Every 1-planar graph without 4-cycles or adjacent 5-vertices is 5-colorable*

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Abstract A graph is 1-planar if it can be drawn on the plane so that each edge is crossed by at most one other edge. In this paper, we prove that every 1-planar graph without 4-cycles or adjacent 5-vertices is 5-colorable.

Keywords: 5-colorable; 1-planar graph; cross vertices; cross faces.

1. Introduction

All graphs considered in this paper are finite, simple and undirected. Any undefined notation follows that of Bondy and Murty[1]. Let G be a 1-planar graph. We use V(G), E(G), F(G) and δ(G) to denote vertex set, edge set, face set and the minimum degree of G, respectively. For an element $x \in$ V(G)$\cup$E(G), $d(x)$ denotes the degree of x in G.

A *proper vertex coloring* of G is an assignment φ of integers (or labels) to the vertices of G in such a way that: $\varphi(u) \neq \varphi(v)$ if the vertex u and v are adjacent in G. A *k-coloring* is a proper vertex coloring with k colors. A coloring of graph G is called *improperly* $(d_1, \ldots, d_k)$*-colorable* or just $(d_1, \ldots, d_k)$*-colorable* if the vertex set of G can be partitioned into subsets $V_1, \ldots, V_k$ and the subgraph $G[V_i]$ induced by V_i has maximum degree at most d_i for $1 \leq i \leq k$. When $d_1 = \ldots = d_k = 0$, it is a *proper vertex coloring* of G. When $d_1 = \ldots = d_k = d \geq 1$, it is a *d-improper coloring*.

The *proper* and *d-improper coloring* of planar graphs have been widely investigated. In 1976-1977, the Four Color Problem was proved by Appel and Haken ([2-4]). In other words, every planar graph is $(0,0,0,0)$-*colorable*. Later in 1984, Bordin[5] showed that every 1-planar graph is 6-colorable. For *d-improper coloring*, Wang[11] proved planar graphs without

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4-cycles or 6-cycles are $(2,0,0)$-*colorable*; Bu[6] proved that planar graphs without cycles of length 4 or 6 are $(1,1,0)$-*colorable*; Wang[10] proved that planar graphs without cycles of length 4, 5 or 9 are $(1,0,0)$-*colorable*. For 1-planar graphs, there are few research results. In 2011, Fabiric[8] studied structures of 1-planar graphs. Later, Zhang studied edge colorings of 1-planar graphs in [12-13]. As far as we know, there is no result on improper vertex coloring of 1-planar graph. In this paper, we consider 1-improper coloring of 1-planar graphs without 4-cycles, and prove Theorem 2 that 1-planar graphs without cycles of length 4 are (1,1,1,1,1)-colorable. We also obtain the following Theorem.

Theorem 1. Let G be a 1-planar graph. If G contains no cycles of length 4 and 5-vertex is not adjacent to 5-vertex in G, then G is 5-colorable.

2. Notations

A vertex of degree k (*resp.* at least k, at most k) is called a *k-vertex* (*resp. k^+-vertex, k^--vertex*). The notation will be used for cycles and faces similarly. For $f \in F(G)$, we use $f = [u_1 u_2 \dots u_n]$ to denote the face f if $u_1, u_2, \dots, u_n$ are the boundary vertices of f in cyclic order. A 3-*face* $[u_1 u_2 u_3]$ is called an (m_1, m_2, m_3)-*face* if $d(u_i) = m_i$, for $i = 1, 2, 3$. Specially, a 3-*cycle* is synonymous with a triangle.

For a 1-planar graph G, we assume that G is the *best*. That is to say, we have drawn the graph G in the plane so that the number of the cross vertices is minimized. So if z is a cross vertex formed by the intersection of two edges x_1y_1 and x_2y_2, the four vertices are different. We use $C(G)$ to denote the set of cross vertex in G. $E_0(G)$ is the edge set in which the edge is not crossed by the other edges. The *associated plane graph* G^* of 1-planar graph G is the plane graph which is obtained from G by turning all crossings into new 4-vertices. We call these new 4-vertices *cross vertices*. If a vertex in G^* is not a cross vertex, we call it *true*. Similarly, we call a face in G^* *cross face* if there are some cross vertices on it. Otherwise it is a *normal face*. Specially, a 4-face is called a *bad 4-face* if it has two cross vertices which are not adjacent each other.

Here, we introduce some notations which will be used in this paper.
$m_{3f}(v)$: the number of cross 3-faces adjacent to v.
$m_3(v)$: the number of 3-faces adjacent to v.
$m_{4b}(v)$: the number of bad 4-faces adjacent to v.
$n_{5^+}(v)$: the number of 5^+-vertices adjacent to v.
These marks are also quite applicable to faces in G^*.

3.Structural Properties

Next, let $C = \{1,2,3,4,5\}$ denote the color set with five colors. The proof of Theorem 1 is given by contraction. Let G be a counterexample with the least number of vertices and edges embedded in the 1-plane. Obviously, G is connected. Moreover, every subgraph G' of G with fewer vertices is 5-*colorable*. In other words, $V(G')$ can be partitioned into five subsets $V_1, V_2, \dots, V_5$ such that $\Delta(G[V_1])=\Delta(G[V_2])= \dots =\Delta(G[V_5])=0$. Now, suppose that the vertices of $G[V_i]$ are colored with color i, $i = 1, 2, \ldots, 5$.

Claim 1. $\delta(G) \geq 5$.

proof: Suppose to the contrary that G contains a 4^--vertex v and $v_1, \ldots, v_k$ and the neighbors of v in cyclic order. Note that $1 \leq k \leq 4$. Let $G' = G - \{v\}$. By the minimality of G, G' is 5-*colorable* with the color set C. We call the coloring above φ. Since the color set C has 5 colors and v is a 4^--vertex, there is one color left. We may easily extend φ to G by properly coloring v with the color which is different from the color of vertices $v_1, v_2, \ldots, v_k$. So G is 5-*colorable*. This contradicts the choice of G. So $\delta(G) \geq 5$.

Claim 2[12]**.** Let G be a 1-planar graph and G^* be the *associated plane graph* of G. Then for any two cross vertices u and v in G^*, $uv \notin E(G^*)$.

4. Completing the proof of Theorem 1

By contradiction, we assume that G is not 5-*colorable*. Define a weight function ω on the vertices and faces of G^*. Let $\omega(x) = d(x) - 4$ if $x \in V(G^*) \cup F(G^*)$. It follows from Euler formula $|V(G^*)| - |E(G^*)| + |F(G^*)| = 2$ and the relation $\sum_{v \in V(G^*)} d(v) = \sum_{f \in F(G^*)} d(f) = 2|E(G^*)|$. So the total sum of weights of the vertices and faces in G^* is equal to

$$\sum_{v \in V(G^*)} (d(v) - 4) + \sum_{f \in F(G^*)} (d(f) - 4) = -8.$$

We shall design appropriate discharging rules and redistribute weights accordingly. Once the discharging is finished, a new function ω^* will be produced. The total sum of weight remains unchanged during the process of discharging. Nevertheless, the new weight function satisfies that $\omega^*(x) \geq 0$ for all $x \in V(G^*) \cup F(G^*)$ when the discharging is completed. This leads to the obvious contradiction.

$$-8 = \sum_{x \in V(G^*) \cup F(G^*)} \omega(x) = \sum_{x \in V(G^*) \cup F(G^*)} \omega^*(x) \geq 0.$$

So no such counterexample exists.

For $x, y \in V(G^*) \cup F(G^*)$, let $\tau(x \to y)$ denote the amount of weights transferred from x to y. The discharge rules are defined as follows:

(R1)Every bad 4-face sends $\frac{1}{3}$ to each adjacent cross 3-face;

(R2)Every 5^+-vertex sends $\frac{1}{3}$ to each adjacent 3-face;

(R3)Every 5^+-vertex sends $\frac{1}{6}$ to each adjacent bad 4-face;

(R4)Every cross 5^+-face sends $\frac{1}{3}$ to each cross 3-face which has a common cross vertex;

(R5)Every 6^+-face sends $\frac{1}{3}$ to a 5-vertex embedded in it;

(R6)Every cross 5-face sends $\frac{1}{6}$ to each 5-vertex embedded in it.

In the following part, we will prove that $\omega^*(x) \geq 0$ for all $x \in V(G^*) \cup F(G^*)$. Let $v_1, v_2, \ldots, v_{d(v)}$ denote the neighbors of v in G^* in cyclic order for any $v \in V(G^*)$.

Vertices.

Case 1. $d(v) = 4$.
In this case, v must be a cross vertex. Otherwise $\delta(G) \geq 5$ by claim 1. So $\omega(v) = 4 - 4 = 0$. It is trivial that $\omega^*(v) = \omega(v) = 0$ since v does not send any charge out according to discharge rules (R1)-(R6).

Case 2. $d(v) = 5$.
We can know that v must be *true* as the degree of cross vertex is 4. Now we consider if there are any cross vertices in the neighbors of v. If there are, we want to know how many they are and how they are generated. So we have the following two subcases.

Case2.1. If there is no cross vertex in the neighbors of v, then $d(v_i) \geq 5, i = 1, 2, \cdots, 5$. According to Claim 1 and the condition that G has no 4-cycles, $m_3(v) \leq \lfloor \frac{d(v)}{2} \rfloor = 2$. Since both v and its adjacent vertices are true, we have $m_{4b}(v) = 0$. By discharge rules (R2) and (R3),

$$\omega^*(v) = \omega(v) - \tfrac{1}{3}m_3(v) - \tfrac{1}{6}m_{4b}(v) \geq 5 - 4 - \tfrac{1}{3} \times 2 - \tfrac{1}{6} \times 0 = \tfrac{1}{3} > 0.$$

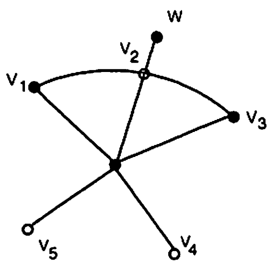

Fig 1

Case2.2. There are some cross vertices in the neighbors of v.

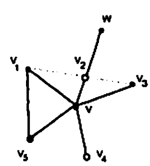
Fig 1.1

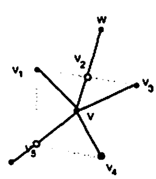
Fig 1.2

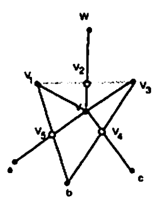
Fig 1.3

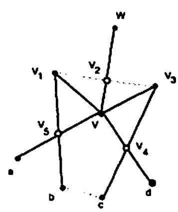
Fig 1.4

(a) See the graph in Fig.1. v is a 5-vertex, $v_1, v_2, \ldots, v_5$ are its neighbors in cyclic order and v_2 is a cross vertex generated by the crossing of edge v_1v_3 and vw. According to observations and the hypothesis that G contains no cycles of length 4, we can find that the graphs in Fig.1.1-Fig.1.4 do not exist. The difficult cases are illustrated in Fig.1.5 and Fig.1.6. In the Fig.1.5, we have $m_3(v) = 4$, $m_{4b}(v) = 0$ and v will be adjacent to a 6^+-face $f = [bv_5vv_4c \ldots b]$. So by (R2), (R3) and (R5), we can obtain that

$$\omega^*(v) = \omega(v) + \tfrac{1}{3}m_{6^+f}(v) - \tfrac{1}{3}m_3(v) - \tfrac{1}{6}m_{4b}(v) \geq$$
$$5 - 4 + \tfrac{1}{3} \times 1 - \tfrac{1}{3} \times 4 - \tfrac{1}{6} \times 0 = 0.$$

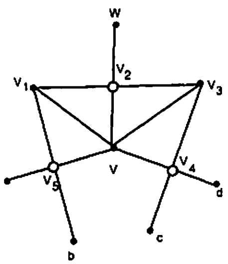

Fig1.5

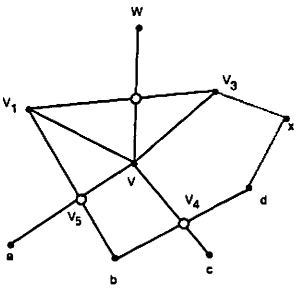

Fig 1.6

Also, the graph in Fig.1.6 may occur. In this situation, $m_3(v) = 3$, $m_{4b}(v) = 1$ and v may be adjacent to a cross 5-face $f = [vv_4dxv_3v]$. By (R2), (R3)and (R6), we can get

$$\omega^*(v) = \omega(v) + \tfrac{1}{6}m_{5^+f}(v) - \tfrac{1}{3}m_3(v) - \tfrac{1}{6}m_{4b}(v) \geq$$
$$5 - 4 + \tfrac{1}{6} \times 1 - \tfrac{1}{3} \times 3 - \tfrac{1}{6} \times 1 = 0.$$

If v is not adjacent to a cross 5^+-face, v will be adjacent to a 6^+-face, then v will get $\frac{1}{3}$.

(b) See the graph in Fig.2. v_2 is a cross vertex produced by the crossing of uv_3 and vw. Then $d(v_2) = 4$. There is no edges between v_2 and the other neighbors of v. According to (R2) and (R3), the more 3-faces and bad 4-faces appear, the worse the case is. Similarly, the cases in Fig.2.1 and Fig.1.4 will not happen. But the case showed in the graph Fig.2.2 is more complicated where $m_3(v) = 4$, $m_{4b}(v) = 0$, $m_{6^+f}(v) = 1$. So we can obtain

$$\omega^*(v) = \omega(v) + \tfrac{1}{3}m_{6^+f}(v) - \tfrac{1}{3}m_3(v) - \tfrac{1}{6}m_b4(v) \geq$$
$$5 - 4 + \tfrac{1}{3} \times 1 - \tfrac{1}{3} \times 4 - \tfrac{1}{6} \times 0 = 0.$$

Also, the graph in Fig.1.6 may occur. This is similar to Case 2.2(a).

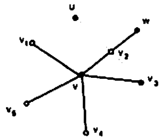

Fig2

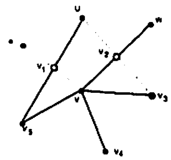

Fig2.1

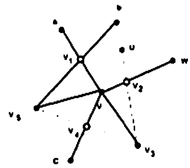

Fig2.2

(c) See the graph in Fig.3. v_2 is a cross vertex produced by the crossing of u_1u_2 and vw. Then $d(v_2) = 4$. There is no edge between v_2 and the other neighbors of v. The difficult cases are shown in Fig.3.1 and Fig.3.2.

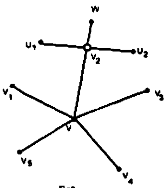

Fig3

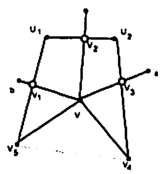

Fig3.1

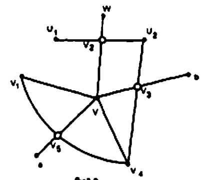

Fig3.2

In Fig.3.1, we have $m_3(v) = 2$, $m_{4b}(v) = 2$, $m_{5^+f}(v) \geq 0$. By (R2), (R3) and (R5), we can obtain

$$\omega^*(v) = \omega(v) + \tfrac{1}{6}m_{5^+f}(v) - \tfrac{1}{3}m_3(v) - \tfrac{1}{6}m_{4b}(v) \geq$$
$$5 - 4 + \tfrac{1}{6} \times 0 - \tfrac{1}{3} \times 2 - \tfrac{1}{6} \times 2 = 0.$$

In Fig.3.2, we have $m_3(v) = 3$, $m_{4b}(v) = 1$. There is no edge between v_1u_1. Otherwise $[v_1u_1u_2v_4v_1]$ is a cycle of length 4 which contradicts to the choice of G. So v must be adjacent to a cross 5^+-face. By (R2), (R3) and (R6), we have

$$\omega^*(v) = \omega(v) + \tfrac{1}{6}m_{5^+f}(v) - \tfrac{1}{3}m_3(v) - \tfrac{1}{6}m_{4b}(v) \geq$$
$$5 - 4 + \tfrac{1}{6} \times 1 - \tfrac{1}{3} \times 3 - \tfrac{1}{6} \times 1 = 0.$$

Case3. $d(v) = k (k \geq 6)$.

According to the discharge rules (R2) and (R3), v will send $\frac{1}{3}$ to the adjacent 3-faces and v will send $\frac{1}{6}$ to the adjacent bad 4-faces. So

$$\omega^*(v) = \omega(v) - \tfrac{1}{3}m_3(v) - \tfrac{1}{6}m_{4b}(v) \geq k - 4 - \tfrac{1}{3} \times k = \tfrac{2k}{3} - 4 \geq 0.$$

We have proved that $\omega^*(v) \geq 0$ for any vertex in G^* by now.

Faces.

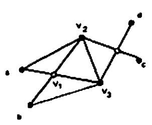

Fig4.1

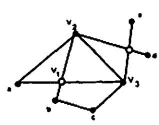

Fig4.2

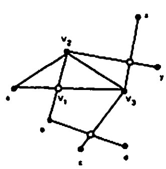

Fig4.3

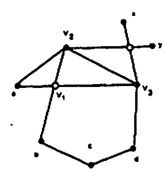

Fig4.4

Case1. $d(f) = 3$. We assume that $f = [v_1v_2v_3]$, $f \in$F(G*). If there is no cross vertex on f, then $d(v_i) \geq 5$ $(i = 1, 2, 3)$, $n_{5^+}(v) = 3$. By (R2), we can get

$$\omega^*(f) = \omega(f) + \tfrac{1}{3} \times n_{5^+}(v) = 3 - 4 + \tfrac{1}{3} \times 3 = 0.$$

Otherwise, there is just one *false vertex* on f by Claim2. Without loss of generality, assume v_1 is a cross vertex, then $n_{5^+}(f) \geq 2$. We can easily conclude that the graphs showed in Fig.4.1 and Fig.4.2 don't exist as G contains no cycles of length 4. So f must be adjacent to a *bad* 4-face or a cross 5^+-face as shown in Fig.4.3 and Fig.4.4. By discharge rules (R1) and (R2), v can get $\frac{1}{3}$. So we can get

$$\omega^*(f) = \omega(f) + \tfrac{1}{3} \times n_{5^+}(f) + \tfrac{1}{3} \times 1 = 3 - 4 + \tfrac{1}{3} \times 2 + \tfrac{1}{3} = 0.$$

Case2. $d(f) = 4$.

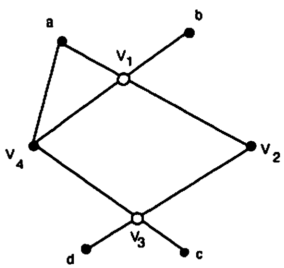

Fig 5

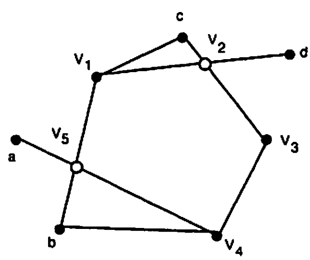

Fig 6

Case2.1. If f is not a *bad* 4-*face*, it does not send any charge out according to (R1)-(R6). So

$$\omega^*(f) = \omega(f) = 4 - 4 = 0.$$

Case2.2. f is a *bad* 4-*face*, as shown in Fig.5 $f = [v_1v_2v_3v_4]$ and $d(v_1) = d(v_3) = 4$. Suppose that $f = [av_1v_4]$ is a cross 3-face, then $bv_2 \notin$ E(G), $dv_4 \notin$ E(G), $cv_2 \notin$ E(G). Otherwise, it will form 4-cycles. So $m_{3f}(f) \leq 1$, $n_{5^+}(f) = 2$. By (R1) and (R3)

$$\omega^*(f) = \omega(f) + \tfrac{1}{6} \times n_{5^+}(f) - \tfrac{1}{3} \times m_{3f}(f) \geq 4 - 4 + \tfrac{1}{6} \times 2 - \tfrac{1}{3} \times 1 = 0.$$

Case3. $d(f) = 5$.

Case3.1. If f is a *normal* 5-face, there is no cross vertex on the boundary of f. We have $\omega^*(f) = \omega(f) = 5 - 4 = 1 > 0$ by discharge rules.

Case3.2. If f is a cross 5-face, the number of cross vertices is at most 2 by Claim 2. By the graph in Fig.6, $f = [v_1v_2v_3v_4v_5]$ where v_2 and v_5 are cross vertices. Then v_1, v_3, v_4 must be *true* vertices. If $bv_4 \in$ E(G), then $av_1 \notin$ E(G). The number of 3-faces which have a common cross vertex with f, $m_{3f}(f)$ satisfies that $m_{3f}(f) \leq 2$. By the hypotheses, 5-vertex is not adjacent to 5-vertex, we can obtain $n_5(f) \leq 2$. By discharge rules (R4) and (R6), we have

$$\omega^*(f) = \omega(f) - \frac{1}{3} \times m_{3f}(f) - \frac{1}{6} \times n_5(f) \geq 5 - 4 - \frac{1}{3} \times 2 - \frac{1}{6} \times 2 \geq 0.$$

Case4. $d(f) = k$ $(k \geq 6)$.

Case4.1 If f is a *normal* k-face $(k \geq 6)$, $n_5(f) \leq k$. By (R5), we can get

$$\omega^*(f) = \omega(f) - \frac{1}{3} \times n_5(f) \geq k - 4 - \frac{1}{3} \times k = \frac{2k}{3} - 4 \geq 0 (k \geq 6).$$

Case4.2 f is a cross k-face. The worst case is that all the vertices on the boundary are cross vertices or 5-vertices. When k=7, see the graphs shown in Fig.7. However, every cross vertex produces at most one cross 3-face. So $m_{3f}(f) + n_5(f) \leq k$. Then

$$\omega^*(f) = \omega(f) - \frac{1}{3} \times m_{3f}(f) - \frac{1}{3} \times n_5(f) = k - 4 - \frac{1}{3} \times k = \frac{2k}{3} - 4 \geq 0.$$

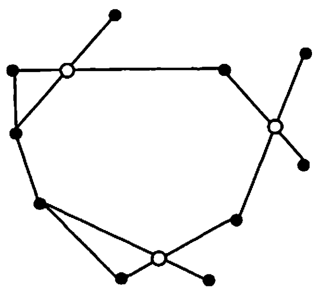

Fig7 k=7

By now, we have proved that $\omega^*(f) \geq 0$ for any face in G*.

Through the discussion above, we have proved $\sum_{x \in V(G^*) \cup F(G^*)} \omega^*(x) \geq 0$. This leads to the obvious contradiction.

$$-8 = \sum_{x \in V(G^*) \cup F(G^*)} \omega(x) = \sum_{x \in V(G^*) \cup F(G^*)} \omega^*(x) \geq 0.$$

The proof is completed .

Theorem 2. 1-planar graphs without cycles of length 4 are (1,1,1,1,1)-colorable.

Assume that G is a minimum counterexample. Then 5-vertex is not adjacent to 5-vertex in G.

Proof. Suppose to the contrary that a 5-vertex v is adjacent to a 5-vertex x. Let v_1,v_2,v_3,v_4 denote the other neighbors of v. Let G$'$=G - $\{v, x\}$. Clearly, G$'$ is $(1,1,1,1,1)$-*colorable* by the minimality of G. let φ denote the $(1,1,1,1,1)$-*coloring* with the colors in C. First, we can properly color x. If $\{\varphi(x), \varphi(v_1), \varphi(v_2), \varphi(v_3), \varphi(v_4)\} \neq C$, we may color v with a color in $C \setminus \{\varphi(x), \varphi(v_1), \varphi(v_2), \varphi(v_3), \varphi(v_4)\}$. Otherwise, $\{\varphi(x), \varphi(v_1), \varphi(v_2), \varphi(v_3), \varphi(v_4)\}=C$. We can color v with the color of x. It is not difficult to check that the resulting coloring of G is a (1,1,1,1,1)-*colorable* which is a contraction. So 5-vertex is not adjacent to 5-vertex in G.

In the following part, the proof is similar to the one in Theorem 1. So 1-planar graph without cycles of length 4 is (1,1,1,1,1)-*colorable*.

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