

MINIMAL ZERO-SUM SEQUENCES OF LENGTH FIVE OVER CYCLIC GROUPS OF PRIME POWER ORDER

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ABSTRACT. Let G be a finite cyclic group. Every sequence S of length l over G can be written in the form $S = (x_1g) \cdot \dots \cdot (x_lg)$ where $g \in G$ and $x_1, \dots, x_l \in [1, \text{ord}(g)]$, and the index $\text{ind}(S)$ of S is defined to be the minimum of $(x_1 + \dots + x_l)/\text{ord}(g)$ over all possible $g \in G$ such that $\langle g \rangle = G$. Recently the second and the third authors determined the index of any minimal zero-sum sequence S of length 5 over a cyclic group of a prime order where $S = g^2 \cdot (x_2g) \cdot (x_3g) \cdot (x_4g)$. In this paper, we determine the index of any minimal zero-sum sequence S of length 5 over a cyclic group of a prime power order. It is shown that if $G = \langle g \rangle$ is a cyclic group of prime power order $n = p^\mu$ with $p \geq 7$ and $\mu \geq 2$, and $S = (x_1g) \cdot (x_2g) \cdot (x_3g) \cdot (x_4g) \cdot (x_5g)$ with $x_1 = x_2$ is a minimal zero-sum sequence with $\gcd(n, x_1, x_2, x_3, x_4, x_5) = 1$, then $\text{ind}(S) = 2$ if and only if $S = (mg) \cdot (mg) \cdot (m\frac{n-1}{2}g) \cdot (m\frac{n+3}{2}g) \cdot (m(n-3)g)$ where m is a positive integer such that $\gcd(m, n) = 1$.

1. INTRODUCTION

Throughout the paper G is assumed to be a finite cyclic group of order n written additively. Denote by $\mathcal{F}(G)$, the free abelian monoid with basis G and elements of $\mathcal{F}(G)$ are called *sequences* over G . A sequence of length l of not necessarily distinct elements from G can be written in the form $S = (x_1g) \cdot \dots \cdot (x_lg)$ for some $g \in G$. S is called a *zero-sum sequence* if the sum of S is zero (i.e. $\sum_{i=1}^l x_i g = 0$). If S is a zero-sum sequence,

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but no proper nontrivial subsequence of S has sum zero, then S is called a *minimal zero-sum sequence*. The index of a sequence S over G is defined as follows.

Definition 1.1. For a sequence over G

$$S = (x_1g) \cdot \dots \cdot (x_lg), \text{ where } 1 \leq x_1, \dots, x_l \leq \text{ord}(g),$$

the index of S is defined by $\text{ind}(S) = \min\{\|S\|_g \mid g \in G \text{ with } G = \langle g \rangle\}$ where

$$\|S\|_g = \frac{x_1 + \dots + x_l}{\text{ord}(g)}.$$

Clearly, S has sum zero if and only if $\text{ind}(S)$ is an integer. There are also slightly different definitions of the index in the literature, but they are all equivalent (see Lemma 5.1.2 in [7]).

The index of a sequence is a crucial invariant in the investigation of (minimal) zero-sum sequences (resp. of zero-sum free sequences) over cyclic groups. It was first addressed by Kleitman-Lemke (in the conjecture [9, page 344]), used as a key tool by Geroldinger ([6, page 736]), and then investigated by Gao [3] in a systematical way. Since then it has received a great deal of attention (see for example [1, 2, 4, 5, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]).

A main focus of the investigation of index is to determine minimal zero-sum sequences of index 1. If S is a minimal zero-sum sequence of length $|S|$ such that $|S| \leq 3$ or $|S| \geq \lfloor \frac{n}{2} \rfloor + 2$, then $\text{ind}(S) = 1$ (see [1, 14, 20]). In contrast to that, it was shown that for each l with $5 \leq l \leq \lfloor \frac{n}{2} \rfloor + 1$, there is a minimal zero-sum sequence S of length $|S| = l$ with $\text{ind}(S) \geq 2$ ([14, 20]) and that the same is true for $l = 4$ and $\text{gcd}(n, 6) \neq 1$ ([13]). In recent papers [10, 11, 18], the authors proved that $\text{ind}(S) = 1$ if $|S| = 4$ and $\text{gcd}(n, 6) = 1$ when n is a prime power or a product of two prime powers. When n is a product of at least 3 prime powers, some partial results were also obtained in [15, 16, 17]. However, the general case is still open.

It was mentioned in [12], in order to further investigate the index of a general minimal zero-sum sequence of length 4, it is helpful to determine the index of certain minimal zero-sum sequences of length 5. It is routine to check that if S is a minimal zero-sum sequence over G of length 5, then $1 \leq \text{ind}(S) \leq 2$. Let $h(S)$ be the maximal repetition of an element in S .

In [12], the index of any minimal zero-sum sequence S of length 5 over a cyclic group of a prime order with $h(S) \geq 2$ was completely determined. In this paper, we continue the investigation on the index of minimal zero-sum sequences of length 5 over a cyclic group of prime power order. When G is a cyclic group of prime power order $n = p^\mu$ with $p \geq 7$ and $\mu \geq 2$, and S is a minimal zero-sum sequence over G with $h(S) \geq 2$, we were able to determine completely the index of S . Our main result is as follows.

Theorem 1.2. *Let G be a cyclic group of prime power order $n = p^\mu$ with $p \geq 7$ and $\mu \geq 2$, and $S = (x_1g) \cdot (x_2g) \cdot (x_3g) \cdot (x_4g) \cdot (x_5g)$ be a minimal zero-sum sequence over G with $\gcd(n, x_1, x_2, x_3, x_4, x_5) = 1$ and $x_1 = x_2$. Then $\text{ind}(S) = 2$ if and only if $S = (mg) \cdot (mg) \cdot (m\frac{n-1}{2}g) \cdot (m\frac{n+3}{2}g) \cdot (m(n-3)g)$ where m is a positive integer such that $\gcd(m, n) = 1$.*

The paper is organized as follows. In the next section, we provide some preliminary results. In Section 3, we state three main propositions and use them together with some preliminary results to give a proof for our main result. The proofs of the main propositions are given in the last section.

2. PRELIMINARIES

We first prove some preliminary results which will be needed in the sequel. Let $G = \langle g \rangle$ be a cyclic group of order n . Suppose that $S = (x_1g) \cdot \dots \cdot (x_lg)$. Let $\|S\|'_g = \text{ord}(g)\|S\|_g = \sum_{i=1}^l x_i \in \mathbb{N}_0$ and denote by $|x|_n$ the least positive residue of x modulo n , where $n \in \mathbb{N}$ and $x \in \mathbb{Z}$. Let mS denote the sequence $(mx_1g) \cdot \dots \cdot (mx_lg)$. Since $|g| = n$, we have $mS = (|mx_1|_ng) \cdot \dots \cdot (|mx_l|_ng)$. We note that if $\gcd(n, m) = 1$, then the multiplication by m is a group automorphism of G and hence $\text{ind}(S) = \text{ind}(mS)$. Two sequences S and S' are called equivalent, denoted by $S \sim S'$, if $S = mS'$ for some m with $\gcd(n, m) = 1$. Clearly, if $S \sim S'$, then $\text{ind}(S) = \text{ind}(S')$. For all real numbers $a < b$, define $[a, b] = \{k \in \mathbb{Z} \mid a \leq k \leq b\}$.

From now on we always assume that $G = \langle g \rangle$ is a cyclic group of order $n = p^\mu$ and $S = (x_1g) \cdot \dots \cdot (x_lg)$ is a minimal zero-sum sequence over G . We remark that if $\gcd(n, x_1, x_2, x_3, x_4, x_5) = p^w > 1$, let $h = p^w g$ and $x'_i = x_i/p^w$ for $1 \leq i \leq 5$. Then a minimal zero-sum sequence S over G can

be rewritten as $S = (x'_1 h) \cdot \dots \cdot (x'_1 h)$, which is a minimal zero-sum sequence over $H = \langle h \rangle$. Note that H is a cyclic group of prime power order with $|H||G|$ and $\gcd(n, x'_1, x'_2, x'_3, x'_4, x'_5) = 1$. In what follows we may always assume that $\gcd(n, x_1, x_2, x_3, x_4, x_5) = 1$. Let $U(n)$ denote the unit group of $n = p^\mu$, i.e. $U(n) = \{m \mid 1 \leq m \leq n-1, \gcd(m, n) = 1\}$. It is well known that $1 + t\alpha \in U(n)$ for any $1 < \alpha = p^\lambda < n$.

Lemma 2.1. *Let $n = p^\mu$ with $p \geq 5$, $\mu \geq 2$ and $\alpha = \frac{n}{p}$. If v is an integer such that $1 \leq v \leq n-1$ and $\gcd(v, p) = 1$, then there exists $y = 1 + t\alpha \in U(n)$ such that $|vy|_n < \frac{n}{p}$.*

Proof. Let $v^{-1} \in U(n)$ be the inverse of v and $v = k\alpha + v_0$ where $1 \leq v_0 \leq \alpha - 1$ and $k \geq 0$. Choose $y = 1 + t\alpha$ with $t = |v^{-1}(p-k)|_p$. Clearly $y \in U(n)$ and we have $|vy|_n = |v + vv^{-1}(p-k)\alpha|_n = v_0 < \alpha$ as desired. \square

The following proposition is a generalization of [12, Proposition 2.2].

Proposition 2.2. *Let $G = \langle g \rangle$ be a cyclic group of order $n = p^\mu$ with $p \geq 5$. If $S = g^2 \cdot (\frac{p^\mu-1}{2}g) \cdot (\frac{p^\mu+3}{2}g) \cdot ((p^\mu-3)g)$ is zero-sum sequence over G , then $\text{ind}(S) = 2$.*

Proof. This proposition was proved in [12] for $n = p$ and the same proof works for the prime power case. \square

Proposition 2.3. *Let G be a cyclic group of prime power order $n = p^\mu$ with $p \geq 5$ and $\mu \geq 2$, and S be a minimal zero-sum sequence of length 5. If $h(S) \geq 3$, then $\text{ind}(S) = 1$.*

Proof. If $h(S) \geq 4$, then we may assume that $x_1 = x_2 = x_3 = x_4 = \gcd(n, x_1)$. So we have that $\|S\|'_g = \sum_{i=1}^5 x_i = 4x_1 + x_5 < 2n$. Therefore $\text{ind}(S) = 1$.

If $h(S) = 3$, then we may assume that $x_1 = x_2 = x_3 = \gcd(n, x_1)$. If $x_1 = 1$, then $x_4, x_5 \leq n-4$ and $\|S\|'_g = \sum_{i=1}^5 x_i \leq 3 + 2(n-4) < 2n$, so we infer that $\|S\|'_g = \sum_{i=1}^5 x_i = n$ and $\text{ind}(S) = 1$.

If $x_1 = p^s > 1$, then $\gcd(n, x_4) = 1$. Let $\alpha = p^{\mu-1}$. Then by Lemma 2.1 there exists $y = 1 + t\alpha \in U(n)$ such that $|yx_4|_n < \alpha$. Since $x_1 \leq \alpha$ and y fixes x_1 (i.e. $|yx_1|_n = |x_1|_n$), we have $\|yS\|'_g = |yx_1|_n + \dots + |yx_5|_n < 4\alpha + x_5 < 2n$, so $\text{ind}(S) = 1$ and we are done. \square

By using a computer software (MATLAB), we were able to verify the following proposition.

Proposition 2.4. *Let $G = \langle g \rangle$ be a cyclic group of prime power order $n = p^\mu$ with $p \geq 5$ and $\mu \geq 2$, and $S = g^2 \cdot (x_3g) \cdot (x_4g) \cdot (x_5g)$ be a minimal zero-sum sequence of length 5. If $n < 289$, then $\text{ind}(S) = 2$ if and only if $S = g^2 \cdot (\frac{n-1}{2}g) \cdot (\frac{n+3}{2}g) \cdot ((n-3)g)$.*

3. MAIN PROPOSITIONS

In this section we present three main propositions and then use them together with some preliminary results to provide a proof for our main theorem. In terms of Proposition 2.3, from now on we may always assume that $h(S) = 2$ and $x_1 = x_2 = \text{gcd}(n, x_1)$. The following proposition takes care of the case when $x_1 > 1$.

Proposition 3.1. *Let $G = \langle g \rangle$ be a cyclic group of prime power order $n = p^\mu$ with $p \geq 7$ and $\mu \geq 2$, and $S = (x_1g) \cdot (x_2g) \cdot (x_3g) \cdot (x_4g) \cdot (x_5g)$ be a minimal zero-sum sequence of length 5. If $x_1 > 1$, then $\text{ind}(S) = 1$.*

Next we assume that $x_1 = x_2 = \text{gcd}(n, x_1) = 1$ and $S = g^2 \cdot (x_3g) \cdot (x_4g) \cdot (x_5g)$ such that $2 + x_3 + x_4 + x_5 = 2n$. The following two propositions are crucial to prove the main theorem.

Proposition 3.2. *Let G be a cyclic group of order n with $n = p^\mu \geq 289$, $p \geq 7$ and $\mu \geq 2$. Let $S = g^2 \cdot (cg) \cdot ((n-b)g) \cdot ((n-a)g)$ be a minimal zero-sum sequence with $4 \leq a \leq b < c < \frac{n}{2}$ and $c = b + a - 2$. Then $\text{ind}(S) = 1$.*

Proposition 3.3. *Let G be a cyclic group of order n with $n = p^\mu \geq 289$, $p \geq 7$ and $\mu \geq 2$. Let $S = g^2 \cdot (cg) \cdot ((n-b)g) \cdot ((n-a)g)$ be a minimal zero-sum sequence with $3 = a \leq b < b + 1 = c < \frac{n}{2}$. Then $\text{ind}(S) = 2$ if and only if $c = \frac{n-1}{2}$.*

We are now ready to present a proof for our main result.

Proof of Theorem 1.2.

In terms of Proposition 2.3, we may assume that $h(S) = 2$ and $x_1 = x_2 = \text{gcd}(n, x_1)$. If $x_1 > 1$, then the result follows from Proposition 3.1.

Next we assume $x_1 = x_2 = \gcd(n, x_1) = 1$ and $S = g^2 \cdot (x_3g) \cdot (x_4g) \cdot (x_5g)$. As discussed in [12], we may assume that $S = g^2 \cdot (cg) \cdot ((n-b)g) \cdot ((n-a)g)$ with $3 \leq a \leq b < c < \frac{n}{2}$ and $c = a + b - 2$, for otherwise, it is easy to show that $\text{ind}(S) = 1$. If $n < 289$, the result follows from Proposition 2.4. So we may assume that $n \geq 289$. Now the result follows immediately from Propositions 3.2 and 3.3, and we are done.

4. PROOFS FOR MAIN PROPOSITIONS

We now give the proofs for the three crucial propositions stated in previous section. The proofs will be presented in the following three subsections.

4.1. Proof of Proposition 3.1. .

Let $x_1 = p^s > 1$ and $\alpha' = \frac{n}{p^s} = \frac{n}{x_1}$. We first assume that $\gcd(n, x_i) > 1$ for some $i = 3, 4, 5$, say $\gcd(n, x_3) > 1$. We divide the proof into 3 cases.

Case 1. If $x_1 > \gcd(n, x_3) > 1$, let $x'_3 = \frac{x_3}{\gcd(n, x_3)}$, $n' = \frac{n}{\gcd(n, x_3)}$ and $\alpha' = \frac{n'}{p}$. Then by Lemma 2.1, we may find a $y = 1 + t\alpha' \in U(n')$ such that $|yx'_3|_{n'} < \alpha'$. Thus $|yx_3|_n = \gcd(n, x_3)|yx'_3|_{n'} < \gcd(n, x_3)\alpha' = \frac{n}{p}$. Since $x_1 > \gcd(n, x_3)$, y fixes $x_1 = x_2$, so by multiplying S with y , we may assume that $x_1, x_2, x_3 \leq \frac{n}{p}$. Since $\gcd(n, x_1, x_2, x_3, x_4, x_5) = 1$, we have $\gcd(n, x_4) = 1$. Thus by Lemma 2.1, there exists $y' = 1 + t\frac{n}{p} \in U(n)$ such that $|y'x_4|_n < \frac{n}{p}$. Since y' fixes x_1, x_2 and x_3 , again we may assume that $x_i \leq \frac{n}{p}$ for $i = 1, 2, 3, 4$. Thus $\|S\|'_g = \sum_{i=1}^5 x_i \leq \frac{4n}{p} + x_5 < 2n$, so $\text{ind}(S) = 1$ and we are done.

Case 2. If $x_1 < \gcd(n, x_3)$, then there exists $u \in U(n)$ such that $|ux_3|_n = \gcd(n, x_3)$. So we may assume that $x_3 = \gcd(n, x_3)$. Note that x_1 may not be $\gcd(n, x_1)$ any longer. However, as in Case 1 we may find $y = 1 + t\alpha'$, where $\alpha' = \frac{n}{\gcd(n, x_3)}$, such that $|yx_1|_n < \alpha' \leq \frac{n}{p}$. Since y fixes x_3 , we may assume that $x_1, x_2, x_3 \leq \frac{n}{p}$. Since $\gcd(n, x_4) = 1$, there exists $y' = 1 + t\frac{n}{p} \in U(n)$ such that $|y'x_4|_n < \frac{n}{p}$. Since y' fixes x_1, x_2 and x_3 , so we may assume that $x_i \leq \frac{n}{p}$ for $i = 1, 2, 3, 4$. Thus $\sum_{i=1}^5 x_i \leq \frac{4n}{p} + x_5 < 2n$, so $\text{ind}(S) = 1$.

Case 3. If $x_1 = \gcd(n, x_3)$, let $x_3 = n - kx_1 = n - w$. Since S is a minimal zero-sum sequence, we have $k \geq 3$. (Otherwise, $x_1 + x_3$ or $x_1 + x_2 + x_3$ has zero-sum.) If $w < \frac{n}{2}$, then $2w < n$ and $|2x_3|_n = n - 2w$ and $|2x_1|_n = 2x_1$. We may replace S by the equivalent sequence $S' = (2x_1g) \cdot (2x_2g) \cdot ((n-w')g) \cdot (|2x_4|_ng) \cdot (|2x_5|_ng)$. By repeating this process, we may assume that $w > \frac{n}{2}$. As before, since $\gcd(n, x_4) = 1$, we may assume that $x_4 < \frac{n}{p}$. Since $k \geq 3, p > 6$ and $kp - 2p - 2k = (k - 2)(p - 2) - 4 > 0$, $\frac{k-2}{k}w - x_4 > \frac{k-2}{k} \frac{n}{2} - \frac{n}{p} = \frac{n}{2kp}(kp - 2p - 2k) > 0$, so we have

$$\|S\|'_g = \sum_{i=1}^5 x_i = n - (k - 2)x_1 + x_4 + x_5 = n + x_5 - \left(\frac{k - 2}{k}w - x_4\right) < 2n.$$

Therefore, $\text{ind}(S) = 1$ and we are done.

Next we assume that $\gcd(n, x_i) = 1$ for all $i = 3, 4, 5$. Let $x_1 = p^s > 1$ and $\alpha' = \frac{n}{p^s}$. As before, we may use a unit $1 + t\alpha' (0 \leq t \leq x_1 - 1)$ to move x_3 so that $x_3 < \alpha'$. Thus we may assume that $x_3 < \alpha'$. Let $\beta = \frac{n}{p} \geq \max(x_1, \alpha')$. Then $y_1 = 1 + t_1x_3^{-1}\beta \in U(n)$ and $y_2 = 1 + t_2x_3^{-1}\beta \in U(n)$ for $t_1 \neq t_2 \in [0, p - 1]$ are distinct units. Note that every such y fixes x_1 . Note also that $|(1 + tx_3^{-1}\beta)x_i|_n = |x'_i + (n - t'\beta)|_n \leq x'_i + (n - t'\beta)$ where $1 \leq t' \leq p$ and $x'_i = x_i \bmod \beta$. If $|(1 + t_1x_3^{-1}\beta)x_i|_n \neq |(1 + t_2x_3^{-1}\beta)x_i|_n$, then $|x'_i + (n - t'_1\beta)|_n \neq |x'_i + (n - t'_2\beta)|_n$, implying $t'_1 \neq t'_2$. Thus $\sum_{t=0}^6 |(1 + tx_3^{-1}\beta)x_i|_n = \sum_{t=0}^6 |x'_i + (n - t'\beta)|_n \leq 7x'_i + \sum_{t'=1}^7 (n - t'\beta) = 7x'_i + 7n - 28\beta$. We now compute the following sum

$$\begin{aligned} \sum_{i=1}^5 \sum_{t=0}^6 |(1 + tx_3^{-1}\beta)x_i|_n &= 7x_1 + 7x_2 + \sum_{t=0}^6 (x_3 + t\beta) + \\ \sum_{i=4,5} \sum_{t=0}^6 |(1 + tx_3^{-1}\beta)x_i|_n &\leq 14x_1 + 7x_3 + 21\beta + 7x'_4 + 7x'_5 + \\ 2 \sum_{t=0}^6 (n - (1 + t)\beta) &= 7(2x_1 + x_3 + x'_4 + x'_5) + 14n - 35\beta. \end{aligned}$$

Since $2x_1 + x_3 + x'_4 + x'_5 < 5\beta$, we have

$$\sum_{i=1}^5 \sum_{t=0}^6 |(1 + tx_3^{-1}\beta)x_i|_n < 14n + 35\beta - 35\beta = 14n.$$

Then there must exist $t \in [0, 6]$ such that

$$\sum_{i=1}^5 |(1 + tx_3^{-1}\beta)x_i|_n < 2n.$$

So

$$\sum_{i=1}^5 |(1 + tx_3^{-1}\beta)x_i|_n = n,$$

and thus $\text{ind}(S) = 1$. This completes the proof.

4.2. Proof of Proposition 3.2. .

In this subsection we will prove Proposition 3.2 through the following lemmas. Let $s = \lfloor \frac{b}{a} \rfloor$. Recall that from now on we always assume that $n \geq 289$.

Lemma 4.1. *Let $S = g^2 \cdot (cg) \cdot ((n-b)g) \cdot ((n-a)g)$ be the same sequence described in Proposition 3.2. If there exist integers M and k such that $\text{gcd}(n, M) = 1$, $\frac{kn}{c} < M < \frac{kn}{b}$ and $Ma < n$, then $\text{ind}(S) = 1$.*

Proof. Note that

$$\begin{aligned} \|MS\|'_g &= M + M + |Mc|_n + |M(n-b)|_n + |M(n-a)|_n \\ &\leq M + M + Mc - kn + (kn - Mb) + (n - Ma) \\ &= n + M(1 + 1 + c - b - a) = n. \end{aligned}$$

We conclude that $\text{ind}(S) = 1$. □

Lemma 4.2. *If $S = g^2 \cdot (cg) \cdot ((n-b)g) \cdot ((n-a)g)$ is a minimal zero-sum sequence such that $3 \leq a \leq b < c < \frac{n}{2}$, then $\text{ind}(S) = 1$ provided that one of the following two conditions holds:*

- (1) $\frac{s(a-2)n}{bc} > 2$.
- (2) $\frac{(s-1)(a-2)n}{bc} > 1$ and $\frac{n}{c} < p - 2$.

Proof. (1). If $\frac{s(a-2)n}{bc} > 2$, then there exist two integers $M_1, M_2 \in [\frac{sn}{c}, \frac{sn}{b}]$, so one of them, say M , is co-prime to n . Since $Ma < n$, by Lemma 4.1 we have $\text{ind}(S) = 1$.

(2). If $\frac{(s-1)(a-2)n}{bc} > 1$, then we can find two integers M_1 and M_2 such that $M_1 \in [\frac{(s-1)n}{c}, \frac{(s-1)n}{b}]$ and $M_2 \in [\frac{sn}{c}, \frac{sn}{b}]$. We may assume that

$\frac{(s-1)n}{b} - M_1 \leq 1$ and $M_2 - \frac{sn}{c} \leq 1$; for otherwise, by the proof in (1) we infer $\text{ind}(S) = 1$. Note that

$$M_2 - M_1 = (M_2 - \frac{sn}{c}) + (\frac{sn}{c} - \frac{(s-1)n}{b}) + (\frac{(s-1)n}{b} - M_1) \leq 2 + \frac{n}{c} < p.$$

Thus one of M_1 and M_2 is co-prime to n , so by Lemma 4.1 we have $\text{ind}(S) = 1$. \square

Lemma 4.3. *If the sequence $S = g^2 \cdot (cg) \cdot ((n-b)g) \cdot ((n-a)g)$ is a minimal zero-sum sequence such that $6 \leq a \leq b < c < \frac{n}{2}$ and $s \geq 4$, then $\text{ind}(S) = 1$.*

Proof. We divide the proof into the following 3 cases.

Case 1. $\frac{n}{c} \geq 4$. Since $\frac{s(a-2)n}{bc} > \frac{s(a-2)n}{(s+1)ac} \geq (\frac{4}{5})(\frac{4}{6})4 > 2$, by Lemma 4.2 we infer $\text{ind}(S) = 1$.

Case 2. $3 \leq \frac{n}{c} < 4$. Then $\frac{(s-1)(a-2)n}{bc} > \frac{(s-1)(a-2)n}{(s+1)ac} \geq (\frac{3}{5})(\frac{4}{6})3 > 1$. Since $\frac{n}{c} < 4 < p - 2$, it follows from Lemma 4.2 that $\text{ind}(S) = 1$.

Case 3. $2 < \frac{n}{c} < 3$. If $\frac{n}{c} < 3 < \frac{n}{b}$, then $3a < n$ and thus by Lemma 4.1 $\text{ind}(S) = 1$. So we may assume that $2 < \frac{n}{c} < \frac{n}{b} < 3$. If $s \geq 7$, then $\frac{(s-1)(a-2)n}{bc} > \frac{(s-1)(a-2)n}{(s+1)ac} \geq (\frac{6}{8})(\frac{4}{6})2 = 1$. By Lemma 4.2 $\text{ind}(S) = 1$. If $4 \leq s \leq 6$, $n < 3b < 3(s+1)a \leq 21a$, so $a \geq 14$. Again, $\frac{(s-1)(a-2)n}{bc} > \frac{2(s-1)(a-2)}{(s+1)a} \geq (\frac{3}{5})(\frac{12}{14})2 > 1$. By Lemma 4.2 we infer $\text{ind}(S) = 1$. \square

Lemma 4.4. *If $S = (g) \cdot (g) \cdot (cg) \cdot ((n-b)g) \cdot ((n-a)g)$ is a minimal zero-sum sequence such that $6 \leq a \leq b < c < \frac{n}{2}$ and $s \leq 3$, then $\text{ind}(S) = 1$.*

Proof. We divide the proof into two cases.

Case 1. $\frac{n}{c} \geq 5$. If $s \geq 2$, then $\frac{s(a-2)n}{bc} > \frac{s(a-2)n}{(s+1)ac} > (\frac{2}{3})(\frac{4}{6})5 > 2$, so by Lemma 4.2, $\text{ind}(S) = 1$.

Next assume that $s = 1$. If $\frac{n}{c} \geq 6$, then $\frac{(a-2)n}{bc} > \frac{(a-2)n}{2ac} \geq 2$, so again we infer $\text{ind}(S) = 1$. If $5 \leq \frac{n}{c} < 6$, then $3a - 3 \geq c > \frac{n}{6}$, so $a \geq 18$. Thus $\frac{(a-2)n}{bc} > \frac{(a-2)n}{2ac} \geq (\frac{16}{36})5 > 2$, so $\text{ind}(S) = 1$.

Case 2. $\frac{n}{c} < 5$. We may assume that $m < \frac{n}{c} < \frac{n}{b} < m + 1$ (*) with $m \in [2, 4]$ (for otherwise, it is easy to show that $\text{ind}(S) = 1$).

Subcase 2.1. $s = 1$. If $m \geq 3$, we have

$$\frac{n}{b} - \frac{n}{c} = \frac{(a-2)n}{bc} > \frac{3(a-2)}{2a} \geq 1,$$

giving a contradiction to (*).

Next assume $m = 2$. If $b \leq 2a - 4$, then $\frac{n}{b} - \frac{n}{c} = \frac{(a-2)n}{bc} \geq \frac{n}{2c} > 1$, giving a contradiction. If $b \geq 2a - 3$, then $n \geq 2c + 1 = 6a - 9$. Since $\frac{2n}{b} - \frac{2n}{c} = \frac{2(a-2)n}{bc} \geq \frac{4(a-2)}{2a-1} > 1$, we have $\frac{2n}{c} < 5 < \frac{2n}{b}$. Note that $3a - 3 \geq c > \frac{n}{3}$, so $a \geq 34$. Thus $5a < 6a - 9 < n$, so by Lemma 4.1 $\text{ind}(S) = 1$.

Subcase 2.2. $s = 2$. If $m = 2$, then $2 < \frac{n}{c} < \frac{n}{b} < 3$ and thus $a > \frac{n}{9} > 32$. Since $\frac{2n}{b} - \frac{2n}{c} = \frac{2(a-2)n}{bc} \geq \frac{4(a-2)}{3a-1} > 1$, we have $\frac{2n}{c} < 5 < \frac{2n}{b}$, implying $\text{ind}(S) = 1$.

Next assume $m = 3$. Thus $a > \frac{b}{3} > \frac{n}{12} > 24$. So $\frac{2n}{b} - \frac{2n}{c} > \frac{2(a-2)n}{3ac} > \frac{(2 \times 22)n}{(3 \times 24)c} = \frac{11n}{18c}$. If $\frac{n}{c} > \frac{36}{11}$, then $\frac{2(a-2)n}{bc} > 2$, implying $\text{ind}(S) = 1$. Next assume that $\frac{n}{c} \leq \frac{36}{11}$. Then $\frac{3n}{c} \leq \frac{108}{11} < 10$. Since $\frac{3n}{b} - \frac{3n}{c} \geq \frac{3(a-2)n}{3ac} > 1$. We have $\frac{3n}{c} < 10 < \frac{3n}{b}$. We next show that $10a < n$ and thus $\text{ind}(S) = 1$ (by Lemma 4.1). Note that $3 < \frac{n}{c} < \frac{n}{b} < 4$ and $a > 24$. Then $a \leq \frac{24}{22}(a-2) \leq \frac{24}{22}(c-b) < \frac{24}{22}(\frac{n}{3} - \frac{n}{4}) = \frac{n}{11}$, so $10a < n$ as required.

If $m = 4$, then $a \geq 20$ and thus $\frac{2(a-2)n}{bc} > \frac{8(a-2)}{3a} > 2$. By Lemma 4.2 $\text{ind}(S) = 1$.

Subcase 2.3. $s = 3$. If $m \geq 4$, then $\frac{3n}{b} - \frac{3n}{c} > \frac{3(a-2)n}{4ac} > 2$. It follows from Lemma 4.2 that $\text{ind}(S) = 1$. If $m = 3$, then $\frac{2n}{b} - \frac{2n}{c} > \frac{2(a-2)n}{4ac} > 1$, so by Lemma 4.2 $\text{ind}(S) = 1$.

Next we assume that $m = 2$ and $2 < \frac{n}{c} < \frac{n}{b} < 3$. If $b \leq 4a - 8$, then $\frac{2n}{b} - \frac{2n}{c} \geq \frac{2(a-2)n}{4(a-2)c} > 1$, so $\text{ind}(S) = 1$. Now assume that $4a - 7 \leq b \leq 4a - 1$. Thus $c \geq 5a - 9$, so $n \geq 2c + 1 \geq 10a - 17 > 9a$ (as $a > \frac{n}{12} \geq 24$). Since $\frac{3n}{b} - \frac{3n}{c} > \frac{3(a-2)}{2a} > 1$, we have $6 < \frac{3n}{c} < M < \frac{3n}{b} < 9$. If $M = 8$, since $8a < n$, we have $\text{ind}(S) = 1$. So we may assume $6 < \frac{3n}{c} < 7 < \frac{3n}{b} < 8$. Thus $\frac{2n}{c} < \frac{14}{3} < 5$. If $\frac{2n}{c} < 5 < \frac{2n}{b}$, since $5a < n$, we have $\text{ind}(S) = 1$. Thus $4 < \frac{2n}{c} < \frac{2n}{b} < 5$. Since $\frac{4n}{b} - \frac{4n}{c} > 1$, we have $8 < \frac{4n}{c} < 9 < \frac{4n}{b} < 10$. Recall that $9a < n$, so by Lemma 4.1 $\text{ind}(S) = 1$ and we are done.

□

Lemma 4.5. *If $S = g^2 \cdot (cg) \cdot ((n-b)g) \cdot ((n-a)g)$ is a minimal zero-sum sequence such that $a \in [4, 5]$, $a \leq b < c < \frac{n}{2}$ and $s \leq 10$, then $\text{ind}(S) = 1$.*

Proof. Note that $b \leq sa + 4$ and $c \leq (s+1)a + 2$. If $s \leq 3$, then $\frac{n}{c} \geq \frac{289}{(s+1)a+2} \geq \frac{289}{22} > 13$. Thus $\frac{s(a-2)n}{bc} > \frac{13s(a-2)}{(s+1)a} \geq \frac{13 \times 2}{8} > 2$. If $s \geq 4$, then $\frac{n}{c} \geq \frac{289}{(s+1)a+2} \geq \frac{289}{57} > 5$. Thus $\frac{s(a-2)n}{bc} > \frac{5s(a-2)}{(s+1)a} \geq \frac{5 \times 4 \times 2}{5 \times 4} = 2$. It follows from Lemma 4.2 that $\text{ind}(S) = 1$. \square

Lemma 4.6. *If $S = g^2 \cdot (cg) \cdot ((n-b)g) \cdot ((n-a)g)$ is a minimal zero-sum sequence such that $a \in [4, 5]$, $a \leq b < c < \frac{n}{2}$ and $s \geq 11$, then $\text{ind}(S) = 1$.*

Proof. We divide the proof into two cases according to $a = 5$ and $a = 4$.

Case 1. $a = 5$. In terms of Lemma 4.2, we may assume that $\frac{s(a-2)n}{(s+1)ac} \leq 2$. Thus $\frac{3sn}{5(s+1)c} \leq 2$, so $\frac{n}{c} \leq \frac{10(s+1)}{3s} \leq \frac{120}{33} < 4 < p-2$. Since $\frac{(s-1)(a-2)n}{bc} > \frac{(s-1)(a-2)n}{(s+1)ac} > \frac{10 \times 3 \times 2}{12 \times 5} = 1$, by Lemma 4.2 we have $\text{ind}(S) = 1$.

Case 2. $a = 4$. In terms of Lemma 4.2, we may assume that $\frac{s(a-2)n}{(s+1)ac} \leq 2$. Thus $\frac{n}{c} \leq \frac{8(s+1)}{2s} \leq \frac{96}{22} < 5 \leq p-2$.

If $\frac{n}{c} \geq 3$, then $\frac{(s-1)(a-2)n}{bc} > \frac{2(s-1)n}{(s+1)4c} > \frac{2 \times 10 \times 3}{12 \times 4} > 1$, by Lemma 4.2 we have $\text{ind}(S) = 1$. Next as before we may assume $2 < \frac{n}{c} < \frac{n}{b} < 3$.

Since $2 < \frac{n}{b} < 3$, we have $b > \frac{n}{3} > 96$ and $b \geq 97$. Assume that $n = 2c + j = 2b + j + 4$. If $j \geq 17$, we have $2b^2 + (j-10)b - 7(j+4) - (2b^2 + 4b) = (j-14)b - 7(j+4) = j(b-7) - 14(b+2) = (j-14)(b-7) - 14 \times 9 \geq 3 \times 90 - 14 \times 9 = 144 > 0$ and

$$\begin{aligned} \frac{(s-1)n}{b} - \frac{(s-1)n}{c} &= \frac{2(s-1)n}{bc} \geq \frac{(b-7)(2b+j+4)}{2b(b+2)} \\ &= \frac{2b^2 + (j-10)b - 7(j+4)}{2b^2 + 4b} > 1. \end{aligned}$$

By Lemma 4.2 we have $\text{ind}(S) = 1$.

Next assume that $j \leq 15$ (j is odd as n is odd). Let $n = (j+3)t + r$, $r \in [0, j+2]$.

First we claim that $r < t$. Since $t = \lfloor \frac{n}{j+3} \rfloor \geq \lfloor \frac{n}{18} \rfloor \geq 16$ and $r \leq 17$. If $r = 17 = t + 1$, we have $n = 18 \times 16 + 17 = 305$, which is not a prime power. If $r = 17 = t$, then $n = 18 \times 17 + 17 = 17 \times 19$, which is not a prime

power. If $r = 16 = t$, then $n = 18 \times 16 + 16 = 16 \times 19$, which is not a prime power. Hence our claim holds.

Subcase 2.1. If $j = 1$, let $n = 5l + k$ with $k \in [1, 4]$, so $l \geq 57$ and $\gcd(n, l) = 1$. Then

$$\begin{aligned}
 S &= (g) \cdot (g) \cdot \left(\frac{n-j}{2}g\right) \cdot \left(\frac{n+j+4}{2}g\right) \cdot ((n-4)g) \\
 &\sim (2g) \cdot (2g) \cdot ((n-1)g) \cdot (5g) \cdot ((n-8)g) \quad (\text{multiplied by } 2) \\
 &\sim (4lg) \cdot (4lg) \cdot ((3l+k)g) \cdot ((5l-k)g) \cdot ((4l+4k)g) \quad (\text{multiplied by } 2l) \\
 &\sim ((l+k)g) \cdot ((l+k)g) \cdot ((l-k)g) \cdot ((l+3k)g) \cdot ((l-3k)g) \\
 &\quad (\text{multiplied by } n-1).
 \end{aligned}$$

Since $(l+k) + (l+k) + (l-k) + (l+3k) + (l-3k) = 5l+k = n$, we have $\text{ind}(S) = 1$.

Subcase 2.2. If $j = 3$, let $n = 7l + k$ with $k \leq 6$, then $l \geq 41 > k$ and $\gcd(n, l) = 1$. If $k > 0$, then

$$\begin{aligned}
 S &= (g) \cdot (g) \cdot \left(\frac{n-j}{2}g\right) \cdot \left(\frac{n+j+4}{2}g\right) \cdot ((n-4)g) \\
 &\sim (2g) \cdot (2g) \cdot ((n-3)g) \cdot (7g) \cdot ((n-8)g) \quad (\text{multiplied by } 2) \\
 &\sim (6lg) \cdot (6lg) \cdot ((5l+2k)g) \cdot ((7l-2k)g) \cdot ((4l+4k)g) \\
 &\quad (\text{multiplied by } 3l) \\
 &\sim ((l+k)g) \cdot ((l+k)g) \cdot ((2l-k)g) \cdot (3kg) \cdot ((3l-3k)g) \\
 &\quad (\text{multiplied by } n-1).
 \end{aligned}$$

Since $(l+k) + (l+k) + (2l-k) + 3k + (3l-3k) = 7l+k = n$, we have $\text{ind}(S) = 1$.

If $k = 0$, then n is a power of 7.

$$\begin{aligned}
 S &= (g) \cdot (g) \cdot \left(\frac{n-j}{2}g\right) \cdot \left(\frac{n+j+4}{2}g\right) \cdot ((n-4)g) \\
 &\sim (2g) \cdot (2g) \cdot ((n-3)g) \cdot (7g) \cdot ((n-8)g) \quad (\text{multiplied by } 2) \\
 &\sim (6(l-1)g) \cdot (6(l-1)g) \cdot ((5l+9)g) \cdot ((7l-21)g) \cdot ((4l+24)g) \\
 &\quad (\text{multiplied by } 3(l-1)) \\
 &\sim ((l+6)g) \cdot ((l+6)g) \cdot ((2l-9)g) \cdot (21g) \cdot ((3l-24)g) \\
 &\quad (\text{multiplied by } n-1),
 \end{aligned}$$

where $3l - 24 = 3(l - 8) > 0$. Since $(l + 6) + (l + 6) + (2l - 9) + 21 + (3l - 24) = 7l = n$, again we have $\text{ind}(S) = 1$.

Subcase 2.3. If $\text{gcd}(n, t) = 1$ and $j \geq 5$, we have

$$\begin{aligned} S &= (g) \cdot (g) \cdot \left(\frac{n-j}{2}g\right) \cdot \left(\frac{n+j+4}{2}g\right) \cdot ((n-4)g) \\ &\sim (2g) \cdot (2g) \cdot ((n-j)g) \cdot ((j+4)g) \cdot ((n-8)g) \quad (\text{multiplied by } 2) \\ &\sim (2tg) \cdot (2tg) \cdot ((3t+r)g) \cdot ((t-r)g) \cdot (((j-5)t+r)g) \\ &\quad (\text{multiplied by } t). \end{aligned}$$

Since $2t + 2t + (3t + r) + (t - r) + ((j - 5)t + r) = (j + 3)t + r = n$, we have $\text{ind}(S) = 1$.

If $\text{gcd}(n, t) > 1$ and $j > 5$, we have

$$\begin{aligned} S &= (g) \cdot (g) \cdot \left(\frac{n-j}{2}g\right) \cdot \left(\frac{n+j+4}{2}g\right) \cdot ((n-4)g) \\ &\sim (2g) \cdot (2g) \cdot ((n-j)g) \cdot ((j+4)g) \cdot ((n-8)g) \quad (\text{multiplied by } 2) \\ &\sim (2(t+1)g) \cdot (2(t+1)g) \cdot ((3t+r-j)g) \cdot ((t-r+j+4)g) \cdot \\ &\quad (((j-5)t+r-8)g) \quad (\text{multiplied by } (t+1)), \end{aligned}$$

where $(j - 5)t + r - 9 \geq t + r - 9 > 0$. Since $2(t + 1) + 2(t + 1) + (3t + r - j) + (t - r + j + 4) + ((j - 5)t + r - 8) = (j + 3)t + r = n$, we infer $\text{ind}(S) = 1$.

If $\text{gcd}(n, t) > 1$ and $j = 5$, we have

$$\begin{aligned} S &= (g) \cdot (g) \cdot \left(\frac{n-j}{2}g\right) \cdot \left(\frac{n+j+4}{2}g\right) \cdot ((n-4)g) \\ &\sim (2g) \cdot (2g) \cdot ((n-j)g) \cdot ((j+4)g) \cdot ((n-8)g) \quad (\text{multiplied by } 2) \\ &\sim (2(t-1)g) \cdot (2(t-1)g) \cdot ((3t+r+j)g) \cdot ((t-r-j-4)g) \cdot \\ &\quad (((j-5)t+r+8)g) \quad (\text{multiplied by } (t-1)), \end{aligned}$$

where $t - r - j - 9 \geq t + r - 9 > 0$. Since $2(t - 1) + 2(t - 1) + (3t + r + j) + (t - r - j - 4) + ((j - 5)t + r + 8) = (j + 3)t + r = n$, again we infer $\text{ind}(S) = 1$.

□

Now Proposition 3.2 follows immediately from Lemmas 4.3, 4.4, 4.5 and 4.6.

4.3. Proof of Proposition 3.3.

In this subsection, we will provide a proof for Proposition 3.3. In terms of Proposition 2.2, from now on we may always assume that $S = g^2 \cdot (cg) \cdot ((n-b)g)((n-3)g)$ where $a = 3 \leq b < c \leq \frac{n-3}{2}$ and $n = p^\mu \geq 289$ with $p \geq 7$ and $\mu \geq 2$ and show that $\text{ind}(S) = 1$.

Lemma 4.7. *If S is a minimal zero-sum sequence such that $\frac{n}{c} > 4$, then $\text{ind}(S) = 1$.*

Proof. If $\frac{n}{c} > 8$, we have $\frac{sn}{bc} \geq \frac{sn}{3(s+1)c} > 2$ (for if $s \leq 2$, $\frac{n}{c} > 12$), so by Lemma 4.2 $\text{ind}(S) = 1$.

Next we assume that $4 < \frac{n}{c} < 8$. We divide our proof into two cases.

Case 1. $\frac{n}{c} < m < \frac{n}{b}$ for some integer $m \leq 8$. If $\gcd(m, n) = 1$, by Lemma 4.1, $\text{ind}(S) = 1$. If $\gcd(m, n) > 1$, since $m \leq 8$, we must have $m = p = 7$, so $n \geq 7^3 = 343$. If $\frac{n}{c} < 6$, we may take $m = 6$ and thus $\text{ind}(S) = 1$. Thus we may assume $\frac{n}{c} < 7 < \frac{n}{b}$. Since $n < 7c \leq 21(s+1)$, we have $s \geq 16$. Since $\frac{sn}{bc} > \frac{7s}{3(s+1)} \geq \frac{112}{51} > 2$, it follows from Lemma 4.2 that $\text{ind}(S) = 1$.

Case 2. $m < \frac{n}{c} < \frac{n}{b} < m+1$ with $m \in [4, 7]$. If $\frac{sn}{bc} > 2$, By Lemma 4.2 we have $\text{ind}(S) = 1$. So we may assume that $\frac{sn}{bc} \leq 2$. Since $s \geq 12$, we have $\frac{kn}{b} - \frac{kn}{c} = \frac{kn}{bc} > \frac{4k}{3(s+1)} \geq \frac{4(s-1)}{3(s+1)} > 1$ for $k = s-1, s$. Thus there exist two integers $M_{s-1} \in [\frac{(s-1)n}{c}, \frac{(s-1)n}{b}]$ and $M_s \in [\frac{sn}{c}, \frac{sn}{b}]$. Note that $|M_s - M_{s-1}| \leq 2 + \frac{sn}{c} - \frac{(s-1)n}{b} = 2 + \frac{(c-s)n}{bc} \leq 2 + \frac{(2s+3)n}{bc} \leq 2 + 4 + \frac{3n}{bc} < 6 + \frac{n}{12c} < 7$. We conclude that at least one of M_{s-1}, M_s is co-prime to n . Since $M_s a < n$, it follows from Lemma 4.1 that $\text{ind}(S) = 1$. \square

In terms of Lemma 4.7 and its proof, we may assume that $m < \frac{n}{c} < \frac{n}{b} < m+1$ with $m \in [2, 3]$.

Lemma 4.8. *If S is a minimal zero-sum sequence such that $3 < \frac{n}{c} < \frac{n}{b} < 4$, then $\text{ind}(S) = 1$.*

Proof. Suppose that $n = 3c + j$ and $n = 3b + j + 3$, where $1 \leq j \leq b - 4$ and $\gcd(j, 3) = 1$. If $j \geq 17$, as in Case 2 of Lemma 4.6 we have

$$\frac{kn}{b} - \frac{kn}{c} = \frac{kn}{bc} \geq \frac{(b-5)(3b+j+3)}{3b(b+1)} > 1,$$

for $k = s - 1, s$. By Lemma 4.2 we have $\text{ind}(S) = 1$.

Next assume that $j \leq 16$. Note that

$$\begin{aligned} S &= (g) \cdot (g) \cdot \left(\frac{n-j}{3}g\right) \cdot \left(\frac{n+j+3}{3}g\right) \cdot ((n-3)g) \\ &\sim (3g) \cdot (3g) \cdot ((n-j)g) \cdot ((j+3)g) \cdot ((n-9)g) \quad (\text{multiplied by } 3). \end{aligned}$$

Let $S' = 3S$. Next we show that $\text{ind}(S) = \text{ind}(S') = 1$.

Case 1. $8 \leq j \leq 16$. Let $n = (j+1)t + r$ with $r \in [0, j]$. Since $\frac{n}{j+1} \geq \frac{289}{17} = 17$, we have $t \geq 17 \geq j > r$. If $\gcd(n, t) = 1$, we have

$$\begin{aligned} S' &\sim (3tg) \cdot (3tg) \cdot ((2t+r)g) \cdot ((t-r)g) \cdot (((j-8)t+r)g) \\ &\quad (\text{multiplied by } t). \end{aligned}$$

Since $(3t) + (3t) + (2t+r) + (t-r) + ((j-8)t+r) = (j+1)t + r = n$, we have $\text{ind}(S) = 1$. If $\gcd(n, t) > 1$, we have

$$\begin{aligned} S' &\sim (3(t-1)g) \cdot (3(t-1)g) \cdot ((t+r+j)g) \cdot ((2t-r-j-3)g) \cdot \\ &\quad (((j-8)t+r+9)g) \quad (\text{multiplied by } t-1). \end{aligned}$$

Note that $t \geq 17 \geq j+1 \geq r+2$. We always have $2t-r-j-3 > 0$ for all $j \in [8, 16]$. Again, we have $3(t-1) + 3(t-1) + (t+r+j) + (2t-r-j-3) + ((j-8)t+r+9) = (j+1)t + r = n$, so $\text{ind}(S) = 1$.

Case 2. $j = 7$. Let $n = 9t + r$ with $r \in [1, 8]$. Then $t \geq 32 > r$. If $\gcd(n, t) = 1$, we have

$$S' \sim (3tg) \cdot (3tg) \cdot ((2t+r)g) \cdot ((t-r)g) \cdot (rg) \quad (\text{multiplied by } t).$$

Since $(3t) + (3t) + (2t+r) + (t-r) + r = 9t + r = n$, we have $\text{ind}(S) = 1$.

If $\gcd(n, t) > 1$, then $\gcd(n, t-1) = 1$. So we have

$$\begin{aligned} S' &\sim (3(t-1)g) \cdot (3(t-1)g) \cdot ((2t+r+7)g) \cdot ((t-r-10)g) \cdot ((r+9)g) \\ &\quad (\text{multiplied by } t-1). \end{aligned}$$

Since $(3(t-1)) + (3(t-1)) + (2t+r+7) + (t-r-10) + (r+9) = 9t + r = n$, we again have $\text{ind}(S) = 1$.

Case 3. $j = 5$. Let $n = 5t + r$ with $r \in [1, 4]$. Then $\gcd(n, t) = 1$ and $t \geq 5r > 3r$. We have

$$S' \sim ((t-r)g) \cdot ((t-r)g) \cdot (2rg) \cdot ((t-3r)g) \cdot ((2t+4r)g) \\ \text{(multiplied by } 2t\text{)}.$$

Since $(t-r) + (t-r) + (2r) + (t-3r) + (2t+4r) = 5t + r = n$, we have $\text{ind}(S) = 1$.

Case 4. $j = 4$. Let $n = 7t + r$ with $r \in [0, 6]$. If $r \neq 0$, we have $\gcd(n, t) = 1$

$$S' \sim (6tg) \cdot (6tg) \cdot ((6t+2r)g) \cdot ((7t-r)g) \cdot ((3t+3r)g) \\ \text{(multiplied by } 2t\text{)} \\ \sim ((t+r)g) \cdot ((t+r)g) \cdot ((t-r)g) \cdot (2rg) \cdot ((4t-2r)g) \\ \text{(multiplied by } n-1 = 7t+r-1\text{)}.$$

Since $(t+r) + (t+r) + (t-r) + (2r) + (4t-2r) = 7t + r = n$, $\text{ind}(S) = 1$.

If $r = 0$, we have $p = 7$ and $\gcd(n, t-1) = 1$

$$S' \sim (6(t-1)g) \cdot (6(t-1)g) \cdot ((6t+2r+8)g) \cdot ((7t-r-14)g) \cdot \\ ((3t+3r+18)g) \text{ (multiplied by } 2(t-1)\text{)} \\ \sim ((t+r+6)g) \cdot ((t+r+6)g) \cdot ((t-r-8)g) \cdot ((2r+14)g) \cdot \\ ((4t-2r-18)g) \text{ (multiplied by } n-1 = 7t+r-1\text{)}.$$

Again, we have $\text{ind}(S) = 1$.

Case 5. $j = 2$. Let $n = 5t + r$ with $r \in [1, 4]$. Then $\gcd(n, t) = 1$. We have

$$S' \sim (tg) \cdot (tg) \cdot ((t+r)g) \cdot ((5t-r)g) \cdot ((2t+2r)g) \\ \text{(multiplied by } 2t\text{)} \\ \sim (4tg) \cdot (4tg) \cdot ((4t+4r)g) \cdot ((5t-4r)g) \cdot ((3t+7r)g) \\ \text{(multiplied by } 4\text{)} \\ \sim ((t+r)g) \cdot ((t+r)g) \cdot ((t-3r)g) \cdot ((4r)g) \cdot ((2t-6r)g) \\ \text{(multiplied by } n-1 = 5t+r-1\text{)}.$$

Since $(t+r) + (t+r) + (t-3r) + (4r) + (2t-6r) = 5t + r = n$, $\text{ind}(S) = 1$.

Case 6. $j = 1$. Let $n = 4t + r$ with $r \in [1, 3]$. Then $\gcd(n, t) = 1$. We have

$$\begin{aligned} S' &\sim (3tg) \cdot (3tg) \cdot ((3t+r)g) \cdot ((4t)g) \cdot ((3t+3r)g) \\ &\quad (\text{multiplied by } t) \\ &\sim ((t+r)g) \cdot ((t+r)g) \cdot (tg) \cdot (rg) \cdot ((t-2r)g) \\ &\quad (\text{multiplied by } n-1 = 4t+r-1). \end{aligned}$$

Since $(t+r) + (t+r) + t + r + (t-2r) = 4t+r = n$, we have $\text{ind}(S) = 1$. \square

Lemma 4.9. *If S is a minimal zero-sum sequence such that $2 < \frac{n}{c} < \frac{n}{b} < 3$, then $\text{ind}(S) = 1$.*

Proof. We will show that there exist integers $x, y \in [1, \lfloor \frac{b}{3} \rfloor]$ such that

$$\frac{n}{c} < 2 + \frac{x}{y} < \frac{n}{b}.$$

Then $(2y+x)a < \frac{yn}{b}3 \leq n$. If $\gcd(2y+x, n) = 1$, then by Lemma 4.1 $\text{ind}(S) = 1$ and we are done.

Suppose $n = 2b + b_0$, where $1 \leq b_0 \leq b-1$. Since n is prime power and $c = b+1 < \frac{n-1}{2} = b + \frac{b_0-1}{2}$, we infer that

$$b_0 \equiv 1 \pmod{2} \text{ and } b_0 > 3.$$

It suffices to show there exist $x, y \in [1, \lfloor \frac{b}{3} \rfloor]$ such that

$$\frac{b_0-2}{b+1} < \frac{x}{y} < \frac{b_0}{b} \text{ and } \gcd(2y+x, n) = 1.$$

If it is not easy to determine whether or not $\gcd(2y+x, n) = 1$, we will show that there exist two more integers $z, w \in [1, \lfloor \frac{b}{3} \rfloor]$ such that

$$\frac{b_0-2}{b+1} < \frac{z}{w} < \frac{b_0}{b}, \text{ and } 1 \leq |(2y+x) - (2w+z)| \leq p-1.$$

Thus either $\gcd(2y+x, n) = 1$ or $\gcd(2w+y, n) = 1$, again we have $\text{ind}(S) = 1$. We divide the proof into three cases.

Case 1. $b \equiv 0 \pmod{3}$. Since n is prime power, we infer that $b_0 \not\equiv 0 \pmod{3}$. Suppose $b = 3s$.

If $b_0 = 3t + 1$, then let $x = t$ and $y = s$. We infer that $\frac{3t-1}{3s+1} < \frac{x}{y} < \frac{3t+1}{3s}$. Since $n = 2b + b_0 = 3(2s + t) + 1 = 3(2y + x) + 1$, we have $\gcd(2y + x, n) = 1$ and we are done.

If $b_0 = 3t + 2$, let $x = t$ and $y = s$. Then $\frac{3t}{3s+1} < \frac{x}{y} < \frac{3t+2}{3s}$. Since $n = 2b + b_0 = 3(2s + t) + 2 = 3(2y + x) + 2$, we have $\gcd(2y + x, n) = 1$ and we are done.

Case 2. $b \equiv 1 \pmod{3}$. Since $n = p^\mu$ with a prime $p \geq 7$, we infer that $b_0 \not\equiv 1 \pmod{3}$. Suppose $b = 3s + 1$. Since $289 \leq n < 9s + 3$, $s \geq 32$.

Subcase 2.1. $b_0 = 3t$. Since $b_0 \equiv 1 \pmod{2}$ and $b_0 > 3$, $t \geq 3$.

If $s < 2t - 2$, let $x = t - 1$, $y = s$. Then $\frac{3t-2}{3s+2} < \frac{x}{y} < \frac{3t}{3s+1}$. Since $n = 2b + b_0 = 3(2s + t) + 2 = 3(2y + x) + 5$, we have $\gcd(2y + x, n) = 1$ and we are done.

Next assume that $s \geq 2t - 2$. Choose $y = s - \lceil \frac{s-2t+3}{3t-2} \rceil$ and $x = t - 1$. Then $\frac{3t-2}{3s+2} < \frac{x}{y} < \frac{3t}{3s+1}$. If $s \geq 3t + 1$, let $w = y - 1$, $z = t - 1$, then $\frac{3t-2}{3s+2} < \frac{z}{w} < \frac{3t}{3s+1}$ and $|(2y + x) - (2w + z)| = 2$, so we are done. Now assume that

$$2t - 2 \leq s \leq 3t.$$

Since $32 \leq s \leq 3t$, $t \geq 11$. If $s \leq \frac{11(t-1)}{4}$, let $y = s - \lceil \frac{s-2t+3}{3t-2} \rceil = s - 1$ and $x = t - 1$ as before. Let $w = s - \lceil \frac{4s-2t+5}{3t-2} \rceil = s - 3$ and $z = t - 2$. Then $\frac{3t-2}{3s+2} < \frac{x}{y}, \frac{z}{w} < \frac{3t}{3s+1}$ and $|(2y + x) - (2w + z)| = 5$, so we are done. If $\frac{11(t-1)}{4} < s \leq 3t$, let $y = s - 2$, $x = t - 1$, $w = s - 4$ and $z = t - 2$. Then $\frac{3t-2}{3s+2} < \frac{x}{y}, \frac{z}{w} < \frac{3t}{3s+1}$ and $|(2y + x) - (2w + z)| = 5$, so we are done.

Subcase 2.2. $b_0 = 3t + 2$. Let $x = t$ and $y = s$. We infer that $\frac{3t}{3s+2} < \frac{x}{y} < \frac{3t+2}{3s+1}$. Since $n = 2b + b_0 = 3(2s + t) + 4 = 3(2y + x) + 4$, we have $\gcd(2y + x, n) = 1$ and we are done.

Case 3. $b \equiv 2 \pmod{3}$. Since $n = p^\mu$ with a prime $p \geq 7$, we infer that $b_0 \not\equiv 2 \pmod{3}$. Suppose $b = 3s + 2$. Since $289 \leq 3b - 1 = 9s + 5$, $s \geq 32$.

Subcase 3.1. $b_0 \equiv 0 \pmod{3}$. Suppose $b_0 = 3t$. Recall that $b_0 = 3t \equiv 1 \pmod{2}$ and $b_0 > 3$, we infer that $t \geq 3$ and $t \equiv 1 \pmod{2}$.

If $s < \frac{3t-4}{2}$, then let $x = t - 1$, $y = s$; $z = t - 2$, $w = s - 1$. We infer that $\frac{3t-2}{3s+3} < \frac{z}{w} < \frac{x}{y} < \frac{3t}{3s+2}$ and $|(2y + x) - (2w + z)| = 3$, so we

are done. Next assume that $s \geq \frac{3t-4}{2}$. If $s < 3t - 3$, then $t \geq 13$. Let $x = t - 1, y = s; z = t - 2, w = s - 2$. Then $\frac{3t-2}{3s+3} < \frac{x}{y}, \frac{z}{w} < \frac{3t}{3s+2}$ and $|(2y + x) - (2w + z)| = 5$, so we are done.

If $s \geq 3t - 3$, choose $y = s - \lceil \frac{s-3t+4}{3t-2} \rceil$ and $x = t - 1$. Then $\frac{3t-2}{3s+3} < \frac{x}{y} < \frac{3t}{3s+2}$. Let $w = y - 1, z = t - 1$. Note that if $s < 3t$ then $t > 11$. Thus we have either $s \geq 3t$ or $t \geq 9$. Therefore, $\frac{3t-2}{3s+3} < \frac{z}{w} < \frac{3t}{3s+2}$. Since $|(2y + x) - (2w + z)| = 2$, we are done.

Subcase 3.2. $b_0 \equiv 1 \pmod{3}$. Suppose $b_0 = 3t + 1$. Recall that $b_0 = 3t + 1 \equiv 1 \pmod{2}$ and $b_0 > 3$. Hence $t \equiv 0 \pmod{2}$ and thus $t \geq 2$.

If $s > 5t + 1$, let $x = t, y = s; z = t, w = s - 1$. Then $\frac{3t-1}{3s+3} < \frac{x}{y} < \frac{z}{w} < \frac{3t+1}{3s+2}$ and $|(2y + x) - (2w + z)| = 2$, so we are done. Hence we may assume that $s \leq 5t + 1$.

If $s \geq 3t - 2$, let $x = z = t - 1, y = s - \lceil \frac{2s-3t+4}{3t-1} \rceil, w = y - 1$. If $t \geq 10$, then $\frac{3t-1}{3s+3} < \frac{z}{w}, \frac{x}{y} < \frac{3t+1}{3s+2}$ and $|(2y + x) - (2w + z)| = 2$, so we are done. Hence we may assume that $t \leq 9$ and $3t - 2 \leq s \leq 5t + 1$. Note that $t \equiv 0 \pmod{2}$ and $s \geq 33$. Hence $t = 8$ and then $33 \leq s \leq 41$. Let $x = z = t - 1, y = s - 3, w = s - 4$. Then $\frac{3t-1}{3s+3} < \frac{z}{w}, \frac{x}{y} < \frac{3t+1}{3s+2}$ and $|(2y + x) - (2w + z)| = 2$, so we are done.

If $2t < s < 3t - 2$, let $x = t, y = s; z = t - 1, w = s - 1$. Then $\frac{3t-1}{3s+3} < \frac{x}{y}, \frac{z}{w} < \frac{3t+1}{3s+2}$ and $|(2y + x) - (2w + z)| = 3$, so we are done.

Hence we may assume that $s \leq 2t$. Note that $t \geq \frac{s}{2} \geq \frac{33}{2}$, so $t \geq 17$.

If $s < \frac{6t-7}{5}$, let $x = t - 1, y = s; z = t - 2, w = s - 1$. Then $\frac{3t-1}{3s+3} < \frac{z}{w} < \frac{x}{y} < \frac{3t+1}{3s+2}$ and $|(2y + x) - (2w + z)| = 3$, so we are done. Hence we may assume that $s \geq \frac{6t-7}{5}$.

If $s < \frac{3t-3}{2}$, let $x = t - 1, y = s; z = t - 2, w = s - \lceil \frac{5s-3t+3}{3t-1} \rceil$. Then $\frac{3t-1}{3s+3} < \frac{z}{w}, \frac{x}{y} < \frac{3t+1}{3s+2}$ and $1 \leq |(2y + x) - (2w + z)| \leq 5$, so we are done. Hence we may assume that $s \geq \frac{3t-3}{2}$.

Next assume that $\frac{3t-3}{2} \leq s \leq 2t$.

Let $x = t - 1$ and $y = s - 1$. We infer that $\frac{3t-1}{3s+3} < \frac{t-1}{s-1} < \frac{3t+1}{3s+2}$. Let $z = t - 2, w = s - \lceil \frac{5s-3t+3}{3t-1} \rceil$. Then $\frac{3t-1}{3s+3} < \frac{z}{w}, \frac{x}{y} < \frac{3t+1}{3s+2}$ and $1 \leq |(2y + x) - (2w + z)| \leq 5$, so we are done. \square

Now Proposition 3.3 follows immediately from Lemmas 4.7, 4.8, 4.9 and Proposition 2.2.

Our main result (Theorem 1.2) together with the main results in [12] provides a complete characterization of the index of all minimal zero-sum sequences S of length 5 with $h(S) \geq 2$ over finite cyclic groups of order p^μ for $\mu \geq 1$ below. As mentioned early, we may always assume that $S = (x_1g) \cdot (x_2g) \cdot (x_3g) \cdot (x_4g) \cdot (x_5g)$ with $x_1 = x_2$ and $\gcd(n, x_1, x_2, x_3, x_4, x_5) = 1$. If $\mu = 1$, in addition, we may assume that $x_1 = x_2 = 1$ and $2 \leq x_3 \leq x_4 \leq x_5 \leq p - 3$.

Theorem 4.10. *Let G be a cyclic group of prime power order $n = p^\mu$ with $\mu \geq 1$ and S be a minimal zero-sum sequence of length 5 over G . Then*

(I). *If $\mu \geq 2$ and $p \geq 7$, then $\text{ind}(S) = 2$ if and only if $S = (mg) \cdot (mg) \cdot (m\frac{n-1}{2}g) \cdot (m\frac{n+3}{2}g) \cdot (m(n-3)g)$ where m is a positive integer such that $\gcd(m, n) = 1$.*

(II). *If $\mu = 1$, then $\text{ind}(S) = 2$ if and only if one of the following holds.*

- (1) $p \geq 11$ and $S = g^2 \cdot (\frac{p-1}{2}g) \cdot (\frac{p+3}{2}g) \cdot ((p-3)g)$.
- (2) $p = 17$ and $S = g^2 \cdot (8g) \cdot (11g) \cdot (13g)$.
- (3) $p = 19$, and $S = g^2 \cdot (6g) \cdot (14g) \cdot (16g)$ or $S = g^2 \cdot (9g) \cdot (12g) \cdot (15g)$.
- (4) $p = 23$, and $S = g^2 \cdot (11g) \cdot (15g) \cdot (18g)$ or $S = g^2 \cdot (9g) \cdot (15g) \cdot (20g)$.
- (5) $p = 29$ and $S = g^2 \cdot (14g) \cdot (19g) \cdot (23g)$.

We remark that the case when $h(S) = 1$ is much more complicated and $\text{ind}(S)$ is not yet determined (even when $n = p$ is a prime number). We conclude the paper by listing this case as an open problem.

Open Problem. Let S be a minimal zero-sum sequence of length five over a cyclic group of order n . Determine $\text{ind}(S)$ when the height $h(S) = 1$.

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