SOME PROPERTIES OF k-ORDER GAUSSIAN FIBONACCI AND LUCAS NUMBERS

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ABSTRACT. In this paper we define and study the k-order Gaussian Fibonacci and Lucas Numbers with boundary conditions. We identify and prove the generating functions, the Binet formulas, the summation formulas, matrix representation of k-order Gausian Fibonacci numbers and some significant relationships between k-order Gaussian Fibonacci and k-order Lucas numbers connecting with usual k-order Fibonacci numbers.

1. INTRODUCTION

Fibonacci numbers theory depends on a very interesting recurrence relation that is $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ such that $F_0 = 0$ and $F_1 = 1$. There are a lot of generalizations of Fibonacci numbers defined and studied by some authors. For more information one can see [11, 16].

One of the most interesting generalization is Gaussian Fibonacci numbers introduced by A. F. Horadam [6]. Horadam [7] defined and established some quite general identities about Gaussian Fibonacci numbers. J. R. Jordan [8] extended some relations which are known for the usual Fibonacci sequences to the Gaussian Fibonacci and Gaussian Lucas sequences.

The Gaussian Fibonacci sequence in [8] is $GF_0 = i$, $GF_1 = 1$ and $GF_n = GF_{n-1} + GF_{n-2}$ for $n > 1$. One can see that

$$GF_n = F_n + iF_{n-1} \tag{1.1}$$

where F_n is the usual nth Fibonacci number.

The Gaussian Lucas sequence in [8] is defined similarly to the Gaussian Fibonacci sequence as $GL_0 = 2 - i$, $GL_1 = 1 + 2i$ and $GL_n = GL_{n-1} + GL_{n-2}$ for $n > 1$. Also it can be seen that

$$GL_n = L_n + iL_{n-1} \tag{1.2}$$

where L_n is the usual nth Lucas number.

Asci and Gurel [1] defined and studied the bivariate Gaussian Fibonacci and bivariate Gaussian Lucas polynomials. Asci and Gurel [2] defined

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Gaussian Jacobsthal and Gaussian Jacobsthal Lucas Numbers and transferred significant identities from the usual Jacobsthal numbers to the Gaussian Jacobsthal numbers. Asci and Gurel [3] defined and established some interesting identities about Gaussian Fibonacci $p-$Numbers and Gaussian Lucas $p-$Numbers by matrix methods. Also authors in [4] defined and studied some interesting results about Gaussian Tribonacci numbers and Gaussian Tribonacci polynomials

Another interesting generalization of Fibonacci numbers is order$-k$ Fibonacci Numbers that is defined by Er [5] by the following recurrence relation

$$g_n^i = \sum_{j=1}^{k} g_{n-j}^i, \text{ for } n > 0 \text{ and } 1 \leq i \leq k$$

with boundary conditions for $1 - k \leq n \leq 0$,

$$g_n^i = \begin{cases} 1 & \text{if } i = 1 - n \\ 0 & \text{otherwise} \end{cases} .$$

Kilic and Tasci in [10] extended some relationships about order$-k$ Fibonacci Numbers that were Binet formulas combinatorial representations of order$-k$ Fibonacci numbers. Although the definitions of order$-k$ Fibonacci Numbers by Lee et al. in [12, 13, 14] are different from above, the authors derived a generalized Binet formula for $k-$generalized Fibonacci sequence by using determinants and gave relationships between the Fibonacci numbers and their associated matrices of $k-$generalized Fibonacci numbers.

Generalized order$-k$ Lucas Numbers are defined by many authors different from each other. Kaygisiz and Sahin [9] defined generalized order$-k$ Lucas numbers by the following recurrence relation

$$l_{k,n} = \sum_{j=1}^{k} l_{k,n-j}$$

with boundary conditions

$$l_{k,1-k} = l_{k,2-k} = \ldots = l_{k,-1} = -1 \text{ and } l_{k,0} = k.$$

But Tasci et al. in [10, 15] defined Generalized order$-k$ Lucas Numbers different from these authors and gave new generalizations of order$-k$ Lucas number by matrix methods. Also Lee et al. in [13] generalized between $k-$Fibonacci numbers and Lucas numbers by using different definitions of these numbers

In this article, we define and study the $k-$order Gaussian Fibonacci and Gaussian Lucas Numbers with boundary conditions. We identify and prove the generating functions, the Binet formulas, the summation formulas, the matrix representation of $k-$order Gaussian Fibonacci numbers and some significant relationships between $k-$order Gaussian Fibonacci and $k-$order Lucas numbers connecting with usual $k-$order Fibonacci numbers.

2. The $k-$order Gaussian Fibonacci and $k-$order Gaussian Lucas Numbers

Definition 1. *Let k be an integer. The $k-$order Gaussian Fibonacci numbers $\left\{GF_n^{(k)}\right\}_{n=0}^{\infty}$ are defined by the following recurrence relation*

$$GF_n^{(k)} = \sum_{j=1}^{k} GF_{n-j}^{(k)}, \quad \text{for } n > 0 \text{ and } k \geq 2 \tag{2.1}$$

with boundary conditions for $1-k \leq n \leq 0$,

$$GF_n^{(k)} = \begin{cases} 1-i, & \text{if } k = 1-n \\ i, & \text{if } k = 2-n \\ 0, & \text{otherwise.} \end{cases}$$

It can be easily seen that

$$GF_n^{(k)} = F_n^{(k)} + iF_{n-1}^{(k)}$$

where $F_n^{(k)}$ is the nth $k-$order Fibonacci number.

For later use the first few terms of the sequence $GF_n^{(k)}$ can be seen in the following table

n	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$\cdots$
-5					$1-i$	
-4				$1-i$	i	
-3			$1-i$	i	0	
-2		$1-i$	i	0	0	
-1	$1-i$	i	0	0	0	
0	i	0	0	0	0	
1	1	1	1	1	1	
2	$1+i$	$1+i$	$1+i$	$1+i$	$1+i$	
3	$2+i$	$2+i$	$2+i$	$2+i$	$2+i$	
4	$3+2i$	$4+2i$	$4+2i$	$4+2i$	$4+2i$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Definition 2. *Let k be an integer. The $k-$order Gaussian Lucas numbers $\left\{GL_n^{(k)}\right\}_{n=0}^{\infty}$ are defined by the following recurrence relation*

$$GL_n^{(k)} = \sum_{j=1}^{k} GL_{n-j}^{(k)}, \quad \text{for } n > 0 \text{ and } k \geq 2$$

with boundary conditions for $1-k \leq n \leq 0$,

$$GL_n^{(k)} = \begin{cases} -1+(2k-1)\,i & \text{if } k = 1-n \\ -1-i & \text{otherwise} \\ k-i & \text{if } n = 0. \end{cases}$$

It can be easily seen that

$$GL_n^{(k)} = L_n^{(k)} + iL_{n-1}^{(k)}$$

where $L_n^{(k)}$ is the nth $k-$order Lucas number.

For later use the first few terms of the sequence $GL_n^{(k)}$ can be seen in the following table

n	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$\cdots$
-5					$-1+11i$	
-4				$-1+9i$	$-1-i$	
-3			$-1+7i$	$-1-i$	$-1-i$	
-2		$-1+5i$	$-1-i$	$-1-i$	$-1-i$	
-1	$-1+3i$	$-1-i$	$-1-i$	$-1-i$	$-1-i$	
0	$2-i$	$3-i$	$4-i$	$5-i$	$6-i$	
1	$1+2i$	$1+3i$	$1+4i$	$1+5i$	$1+6i$	
2	$3+i$	$3+i$	$3+i$	$3+i$	$3+i$	
3	$4+3i$	$7+3i$	$7+3i$	$7+3i$	$7+3i$	
4	$7+4i$	$11+7i$	$15+7i$	$15+7i$	$15+7i$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

3. Some Properties of $k-$order Gaussian Fibonacci and Lucas Numbers

Theorem 1. *The generating function for the k−order Gaussian Fibonacci numbers is*

$$g(t) = \sum_{n=0}^{\infty} GF_n^{(k)} t^n = \frac{GF_0^{(k)} + \left(GF_1^{(k)} - GF_0^{(k)}\right)t + (GF_2^{(k)} - GF_1^{(k)} - GF_0^{(k)})t^2}{1-\sum_{j=1}^{k} t^j}$$

and for the k−order Gaussian Lucas numbers is

$$h(t) = \sum_{n=0}^{\infty} GL_n^{(k)} t^n = \frac{GL_0^{(k)} + \sum_{m=1}^{k-1}\left(GL_m^{(k)} - \sum_{j=1}^{m} GL_{j-1}^{(k)}\right)t^m}{1-\sum_{j=1}^{k} t^j}.$$

Proof. Let $g(t)$ be the generating function of the the $k-$order Gaussian Fibonacci numbers $GF_n^{(k)}$ then

$$\begin{aligned} g(t) - tg(t) - \dots - t^k g(t) &= GF_0^{(k)} + t\left(GF_1^{(k)} - GF_0^{(k)}\right) \\ &\quad + t^2\left(GF_2^{(k)} - GF_1^{(k)} - GF_0^{(k)}\right) \\ &\quad + t^3\left(GF_3^{(k)} - GF_2^{(k)} - GF_1^{(k)} - GF_0^{(k)}\right) \\ &\quad + \sum_{n=4}^{\infty} t^n \left(GF_n^{(k)} - \sum_{j=0}^{n-1} GF_j^{(k)}\right) \\ &= GF_0^{(k)} + \left(GF_1^{(k)} - GF_0^{(k)}\right)t \\ &\quad + (GF_2^{(k)} - GF_1^{(k)} - GF_0^{(k)})t^2. \end{aligned}$$

By taking $g(t)$ parenthesis we get

$$g(t) = \frac{GF_0^{(k)} + \left(GF_1^{(k)} - GF_0^{(k)}\right)t + (GF_2^{(k)} - GF_1^{(k)} - GF_0^{(k)})t^2}{1 - \sum_{j=1}^{k} t^j}.$$

The proof for $h(t)$ is similar. □

Corollary 1. *Let $k = 2$. Then the generating function of the usual Gaussian Fibonacci numbers*

$$g(t) = \sum_{n=0}^{\infty} GF_n t^n = \frac{i + (1-i)t}{1 - t - t^2}$$

and Lucas numbers

$$h(t) = \sum_{n=0}^{\infty} GL_n t^n = \frac{2 - i + (i-1)\,t}{1 - t - t^2}.$$

Corollary 2. [4] *Let $k = 3$. Then the generating functions of the Gaussian Tribonacci numbers*

$$g(t) = \sum_{n=0}^{\infty} GT_n t^n = \frac{t + it^2}{1 - t - t^2 - t^3}.$$

Binet's formulas are well known and studied in the theory of Fibonacci numbers.

Let $f(\lambda)$ be the characteristic polynomial of the $k-$order Fibonacci numbers and $x_1, x_2, x_3, \dots, x_{k-1}, x_k$ be the different roots of the characteristic equation of the recurrence relation (2.1). Then the Binet formula of the $k-$order Fibonacci numbers are given in [12].

Now we can give the Binet formula for the $k-$order Gaussian Fibonacci and the $k-$order Gaussian Lucas numbers.

Theorem 2. *For $n \geq 0$*

$$GF_n^{(k)} = c_1 x_1^n + c_2 x_2^n + ... + c_k x_k^n + i\left(c_1 x_1^{n-1} + c_2 x_2^{n-1} + ... + c_k x_k^{n-1}\right)$$

and

$$GL_n^{(k)} = t_1 x_1^n + t_2 x_2^n + ... + t_k x_k^n + i\left(t_1 x_1^{n-1} + t_2 x_2^{n-1} + ... + t_k x_k^{n-1}\right).$$

Proof. We have the relation

$$GF_n^{(k)} = F_n^{(k)} + iF_{n-1}^{(k)}$$

where $F_n^{(k)}$ is the nth $k-$order Fibonacci number. Since the Binet formula of the $k-$order Fibonacci numbers is proved in [12]. Then from above relation it can be seen. The proof for $GL_n^{(k)}$ is similar □

Theorem 3. *For any positive integers m and n*

$$GF_{n+m}^{(k)} = F_{n+1}^{(k)} GF_m^{(k)} + \sum_{j=0}^{k-1}\left(GF_{m-(k-j-1)}^{(k)} \sum_{p=0}^{j} F_{n-p}^{(k)}\right).$$

Proof. Theorem can be proved by mathematical induction on n. □

Corollary 3. *Let $k = 2$. Then*

$$GF_{n+m} = F_{n-1} GF_m + F_n GF_{m+1}.$$

Corollary 4. [4] *Let $k = 3$. Then*

$$GT_{n+m} = T_n GT_{m-2} + (T_n + T_{n-1})\, GT_{m-1} + T_{n+1} GT_m$$

where T_n and GT_n are the usual Tribonacci and Gaussian Tribonacci numbers.

Theorem 4. *For any positive integers m and n*

$$GL_{n+m}^{(k)} = F_{n+1}^{(k)} GL_m^{(k)} + \sum_{j=0}^{k-1}\left(GL_{m-(k-j-1)}^{(k)} \sum_{p=0}^{j} F_{n-p}^{(k)}\right).$$

Proof. Theorem can be proved by mathematical induction on n. □

Corollary 5. *Let $k = 2$. Then*

$$GL_{n+m} = F_{n+1} GL_m + F_n GL_{m-1}.$$

Theorem 5. *The sums of the $k-$order Gaussian Fibonacci and $k-$order Gaussian Lucas numbers are given as:*

$$\sum_{j=1}^{n} GF_j^{(k)} = \frac{1}{k-1}\Bigg(GF_{n+k}^{(k)} - GF_k^{(k)} + \sum_{j=1}^{k-2}(k-j-1)\left(GF_j^{(k)} - GF_{n+j}^{(k)}\right)\Bigg)$$

and

$$\sum_{j=1}^{n} GL_j^{(k)} = \frac{1}{k-1}\left(GL_{n+k}^{(k)} - GL_k^{(k)}\right.$$
$$\left. + \sum_{j=1}^{k-2}(k-j-1)\left(GL_j^{(k)} - GL_{n+j}^{(k)}\right)\right).$$

Proof. By the recurrence relation of $k-$order Gaussian Fibonacci numbers (2.1) we have

$$GF_{n-k}^{(k)} = GF_n^{(k)} - \sum_{j=1}^{k-1} GF_{n-j}.$$

From this equality

$$\begin{aligned}
GF_1^{(k)} &= GF_{k+1}^{(k)} - GF_k^{(k)} - \ldots - GF_3^{(k)} - GF_2^{(k)} \\
GF_2^{(k)} &= GF_{k+2}^{(k)} - GF_{k+1}^{(k)} - \ldots - GF_4^{(k)} - GF_3^{(k)} \\
GF_3^{(k)} &= GF_{k+3}^{(k)} - GF_{k+2}^{(k)} - \ldots - GF_5^{(k)} - GF_4^{(k)} \\
&\vdots \\
GF_{m-1}^{(k)} &= GF_{k+m-1}^{(k)} - GF_{k+m-2}^{(k)} - \ldots - GF_{m+1}^{(k)} - GF_m^{(k)} \\
GF_m^{(k)} &= GF_{k+m}^{(k)} - GF_{k+m-1}^{(k)} - \ldots - GF_{m+2}^{(k)} - GF_{m+1}^{(k)}.
\end{aligned}$$

So we get

$$\begin{aligned}
\sum_{j=1}^{m} GF_j^{(k)} &= GF_{k+m}^{(k)} - GF_2^{(k)} - 2GF_3^{(k)} - 3GF_4^{(k)} \\
&\quad - \ldots - (k-2)\,GF_{k-1}^{(k)} - (k-1)\,GF_k^{(k)} \\
&\quad - (k-2)\sum_{j=k+1}^{m+1} GF_j^{(k)} - (k-3)\,GF_{m+2}^{(k)} \\
&\quad - (k-4)\,GF_{m+3}^{(k)} - \ldots - 3GF_{k+m-4}^{(k)} - 2GF_{k+m-3}^{(k)} \\
&\quad - GF_{k+m-2}^{(k)}.
\end{aligned}$$

Adding and subtracting the following terms in the equation above

$$(k-2)\,GF_1^{(k)} - (k-2)\,GF_1^{(k)} + (k-2)\,GF_2^{(k)} - (k-2)\,GF_2^{(k)}$$
$$+ (k-2)\,GF_3^{(k)} - (k-2)\,GF_3^{(k)} + \ldots + (k-2)\,GF_k^{(k)} - (k-2)\,GF_k^{(k)}$$

we get

$$\sum_{k=1}^{m} GF_j^{(k)} = GF_{k+m}^{(k)} + (k-2)\,GF_1^{(k)} + (k-3)\,GF_2^{(k)} + \ldots$$
$$+2GF_{k-3}^{(k)} + GF_{k-2}^{(k)} - GF_k^{(k)} - (k-2)\sum_{j=1}^{m} GF_j^{(k)}$$
$$-(k-2)\,GF_{m+1}^{(k)} - (k-3)\,GF_{m+2}^{(k)} - \ldots$$
$$-3GF_{k+m-4}^{(k)} - 2GF_{k+m-3}^{(k)} - GF_{k+m-2}^{(k)}.$$

Finally we have

$$(k-1)\sum_{j=1}^{m} GF_j^{(k)} = GF_{k+m}^{(k)} - GF_k^{(k)} + \sum_{j=1}^{k-2}(k-j-1)\,GF_j^{(k)}$$
$$-\sum_{j=1}^{k-2}(k-j-1)\,GF_{m+j}^{(k)}$$

and

$$\sum_{j=1}^{m} GF_j^{(k)} = \frac{1}{k-1}\left(GF_{k+m}^{(k)} - GF_k^{(k)}\right.$$
$$\left.-\sum_{j=1}^{k-2}(k-j-1)\left(GF_j^{(k)} - GF_{m+j}^{(k)}\right)\right).$$

This completes the proof. □

Corollary 6. [8] *For* $k = 2$

$$\sum_{j=1}^{n} GF_j = GF_{n+2} - (1+i)$$

and

$$\sum_{j=1}^{n} GL_j = GL_{n+2} - (3+i)\,.$$

Corollary 7. [4] *For* $k = 3$

$$\sum_{j=1}^{n} GT_j = \frac{1}{2}\left[GT_{n+3} - GT_{n+1} - (1+i)\right].$$

Theorem 6. *For* $n \geq 0$

$$GL_n^{(k)} = kGF_{n+1}^{(k)} - \sum_{j=1}^{k-1}(k-j)\,GF_{n+1-j}^{(k)}.$$

Proof. Theorem can be proved by mathematical induction on n.

If $n = 0$ and $k = 2$, then $GL_0^{(2)} = 2 - i$, $GF_0^{(2)} = i$ and $GF_1^{(2)} = 1$ and then

$$GL_0^{(2)} = 2GF_1^{(2)} - GF_0^{(2)}.$$

Also if $n = 0$ and $k > 2$, then $GL_0^{(k)} = k - i$. By the definition of the $k-$order Gaussian Fibonacci numbers for all $n \in Z^+$, it can be easily seen that

$$\begin{aligned}
kGF_1^{(k)} - \sum_{j=1}^{k-1} (k-j)\, GF_{1-j}^{(k)} &= kGF_1^{(k)} - (k-1)\, GF_0^{(k)} - \ldots - GF_{2-k}^{(k)} \\
&= k - 0 - 0 - \ldots - 0 - i \\
&= k - i \\
&= GL_0^{(k)}.
\end{aligned}$$

Suppose that the equation holds for n, that is

$$GL_n^{(k)} = kGF_{n+1}^{(k)} - \sum_{j=1}^{k-1} (k-j)\, GF_{n+1-j}^{(k)}.$$

Then for $n+1$, by the definition of the $k-$order Gaussian Lucas numbers

$$\begin{aligned}
GL_{n+1}^{(k)} &= GL_n^{(k)} + GL_{n-1}^{(k)} + GL_{n-2}^{(k)} \ldots + GL_{n+1-k}^{(k)} \\
&= \left(kGF_{n+1}^{(k)} - \sum_{j=1}^{k-1} (k-j)\, GF_{n+1-j}^{(k)} \right) \\
&\quad + \left(kGF_n^{(k)} - \sum_{j=1}^{k-1} (k-j)\, GF_{n-j}^{(k)} \right) \\
&\quad + \left(kGF_{n-1}^{(k)} - \sum_{j=1}^{k-1} (k-j)\, GF_{n-1-j}^{(k)} \right) + \ldots \\
&\quad + \left(kGF_{n+2-k}^{(k)} - \sum_{j=1}^{k-1} (k-j)\, GF_{n+1-k-j}^{(k)} \right)
\end{aligned}$$

$$\begin{aligned}
&= k\left(GF_{n+1}^{(k)} + GF_n^{(k)} + \ldots + GF_{n+2-k}^{(k)}\right) \\
&\quad - \sum_{j=1}^{k-1} (k-j) \left(GF_{n+1-j}^{(k)} + GF_{n-j}^{(k)} + GF_{n-1-j}^{(k)} + \ldots GF_{n+1-k-j}^{(k)}\right) \\
&= kGF_{n+2}^{(k)} - \sum_{j=1}^{k-1} (k-j)\, GF_{n+2-j}^{(k)}.
\end{aligned}$$

This completes the proof. □

Corollary 8. [8] *For* $n \geq 0$

$$GL_{n+1} = 2GF_{n+2} - GF_{n+1}.$$

Theorem 7. *For* $n \geq 0$

$$GL_n^{(k)} = \sum_{j=1}^{k} jGF_{n+1-j}^{(k)}.$$

Proof. Theorem can be proved by mathematical induction on n in a similar way to Theorem 6. □

Corollary 9. [8] *For* $n \geq 0$

$$GL_n = GF_n + 2GF_{n-1}.$$

Now we introduce the matrices Q_k, R_k and $E_n^{(k)}$ that plays the role of the Q-matrix. Let Q_k, R_k and $E_n^{(k)}$ denote the $k \times k$ matrices defined as

$$Q_k = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}_{k \times k},$$

$$R_k = \begin{bmatrix} GF_{k-1}^{(k)} & GF_{k-2}^{(k)} & GF_{k-3}^{(k)} & \cdots & GF_2^{(k)} & GF_1^{(k)} & 0 \\ GF_{k-2}^{(k)} & GF_{k-3}^{(k)} & GF_{k-4}^{(k)} & \cdots & GF_1^{(k)} & 0 & 0 \\ GF_{k-3}^{(k)} & GF_{k-4}^{(k)} & GF_{k-5}^{(k)} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ GF_2^{(k)} & GF_1^{(k)} & 0 & \cdots & \vdots & 0 & 0 \\ GF_1^{(k)} & 0 & 0 & \cdots & 0 & 0 & i \\ 0 & 0 & 0 & \cdots & 0 & i & 1-i \end{bmatrix}_{k \times k}$$

and

$$E_n^{(k)} = \begin{bmatrix} GF_{n+k-1}^{(k)} & GF_{n+k-2}^{(k)} & \cdots & GF_{n+1}^{(k)} & GF_n^{(k)} \\ GF_{n+k-2}^{(k)} & GF_{n+k-3}^{(k)} & \cdots & GF_n^{(k)} & GF_{k-3}^{(k)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ GF_{n+1}^{(k)} & GF_n^{(k)} & \cdots & GF_{n-k+3}^{(k)} & GF_{n-k+2}^{(k)} \\ GF_n^{(k)} & GF_{n-1}^{(k)} & \cdots & GF_{n-k+2}^{(k)} & GF_{n-k+1}^{(k)} \end{bmatrix}_{k \times k}.$$

Then we can give the following Lemma without proof and theorem:

Lemma 1. *Let $n \geq 1$. Then*

$$E_{n+1}^{(k)} = Q_k E_n^{(k)}.$$

Theorem 8. *Let $n \geq 1$. Then*

$$Q_k^n R_k = E_n^{(k)}.$$

Proof. By induction method. If $n = 1$, then from the definition of the matrix E_n and $k-$order Gaussian Fibonacci numbers,

$$Q_k R_k = E_1^{(k)}.$$

Assume that the theorem holds for n

$$Q_k^n R_k = E_n^{(k)}.$$

Then for $n+1$ we have

$$\begin{aligned} Q_k^{n+1} R_k &= Q_k Q_k^n R_k \\ &= Q_k E_n^{(k)} \\ &= E_{n+1}^{(k)}. \end{aligned}$$

□

Corollary 10. *Let $k = 2$. Then*

$$Q^n R = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 & i \\ i & 1-i \end{bmatrix} = \begin{bmatrix} GF_{n+1} & GF_n \\ GF_n & GF_{n-1} \end{bmatrix}.$$

Corollary 11. [4] *Let $k = 3$. Then*

$$\begin{aligned} Q_3^n R_3 &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1+i & 1 & 0 \\ 1 & 0 & i \\ 0 & i & 1-i \end{bmatrix} \\ &= \begin{bmatrix} GT_{n+2} & GT_{n+1} & GT_n \\ GT_{n+1} & GT_n & GT_{n-1} \\ GT_n & GT_{n-1} & GT_{n-2} \end{bmatrix}. \end{aligned}$$

4. Conclusion

In this paper we defined and studied the $k-$order Gaussian Fibonacci and Gaussian Lucas Numbers with boundary conditions. We identified and proved the generating functions, the Binet formulas, the summation formulas, the matrix representation of $k-$order Gausian Fibonacci numbers and some significant relationships between $k-$order Gaussian Fibonacci and $k-$order Lucas numbers connecting with usual $k-$order Fibonacci numbers.

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