Lower bounds on the signed k-domination number of graphs

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Abstract

Let G be a graph with vertex set $V(G)$. For any integer $k \geq 1$, a signed k-dominating function is a function $f : V(G) \rightarrow \{-1, 1\}$ satisfying $\sum_{x \in N[v]} f(x) \geq k$ for every $v \in V(G)$, where $N[v]$ is the closed neighborhood of v. The minimum of the values $\sum_{v \in V(G)} f(v)$, taken over all signed k-dominating functions f, is called the signed k-domination number. In this note we present some new lower bounds on the signed k-domination number of a graph. Some of our results improve known bounds.

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1 Terminology and Introduction

Let G be a finite graph with vertex set $V = V(G)$ and edge set $E = E(G)$. We use [4] for terminology and notations which are not defined here. The *order* of G is given by $n = n(G) = |V|$ and its *size* by $m = m(G) = |E|$. If $v \in V(G)$, then $N(v) = N_G(v)$ is the *open neighborhood* of v, and $N[v] = N(v) \cup \{v\}$ is the *closed neighborhood* of v. The *degree* $d(v) = d_G(v)$ of a vertex $v \in V(G)$ is defined by $d(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. If $S \subseteq V(G)$, then $G[S]$ is the subgraph of G induced by S. For disjoint subsets S and T of vertices of a graph G, we let $[S, T]$ denote the set of edges between S and T. Let $S \subseteq V(G)$. For a real-valued function $f : V(G) \to R$ we define $f(S) = \sum_{v \in S} f(v)$. The weight of f is $f(V(G))$.

Let $k \geq 1$ be an integer, and let G be a graph with minmum degree $\delta \geq k-1$. A *signed k-dominating function*, abbreviated SkDF, of G is defined by Changping Wang in [7] as a function $f : V(G) \to \{-1, 1\}$ such that $f(N[v]) \geq k$ for every $v \in V(G)$. The minimum of the values of $f(V(G))$, taken over all signed k-domination functions f, is called the *signed k-domination number*, abbreviated SkDN, of G and is denoted by $\gamma_{sk}(G)$. As the condition $\delta \geq k-1$ is clearly necessary, we will always assume that when we discuss $\gamma_{sk}(G)$ all graphs involved satisfy $\delta \geq k-1$.

If $k = 1$, then $\gamma_{s1}(G) = \gamma_s(G)$ is the classical signed domination number, introduced by Dunbar, Hedetniemi, Henning and Slater [3] and investigated, for example, in [2, 5, 8].

In this paper, we derive some new lower bounds on $\gamma_{sk}(G)$ in terms of several different graph parameters, as order, size, maximum degree and minimum degree. We improve some results of Atapour, Sheikholeslami, Hajypory and Volkmann [1] and Wang [7]. In addition, many of our bounds extend inequalities given by Chen and Song [2], Dunbar, Hedetniemi, Henning and Slater [3], Henning and Slater [5] as well as Zhang, Xu, Li and Liu [8] for $k = 1$.

2 Lower bounds

First, we introduce some notations. Let $k \geq 1$ be an integer, and let G be a graph of order n with minimum degree $\delta \geq k-1$. If $f : V(G) \longrightarrow \{-1, 1\}$ is a minimum SkDF of G, then we define the sets $P = \{v \in V | f(v) = 1\}$ and $M = \{v \in V | f(v) = -1\}$. Therefore $\gamma_{sk}(G) = |P| - |M| = 2|P| - n = n - 2|M|$. Furthermore, let V_o and V_e be the sets of vertices v with the property that $d(v) - k \equiv 0 \pmod 2$ and $d(v) - k \equiv 1 \pmod 2$, respectively.

Our first lemma is important for our investigations.

Lemma 1. *Using the notations above, the following inequalities are valid.*

(a) $\lceil \frac{\delta+k+1}{2} \rceil |M| \leq |[P, M]| \leq \lfloor \frac{\Delta+1-k}{2} \rfloor |P|$,

(b) $2|E(G[P])| \geq 2|E(G[M])| + (k-1)n + 2|M| + |V_o|$,

(c) $2|E(G[P])| + |[M, P]| \geq 4|E(G[M])| + (k-1)n + (k+3)|M| + |V_o|$.

Proof. (a) Let f be a minimum SkDF of G. Let $v \in M$. Since $f(N[v]) \geq k$, we have $2|N(v) \cap P| \geq d(v)+k+1$ and therefore $|N(v) \cap P| \geq \left\lceil \frac{d(v)+k+1}{2} \right\rceil \geq \lceil \frac{\delta+k+1}{2} \rceil$. This leads to $|[M, P]| \geq \lceil \frac{\delta+k+1}{2} \rceil |M|$. Now let $v \in P$. Since $f(N[v]) \geq k$, we see that $2|N(v) \cap M| \leq d(v) + 1 - k$ and therefore $|N(v) \cap M| \leq \left\lfloor \frac{d(v)+1-k}{2} \right\rfloor \leq \lfloor \frac{\Delta+1-k}{2} \rfloor$. This yields $|[P, M]| \leq \lfloor \frac{\Delta+1-k}{2} \rfloor |P|$.

(*b*) Let f be a minimum SkDF of G. First we derive a lower bound on $|[M,P]|$. Let $v \in M$. Since $f(N[v]) \geq k$, we observe that $|N(v) \cap M| \leq |N(v) \cap P| - k - 1$ and $|N(v) \cap M| \leq |N(v) \cap P| - k - 2$ when $v \in M \cap V_o$. We deduce that

$$\begin{aligned}
2|E(G[M])| &= \sum_{v \in M} |N(v) \cap M| \\
&= \sum_{v \in M \cap V_e} |N(v) \cap M| + \sum_{v \in M \cap V_o} |N(v) \cap M| \\
&\leq \sum_{v \in M \cap V_e} (|N(v) \cap P| - k - 1) \\
&+ \sum_{v \in M \cap V_o} (|N(v) \cap P| - k - 2) \\
&= |[M,P]| - (k+1)|M| - |M \cap V_o|.
\end{aligned}$$

This implies

$$|[M,P]| \geq (k+1)|M| + 2|E(G[M])| + |M \cap V_o|. \tag{1}$$

Now let $v \in P$. Since $f(N[v]) \geq 1$, we have $|N(v) \cap P| \geq |N(v) \cap M| + k - 1$ and $|N(v) \cap P| \geq |N(v) \cap M| + k$ when $v \in P \cap V_o$. It follows that

$$\begin{aligned}
2|E(G[P])| &= \sum_{v \in P} |N(v) \cap P| = \sum_{v \in P \cap V_e} |N(v) \cap P| + \sum_{v \in P \cap V_o} |N(v) \cap P| \\
&\geq \sum_{v \in P \cap V_e} (|N(v) \cap M| + k - 1) + \sum_{v \in P \cap V_o} (|N(v) \cap M| + k) \\
&= |[M,P]| + k|P| - |P \cap V_e| \\
&= |[M,P]| + (k-1)|P| + |P \cap V_o|
\end{aligned}$$

Combining this inequality chain with (1), we obtain (b) as follows

$$\begin{aligned}
2|E(G[P])| &\geq |[M,P]| + (k-1)|P| + |P \cap V_o| \\
&\geq 2|E(G[M])| + (k+1)|M| + |M \cap V_o| \\
&+ (k-1)|P| + |P \cap V_o| \\
&= 2|E(G[M])| + (k-1)n + 2|M| + |V_o|.
\end{aligned}$$

(*c*) Using (b) and (1), we deduce that

$$\begin{aligned}
2|E(G[P])| + |[M,P]| &\geq 2|E(G[M])| + (k-1)n + 2|M| + |V_o| \\
&+ 2|E(G[M])| + (k+1)|M| \\
&= 4|E(G[M])| + (k-1)n + (k+3)|M| + |V_o|,
\end{aligned}$$

and (c) is proved. □

Theorem 2. *Let $k \geq 1$ be and integer, and let G be a graph of order n, size m, maximum degree Δ and minimum degree $\delta \geq k-1$. Then*

$$\gamma_{sk}(G) \geq \frac{(\lceil \frac{\delta+k+1}{2} \rceil - \lfloor \frac{\Delta+1-k}{2} \rfloor)n}{\lfloor \frac{\Delta+1-k}{2} \rfloor + \lceil \frac{\delta+k+1}{2} \rceil}, \tag{2}$$

$$\gamma_{sk}(G) \geq \frac{(2k+\delta-\Delta)n + 2|V_o|}{\Delta+\delta+2}, \tag{3}$$

$$\gamma_{sk}(G) \geq \frac{2m+(k+1)n+|V_o|}{\Delta+1} - n, \tag{4}$$

$$\gamma_{sk}(G) \geq n - \frac{2m-(k-1)n-|V_o|}{\delta+1}, \tag{5}$$

$$\gamma_{sk}(G) \geq \left\lceil \frac{(3k+1-3\Delta-2\lfloor \frac{\Delta+1-k}{2} \rfloor)n + 2|V_o| + 8m}{3\Delta+k+3+2\lfloor \frac{\Delta+1-k}{2} \rfloor} \right\rceil, \tag{6}$$

$$\gamma_{sk}(G) \geq \left\lceil \frac{(3k+1+3\delta-2\lfloor \frac{\Delta+1-k}{2} \rfloor)n + 2|V_o| - 4m}{3\delta+k+3+2\lfloor \frac{\Delta+1-k}{2} \rfloor} \right\rceil. \tag{7}$$

Proof. (i) Lemma 1 (a) implies that $\lceil \frac{\delta+k+1}{2} \rceil |M| \leq \lfloor \frac{\Delta+1-k}{2} \rfloor |P|$. Using this inequality and $|P| = \frac{n+\gamma_{sk}(G)}{2}$ and $|M| = \frac{n-\gamma_{sk}(G)}{2}$, the bound (2) is easy to verify.

(ii) We note that

$$2|E(G[P])| = \sum_{v \in P} |N(v) \cap P| = \sum_{v \in P} (d(v) - |N(v) \cap M|) \leq \Delta|P| - |[P, M]| \tag{8}$$

and

$$2|E(G[M])| = \sum_{v \in M} |N(v) \cap M| = \sum_{v \in M} (d(v) - |N(v) \cap P|) \geq \delta|M| - |[P, M]|. \tag{9}$$

By (8), (9) and Lemma 1 (b), we conclude that

$$\begin{aligned} \Delta|P| &\geq 2|E(G[P])| + |[P, M]| \\ &\geq 2|E(G[M])| + (k-1)n + 2|M| + |V_o| + \delta|M| - 2|E(G[M])| \\ &= (k-1)n + (\delta+2)|M| + |V_o|. \end{aligned}$$

Using this inequality chain and again $|P| = \frac{n+\gamma_{sk}(G)}{2}$ and $|M| = \frac{n-\gamma_{sk}(G)}{2}$, we obtain the lower bound (3).

(*iii*) According to (8), (9) and Lemma 1 (b), we conclude that

$$\begin{aligned}\sum_{v\in P} d(v) &= \sum_{v\in M} d(v) + 2|E(G[P])| - 2|E(G[M])| \\ &\geq \sum_{v\in M} d(v) + (k-1)n + 2|M| + |V_o|. \qquad (10)\end{aligned}$$

It follows that

$$\begin{aligned}2\Delta|P| &\geq 2\sum_{v\in P} d(v) \geq \sum_{v\in V} d(v) + (k-1)n + 2|M| + |V_o| \\ &= 2m + (k-1)n + 2(n-|P|) + |V_o|,\end{aligned}$$

and thus

$$2|P| \geq \frac{2m + (k+1)n + |V_o|}{\Delta + 1}.$$

Using this inequality, we obtain (4) as follows

$$\gamma_{sk}(G) = 2|P| - n \geq \frac{2m + (k+1)n + |V_o|}{\Delta + 1} - n.$$

(*iv*) Applying (10), we observe that

$$\begin{aligned}2m &= \sum_{v\in P} d(v) + \sum_{v\in M} d(v) \geq 2\sum_{v\in M} d(v) + (k-1)n + 2|M| + |V_o| \\ &\geq 2\delta|M| + (k-1)n + 2|M| + |V_o|\end{aligned}$$

and therefore

$$2|M| \leq \frac{2m - (k-1)n - |V_o|}{\delta + 1}.$$

Consequently,

$$\gamma_{sk}(G) = n - 2|M| \geq n - \frac{2m - (k-1)n - |V_o|}{\delta + 1},$$

and (5) is proved.

(*v*) Because of $4m = 4|E(G[M])| + 4|[P,M]| + 4|E(G[P])|$ and Lemma 1 (c), we deduce that

$$(k-1)n + (k+3)|M| + |V_o| + 4m \leq 6|E(G[P])| + 5|[P,M]|.$$

Applying (8), we obtain

$$(k-1)n + (k+3)|M| + |V_o| + 4m \leq 3\Delta|P| + 2|[P,M]|.$$

By Lemma 1 (a), we have $2|[P,M]| \leq 2\lfloor\frac{\Delta+1-k}{2}\rfloor|P|$ and therefore

$$(k-1)n+(k+3)|M|+|V_o|+4m \leq \left(3\Delta+2\left\lfloor\frac{\Delta+1-k}{2}\right\rfloor\right)|P|.$$

Using this inequality, it is a simple matter to obtain (6).

(vi) Combining $2m = 2|E(G[M])|+2|[P,M]|+2|E(G[P])|$ with Lemma 1 (c), we find that

$$(k-1)n+(k+3)|M|+6|E(G[M])|+|V_o|+|[P,M]| \leq 2m.$$

Applying (9), we conclude that

$$(k-1)n+(k+3+3\delta)|M|+|V_o|-2|[P,M]| \leq 2m.$$

According to Lemma 1 (a), we have $2|[P,M]| \leq 2\lfloor\frac{\Delta+1-k}{2}\rfloor|P|$ and so the last inequality yields

$$(k-1)n+(k+3+3\delta)|M|+|V_o|-2\left\lfloor\frac{\Delta+1-k}{2}\right\rfloor|P| \leq 2m.$$

This implies inequality (7). □

If G is an r-regular graph of order n, then (2) leads to $\gamma_{sk}(G) \geq \frac{kn}{r+1}$ when $r-k \equiv 1 \pmod 2$ and $\gamma_{sk}(G) \geq \frac{(k+1)n}{r+1}$ when $r-k \equiv 0 \pmod 2$. This is a result by Wang [7]. For the special case $k=1$, these bounds can be found in [3] and [5]. In addition, (3) is slightly better than Corollary 2.9 in the article [1]. We note that (3) and (4) imply results in [8] and [2] for the case $k=1$.

For the complete graph K_n, Wang [7] has proved that $\gamma_{sk}(K_n) = k$ when $n-k \equiv 0 \pmod 2$ and $\gamma_{sk}(K_n) = k+1$ when $n-k \equiv 1 \pmod 2$. It is straightforward to verify that K_n fulfills inequalities (2) - (7) with equality, and therefore all these bounds are sharp.

Moreover, Wang [7] presented the following lower bound on the signed k-domination number of graphs.

Theorem 3. [7] *If G is a graph of order n and size m, then*

$$\gamma_{sk}(G) \geq \frac{(2k+1)n-2m}{k+2},$$

and this bound is sharp.

The special case $k=1$ of Theorem 3 can be found in [8]. Next we will improve the bound of Theorem 3 for $\delta \geq k$.

Theorem 4. *Let $k \geq 1$ be an integer, and let G be a graph of order n, size m and minimum degree $\delta \geq k$. Then*

$$\gamma_{sk}(G) \geq \frac{(3\lceil \frac{\delta+k+1}{2} \rceil + k - 1)n - 4m}{3\lceil \frac{\delta+k+1}{2} \rceil - k + 1}.$$

Proof. Since $|M| = n - |P|$, it follows from Lemma 1 (a) that

$$\|[P, M]\| \geq |M| \left\lceil \frac{\delta + k + 1}{2} \right\rceil = (n - |P|) \left\lceil \frac{\delta + k + 1}{2} \right\rceil. \tag{11}$$

On the other hand $|N(v) \cap M| \leq |N(v) \cap P| - k + 1$ for each vertex $v \in P$, and so

$$\begin{aligned} \|[P, M]\| &= \sum_{v \in P} |N(v) \cap M| \leq \sum_{v \in P} (|N(v) \cap P| - k + 1) \\ &= 2|E(G[P])| - (k-1)|P|. \end{aligned}$$

Combining the last inequality chain with (11), we obtain

$$\begin{aligned} |E(G[P])| &\geq \frac{\|[P, M]\| + (k-1)|P|}{2} \\ &\geq \frac{n\lceil \frac{\delta+k+1}{2} \rceil - |P|\lceil \frac{\delta+k+1}{2} \rceil + (k-1)|P|}{2}. \end{aligned}$$

and hence by (11)

$$\begin{aligned} m &\geq |E(G[P])| + \|[P, M]\| \\ &\geq \frac{n\lceil \frac{\delta+k+1}{2} \rceil - |P|\lceil \frac{\delta+k+1}{2} \rceil + (k-1)|P|}{2} \\ &+ n \left\lceil \frac{\delta + k + 1}{2} \right\rceil - |P| \left\lceil \frac{\delta + k + 1}{2} \right\rceil. \end{aligned}$$

Using $|P| = \frac{n+\gamma_{sk}(G)}{2}$ we deduce the desired bound. □

Note that for $\delta \geq k$

$$\frac{(3\lceil \frac{\delta+k+1}{2} \rceil + k - 1)n - 4m}{3\lceil \frac{\delta+k+1}{2} \rceil - k + 1} \geq \frac{(2k+1)n - 2m}{k+2}$$

is equivalent to

$$2m \left(\left\lceil \frac{\delta + k + 1}{2} \right\rceil - k - 1 \right) \geq (k-1)n \left(\left\lceil \frac{\delta + k + 1}{2} \right\rceil - k - 1 \right).$$

Since $2m \geq (k-1)n$ and $\lceil \frac{\delta+k+1}{2} \rceil \geq k + 1$ for $\delta \geq k$, the last inequality is valid. Therefore Theorem 4 is an improvement of Theorem 3.

Theorem 5. *If $k \geq 1$ is an integer and G a graph of order n and size m, then*

$$\gamma_{sk}(G) \geq \frac{k}{2} + \frac{1}{2}\sqrt{k^2 + 4(2m + (k+1)n + |V_o|)} - n.$$

Proof. Obviously,

$$\sum_{v \in P}(|N(v) \cap P| \leq \sum_{v \in P}(|P| - 1) = |P|(|P| - 1) = |P|^2 - |P|.$$

Using the inequality $2|N(v) \cap P| \geq d(v) + k - 1$ for $v \in P$, we obtain

$$\begin{aligned} 2|P|^2 &\geq 2\sum_{v \in P}|N(v) \cap P| + 2|P| \geq \sum_{v \in P}(d(v) + k - 1) + 2|P| \\ &= \sum_{v \in P} d(v) + (k+1)|P|. \end{aligned} \tag{12}$$

Using (10), it follows that

$$\begin{aligned} 2\sum_{v \in P} d(v) &\geq \sum_{v \in P} d(v) + \sum_{v \in M} d(v) + (k-1)n + 2|M| + |V_o| \\ &= 2m + (k-1)n + 2(n - |P|) + |V_o| \\ &= 2m + (k+1)n - 2|P| + V_o|. \end{aligned}$$

Applying this and inequality (12), we deduce that

$$4|P|^2 \geq 2m + (k+1)n + |V_o| + 2k|P|$$

and so

$$|P|^2 - \frac{k}{2}|P| - \frac{2m + (k+1)n + |V_o|}{4} \geq 0.$$

This implies that

$$|P| \geq \frac{k}{4} + \frac{1}{4}\sqrt{k^2 + 4(2m + (k+1)n + |V_o|)},$$

and we arrive at

$$\gamma_{sk}(G) = 2|P| - n \geq \frac{k}{2} + \frac{1}{2}\sqrt{k^2 + 4(2m + (k+1)n + |V_o|)} - n.$$

□

Theorem 1 (5) in article [2] by Chen and Song is the special case $k = 1$ of Theorem 5.

Theorem 6. *If $k \geq 1$ is an integer and G a graph of order n and minimum degree $\delta \geq k-1$, then*

$$\gamma_{sk}(G) \geq \frac{1}{2}\left(k-1-\delta+\sqrt{(k-1-\delta)^2+8n(\delta+k+1)+8|V_o|}\right)-n.$$

Proof. Using (10), we obtain

$$\begin{aligned} 2|P|(|P|-1) &\geq 2\sum_{v\in P}|N(v)\cap P| \geq \sum_{v\in P}(d(v)+k-1) \\ &= \sum_{v\in P} d(v)+(k-1)|P| \\ &\geq \sum_{v\in M} d(v)+(k-1)n+|V_o|+2|M|+(k-1)|P| \\ &\geq \delta n+(k-3-\delta)|P|+(k+1)n+|V_o|. \end{aligned}$$

This leads to

$$|P|^2+\frac{\delta+1-k}{2}|P|-\frac{(k+1)n+\delta n+|V_o|}{2} \geq 0$$

and thus

$$|P| \geq \frac{1}{4}\left(k-1-\delta+\sqrt{(k-1-\delta)^2+8n(\delta+k+1)+8|V_o|}\right).$$

Combining this inequality with $\gamma_{sk}(G)=2|P|-n$, we arrive at the desired bound. □

A graph is K_p-free if it does not contain the complete graph K_p as a subgraph. For our next lower bound, we use the following well-known Theorem of Turán [6]

Theorem 7. [6] *If G is a K_{r+1}-free graph of order n, then*

$$|E(G)| \leq \frac{r-1}{2r}\cdot n^2.$$

Theorem 8. *Let $k \geq 1$ and $r \geq 2$ be integers, and let G be a K_{r+1}-free graph of order n. If $c=\lceil(\delta(G)+k+1)/2\rceil$, then*

$$\gamma_{sk}(G) \geq \frac{r}{r-1}\left(-(c-k+1)+\sqrt{(c-k+1)^2+4\frac{r-1}{r}cn}\right)-n.$$

Proof. By Lemma 1 (a), we have

$$\|[P,M]\| \geq \left\lceil\frac{\delta(G)+k+1}{2}\right\rceil|M| = c|M| = c(n-|P|). \qquad (13)$$

Furthermore, Theorem 7 leads to

$$\begin{aligned} \|[P,M]\| &= \sum_{v\in P}|N(v)\cap M| \le \sum_{v\in P}(|N(v)\cap P| - k + 1) \\ &= 2|E(G[P])| - (k-1)|P| \le \frac{r-1}{r}|P|^2 - (k-1)|P|. \end{aligned}$$

Combining this inequality chain with (13), we obtain

$$c(n - |P|) \le \frac{r-1}{r}|P|^2 - (k-1)|P|$$

and thus

$$\frac{r-1}{r}|P|^2 + (c-k+1)|P| - cn \ge 0$$

and so

$$|P|^2 + \frac{r}{r-1}(c-k+1)|P| - \frac{r}{r-1}cn \ge 0.$$

It follows that

$$|P| \ge \frac{r}{2(r-1)}\left(-(c-k+1) + \sqrt{(c-k+1)^2 + 4\frac{r-1}{r}cn}\right),$$

and this leads to the desired bound

$$\begin{aligned} \gamma_{sk}(G) &= 2|P| - n \\ &\ge \frac{r}{r-1}\left(-(c-k+1) + \sqrt{(c-k+1)^2 + 4\frac{r-1}{r}cn}\right) - n. \end{aligned}$$

□

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