_3 and u_8, a contradiction. Hence any two adjacent vertices of C must be colored differently. Therefore, there must exist three vertices u, v, w of C such that $c(u) \neq c(v), c(v) \neq c(w)$ and $c(u) = c(w)$, where $uv, vw \in E(C)$. Without loss of generality, assume that $c(u_1) = 1, c(u_2) = 2$ and $c(u_3) = 1$. We have $c(u_4), c(u_8) \in \{2,3\}, c(u_5), c(u_6), c(u_7) \in \{1,2,3\}$ and $c(u_i) \neq c(u_{i+1})$ for $4 \leq i \leq 7$.

Since the coloring c is rainbow vertex 3-connected, we have, for all $1 \leq i \leq 8$, the three disjoint vertex-rainbow $u_i - u_{i+4}$ paths are either $\{u_iu_{i+4}, u_iu_{i+1} \cdots u_{i+4}, u_iu_{i-1} \cdots u_{i-4}\}$ or $\{u_iu_{i+4}, u_iu_{i+1}u_{i+5}u_{i+4}, u_iu_{i-1} u_{i+3}u_{i+4}\}$, with all indices taken modulo 8. By considering the pair $\{u_4, u_8\}$, we have $c(u_5), c(u_7) \in \{2,3\}$. By considering the pair $\{u_1, u_5\}$, we have $(c(u_4), c(u_8)) \neq (2,2)$, and we may assume that $c(u_4) = 3$, which implies $c(u_5) = 2$. If $c(u_6) = 3$, then by considering the pair $\{u_3, u_7\}$, we have $c(u_8) = 2$, but then, $c(u_7) = 1$, a contradiction. Hence $c(u_6) = 1$, and $(c(u_4), c(u_5), c(u_6), c(u_7), c(u_8)) \in \{(3,2,1,2,3), (3,2,1,3,2)\}$. But then, there is no set of three disjoint vertex-rainbow $u_3 - u_4$ paths, a final contradiction.

Hence $rvc_3(M_8) \neq 3$, implying that $rvc_3(M_8) = 4$. □

We now determine the rainbow vertex connectivities of the graph F_1 depicted in Figure 4(a). Notice that $\kappa(F_1) = 3$.

Lemma 2.4. *$rvc(F_1) = 2, rvc_2(F_1) = 3$ and $rvc_3(F_1) = 5$.*

Proof. Evidently, there exists a rainbow vertex connected coloring depicted in Figure 4(b), which follows that $rvc(F_1) \leq 2$. Since $diam(F_1) = 3$, this implies $rvc(F_1) \geq 2$, and so $rvc(F_1) = 2$.

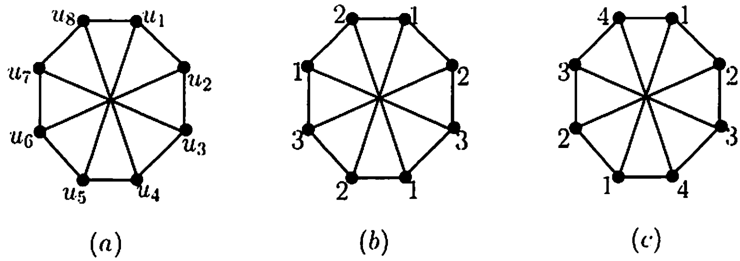


Figure 3: The rainbow vertex 2 and 3-connectivity of M_8.

Next, we prove that $rvc_2(F_1) = 3$. Considering the two vertices w_1 and w_2, any set of two disjoint $w_1 - w_2$ paths contains a path of length at least 4. Thus $rvc_2(F_1) \geq 3$. On the other hand, Figure 4(c) provides a rainbow vertex 2-connected coloring with three colors. Hence $rvc_2(F_1) = 3$.

Finally, we show that $rvc_3(F_1) = 5$. Let c be a rainbow vertex 3-connected coloring with k colors. The following statements must occur.

(i) $c(v_1), c(v_2), c(v_3)$ are distinct. (Consider vertex-rainbow $w_1 - w_2$ paths.)

(ii) $c(w_2) \neq c(v_2)$. (Consider vertex-rainbow $w_1 - v_1$ paths.)

(iii) $c(w_1) \neq c(v_2)$. (Consider vertex-rainbow $w_2 - v_3$ paths.)

(iv) $c(w_1), c(w_2), c(v_3)$ are distinct, and $c(w_1), c(w_2), c(v_1)$ are distinct. (Consider vertex-rainbow $v_1 - v_2$ paths and $v_2 - v_3$ paths, respectively.)

Combining (i), (ii), (iii) and (iv), we obtain that $c(v_1), c(v_2), c(v_3), c(w_1)$, $c(w_2)$ are distinct. Thus $k \geq 5$, implying that $rvc_3(F_1) \geq 5$. On the other hand, there exists a rainbow vertex 3-connected coloring with five colors shown in Figure 4(d). It follows that $rvc_3(F_1) = 5$. □

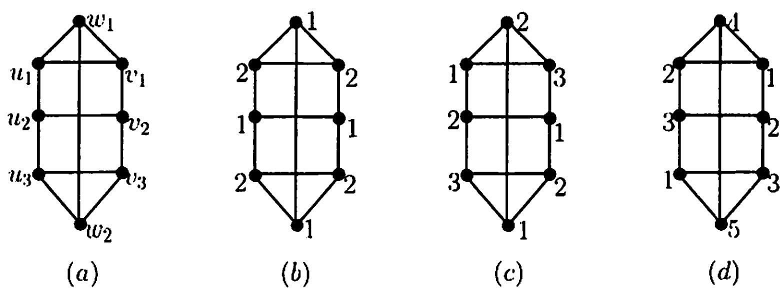


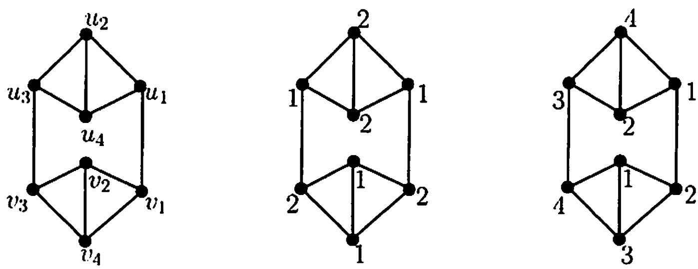
Figure 4: The rainbow vertex connectivities of F_1.

Now, we are in a position to determine the rainbow vertex connectivities of the graph F_2 in Figure 5(a). Since F_2 has connectivity 2, we only consider $rvc(F_2)$ and $rvc_2(F_2)$.

Lemma 2.5. *$rvc(F_2) = 2$ and $rvc_2(F_2) = 4$.*

Proof. Since $diam(F_2) = 3$, this implies $rvc(F_2) \geq diam(F_2) - 1 = 2$. Observe that Figure 5(*b*) shows a rainbow vertex connected coloring. Thus $rvc(F_2) = 2$.

For u_1 and v_1, any set of two disjoint $u_1 - v_1$ paths consists of a path of length 1 and a path of length at least 5. It follows that $rvc_2(F_2) \geq diam_2(F_2) - 1 = 4$. Since there exists a rainbow vertex 2-connected coloring depicted in Figure 5(*c*), we have $rvc_2(F_2) = 4$. □



(*a*) (*b*) (*c*)

Figure 5: The rainbow vertex 1 and 2-connectivity of F_2.

Finally, we determine the rainbow vertex connectivities of the graph F_3 as shown in Figure 6(*a*). Since $diam(F_3) = 2$, it follows that $rvc(F_3) = 1$. Note that $\kappa(F_3) = 3$, we need to consider $rvc_2(F_3)$ and $rvc_3(F_3)$.

Lemma 2.6. *$rvc_2(F_3) = 3$ and $rvc_3(F_3) = 4$.*

Proof. First, we prove that $rvc_2(F_3) = 3$. Considering u_2 and v_2, any set of two disjoint $u_2 - v_2$ paths contains a path of length at least 4. Thus $rvc_2(F_3) \geq 3$. On the other hand, it is easy to check that the vertex-coloring depicted in Figure 6(*b*) is rainbow vertex 2-connected, which follows that $rvc_2(F_3) = 3$.

Next, we show that $rvc_3(F_3) = 4$. Since there exists a rainbow vertex 3-connected coloring, using four colors(see Figure 6(*c*)), we have $3 \leq rvc_3(F_3) \leq 4$. Now we only need to prove that $rvc_3(F_3) \neq 3$. To the contrary, suppose there exists a rainbow vertex 3-connected coloring c with colors 1, 2 and 3. For every pair $\{v_i, v_j\}$, where $i \neq j$ and $1 \leq i, j \leq 3$, we have that $v_i u_i w u_j v_j$ is a vertex-rainbow path for some $w \in \{w_1, w_2\}$. Hence $c(u_i) \neq c(u_j)$. Without loss of generality, assume that $c(u_1) = 1, c(u_2) = 2$ and $c(u_3) = 3$. Considering the pairs $\{u_i, u_j\}$, where $i \neq j$ and $1 \leq i, j \leq 3$, gives that $c(v_1), c(v_2), c(v_3)$ are distinct. By considering the pair $\{u_2, v_2\}$, $u_2 w' u_1 v_1 v_2$ and $u_2 w'' u_3 v_3 v_2$ must be two vertex-rainbow paths, where

$\{w', w''\} = \{w_1, w_2\}$. Hence $c(u_1) \neq c(v_1)$ and $c(u_3) \neq c(v_3)$. Furthermore, we obtain $c(u_2) \neq c(v_2)$ by considering the three disjoint vertex-rainbow paths between u_1 and v_1.

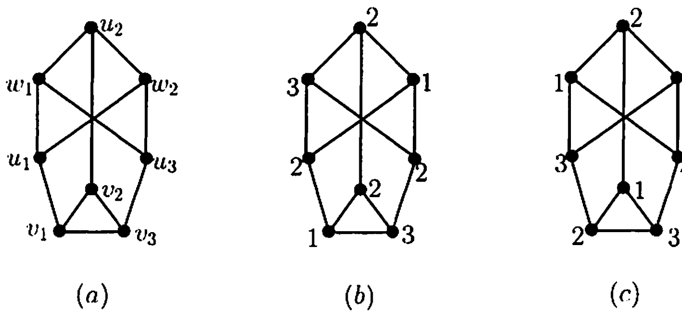


Figure 6: The rainbow vertex 2 and 3-connectivity of F_3.

With the above arguments, we have that $(c(v_1), c(v_2), c(v_3)) = (2, 3, 1)$ or $(3, 1, 2)$. By the obvious symmetry of F_3, it suffices to consider $(c(v_1), c(v_2), c(v_3)) = (2, 3, 1)$. Consider the two pairs vertices $\{u_2, w_i\}$ with $1 \leq i \leq 2$. Since there exist three disjoint vertex-rainbow $u_2 - w_i$ paths, we obtain $c(u_3) \neq c(w_i)$. Hence $c(w_1), c(w_2) \in \{1, 2\}$. However, there is no set of three disjoint vertex-rainbow $u_2 - v_2$ paths, a contradiction.

Therefore, $rvc_3(F_3) \neq 3$, and so $rvc_3(F_3) = 4$. □

By Lemmas 2.1, 2.2, 2.3, 2.4, 2.5 and 2.6, Theorem 1.1 is immediate.

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