A New Combinatorial Identity for Catalan Numbers

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Abstract

In this article, we prove a conjecture about the equality of two generating functions described in "*From Parking Functions to Gelfand Pairs (Aker, Can 2012)*" attached to two sets whose cardinalities are given by Catalan numbers: We establish a combinatorial bijection between the two sets on which the two generating functions were based on.

Keywords: Catalan numbers, parking functions, generating functions, Dyck path, bijection, necklace

1 Introduction

Catalan numbers enumerate a diverse collection of disparate mathematical objects which seem unrelated at first impression. For a nonnegative integer n, the n-th Catalan number, C_n, is $\frac{1}{n+1}\binom{2n}{n}$. A standard combinatorial definition is that the Catalan number, C_n, is the number of Dyck paths in an $n \times n$ box. A Dyck path in $n \times n$ box is a path starting from the corner $(0,0)$ to the corner (n,n) which stays always weakly below the diagonal (or always weakly above).

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For any positive integer n and integer r, define the set $U(n,r)$ as in [1]:

$$U(n,r) = \left\{ (u_i) \in \mathbb{N}^{n+1} : \sum_{i=0}^{n} u_i = n \text{ and } \sum_{i=0}^{n} i\, u_i \equiv r \pmod{n+1} \right\}.$$

The following set $V(n)$ appears as (q^5) in Stanley's Catalan Addendum [2]):

$$V(n) = \left\{ (v_i) \in \mathbb{N}^n : \sum_{i=1}^{n} v_i = n \text{ and } \sum_{i=1}^{j} v_i \geq j \text{ for all } j = 1, \dots, n \right\}.$$

For a sequence $w = (w_0, w_1, \dots, w_n)$ of total n, denote the multinomial coefficient $\binom{n}{w_0, w_1, \dots, w_n}$ by $\binom{n}{w}$.

As in [1], attach to the sets $U(n,r)$ and $V(n)$ generating functions, the sums $\sum_w q^{\binom{n}{w}}$ where q is an indeterminate and the index w runs over the corresponding set. Denote the generating functions by $u(n,r)$ and $v(n)$ respectively.

In [1], Aker and Can conjecture that

Conjecture (Conjecture 1.1 [1]). *For a positive integer n and an integer r, the generating sets $u(n,r)$ and $v(n)$ coincide.*

We prove this conjecture in Theorem 8 as a direct corollary of a bijection established between the sets $U(n,r)$ and $V(n)$ in Theorem 7.

2 The Sets $U(n)$, $U(n,r)$ and the Shift Operator

In this section, we prove that the cardinality of the set $U(n,r)$ is equal to the n-th Catalan number, C_n.

For a positive integer n, the following sets are in bijection:

$$U(n) := \left\{ (u_0, \dots, u_n) \in \mathbb{N}^{n+1} : \sum_{i=0}^{n} u_i = n \right\},$$

$$\overline{U}(n) := \left\{ (u_0, \dots, u_n) \in (\mathbb{Z}/(n+1)\mathbb{Z})^{n+1} : \sum_{i=0}^{n} u_i = n \right\}.$$

Denote the set of n-element subsets of a set X by $\binom{X}{n}$ and the set $\{1, 2, 3, \dots, n\}$ by $[n]$. For $u = (u_0, u_1, \dots, u_n) \in U(n)$, define $F : U(n) \to$

$\binom{[2n]}{n}$ as follows:

$F(u_0, u_1, ..., u_n) := \{u_1 + 1 < u_1 + 1 + u_2 + 1 < ... < u_1 + u_2 + ... + u_n + n\}.$

Then,

Lemma 1. *The map $F : U(n) \to \binom{[2n]}{n}$ is a bijection. The cardinalities of the sets $U(n)$ and $\overline{U}(n)$ are equal to $\binom{2n}{n}$.*

Proof. First, the map F is well–defined: Since $0 \le u_1$, we have $1 \le u_1 + 1$. Similarly, $0 \le u_i$ implies that for $i = 1, \ldots, n$,

$$u_1 + u_2 + \cdots + u_{i-1} + i - 1 < u_1 + u_2 + \cdots + u_i + i.$$

We also have $u_1+u_2+\cdots+u_n+n \le u_0+u_1+u_2+\cdots+u_n+n = n+n = 2n$.

The sequence

$$u_1+1 < u_1+u_2+2 < \cdots < u_1+u_2+\cdots+u_i+i < \cdots < u_1+u_2+\cdots+u_n+n$$

forms an n-element subset of the set $[2n]$.

We prove that F is a bijection by providing an inverse function, G.

For any n-element $a = \{a_1 < a_2 < ... < a_n\}$ subset of $[2n]$, set

$$G(a) := (2n - a_n, a_1 - 1, a_2 - a_1 - 1, \ldots, a_n - a_{n-1} - 1).$$

Let $u = G(a)$. Such u lies in $U(n)$; that is, all entries of u are nonnegative and they add up to n. Because $1 \le a_1$, we have $u_1 = a_1 - 1 \ge 0$. Similarly for $i = 2, \ldots, n$, $a_{i-1} < a_i$, hence $u_i = a_i - a_{i-1} - 1 \ge 0$. Finally, $a_n \le 2n$ implies that $u_0 = 2n - a_n \ge 0$.

Also the sum of all terms telescope and cancel each other:

$$2n - a_n + a_1 - 1 + a_2 - a_1 - 1 + \cdots + a_n - a_{n-1} - 1 = 2n - n = n.$$

Clearly, F and G are inverses of each other, hence F is a bijection. □

Let s be the cyclic shift operator on the set $U(n)$: For $(u_0, \ldots, u_n)$, set

$$s(u_0, \ldots, u_n) := (u_1, \ldots, u_n, u_0).$$

The operator s induces an action of $\mathbb{Z}/(n+1)\mathbb{Z}$ on the set $U(n)$.

Define another map $\psi : U(n) \longrightarrow \mathbb{Z}/(n+1)\mathbb{Z}$. For $(u_0, \ldots, u_n) \in U(n)$, set

$$\psi(u_0, \ldots, u_n) := \sum_{i=0}^{n} i\, u_i.$$

Lemma 2. *1. For any $u \in U(n)$, $\psi(s(u)) = \psi(u) + 1$.*

2. Cyclic shift operator s is a fixed-point free automorphism of $U(n)$.

3. For any $r \in \mathbb{Z}/(n+1)\mathbb{Z}$, shift operator s takes the set $U(n, r)$ bijectively to $U(n, r+1)$.

Proof. *1.* For any $u = (u_0, \ldots, u_n) \in U(n)$,

$$\begin{aligned}\psi(s(u)) &= \sum is(u)_i = \sum iu_{i+1} = \sum_{i=0}^{n}(j-1)u_j = \sum_{i=0}^{n} ju_j - \sum_{i=0}^{n} u_j \\ &= \psi(u) - n \equiv \psi(u) + 1.\end{aligned}$$

2. Suppose the automorphism s fixes some $u = (u_0, \ldots, u_n) \in U(n)$, this implies that all $n+1$ coordinates of u are equal. On the other hand, as an element in $U(n)$, sum of the coordinates of $U(n)$ is equal to n, which is clearly a contradiction. Therefore the automorphism s is fixed-point free.

3. Since it is an automorphism, any restriction of s to a subset of $U(n)$ is a bijection. By (1), the automorphism s maps $U(n, r)$ to $U(n, r+1)$ which shows that the restriction $s : U(n, r) \to U(n, r+1)$ is a bijection. ☐

Corollary 3. *For a positive integer n and an integer r, the cardinality of the set $U(n, r)$ is the n-th Catalan number, C_n.*

Proof. Note that $U(n)$ is a disjoint union of $U(n, r)$'s where $r \in \mathbb{Z}/(n+1)$:

$$U(n) = \bigsqcup_{r \in \mathbb{Z}/(n+1)} U(n, r)$$

and

$$|U(n)/\langle s \rangle| = \frac{1}{|\langle s \rangle|}|U(n)| = \frac{1}{n+1}\binom{2n}{n} = |U(n, r)|.$$

☐

3 Necklaces and the Main Result

In this section, we prove the equality of the generating functions $u(n, r)$ and $v(n)$ introduced in the introduction. We first establish a bijection the sets $U(n)/\langle s \rangle$ and $V(n)$, which in return produces a bijection between $U(n, r)$

and $V(n)$. The equality of the generating functions follows as a direct corollary.

Define a *string of pearls* to be a finite sequence of nonnegative integers. Elements of the sequence are called *pearls*, each with an assigned value in the string. For convenience, we allow such a string to be circular. Such a *circular* string is called a *necklace.*

Definition 4. Given a string of pearls A labelled sequentially $a_1, a_2, \ldots$, by $\ell(A)$ denote the length of string A and by $|A|$ denote the sum $a_1 + a_2 + \cdots$.

A subsequence S of a string A consisting of consecutive pearls is called a *substring.* Write $S \le A$. Denote the set of all substrings of A by $Sub(A)$. Then, $Sub(A, \le)$ is a partially ordered set.

Call a string B a *block* if $b_1 + b_2 + \cdots + b_k \ge k$ for all $k = 1, \ldots, \ell(B)$. *Blocks* of a string A are those substrings which are also blocks. Denote the set of all blocks of A by $Blocks(A)$.

Let $A = (1, 0, 2, 1, 0, 3)$. For instance, $(2, 1, 0, 3)$ is substring, whereas $(2, 1, 3)$ is not. The blocks of A are (1), (2), $(2, 1)$, $(2, 1, 0)$, $(2, 1, 0, 3)$, (1) (this is the 1 to the right of 2) and (3). Note that A is not a block.

If a string A has at least one positive pearl, the set of blocks of A is not empty. The partial order $\le$ on the set of substrings of A induces a partial order on the set of blocks of A.

Notice that in the above example, each positive pearl is contained in a unique block of maximal length.

Now fix a necklace N in $U(n)/\langle s\rangle$, i.e. a circular string of $n + 1$ nonnegative integers whose sum is n. Note that any such necklace contains at least one pearl with label 0.

Fix a clockwise orientation for necklaces. For instance, in the figure is

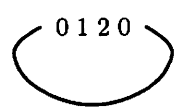


Figure 1: An example of a necklace

the necklace $(0, 1, 2, 0)$ or 0120, which can be equivalently written as 1200, 2001, or as 0012.

Lemma 5. *Suppose B is a maximal block of N. Then,*

1. *Pearls adjacent to B are labelled* 0.
2. $|B| = \ell(B)$.

Proof. Let's analyze the pearls adjacents to the maximal block B in the necklace N.

1. Let's say the pearl P after B has a label ≥ 1. That is, BP is a string of pearls, where B is a maximal block and $P \geq 1$.

Then, $|BP| = |B| + |P| \geq \ell(B) + 1 = \ell(BP)$.

Hence BP is a block which contains B. This contradicts the maximality of B. Reversing the orientation proves the statement for the pearl preceeding the maximal block B. So, any pearl next to B is labelled 0.

2. Assume that $|B| > \ell(B)$.

We proved that a pearl P adjacent to B is labelled 0. (There must be such a pearl, otherwise $|B| \geq \ell(B) \geq n+1$).

Say P follows B. Then BP is a block: Because $|B| \geq \ell(B) + 1$;

$$|BP| = |B| + |P| = |B| \geq \ell(B) + 1 = \ell(BP).$$

Once again, this contradicts the maximality of B. Therefore for any maximal block B is stacked by 0's before and after and $|B| = \ell(B)$. □

Being a poset, the set of blocks of the necklace N must have maximal blocks. In fact,

Lemma 6. *A necklace N contains a unique maximal block B, where $|B| = \ell(B) = n$.*

Proof. Lemma 5 implies that the necklace N consists of (possibly several) maximal blocks $B_1, ..., B_m$ separated by strings of zeros (Figure 2).

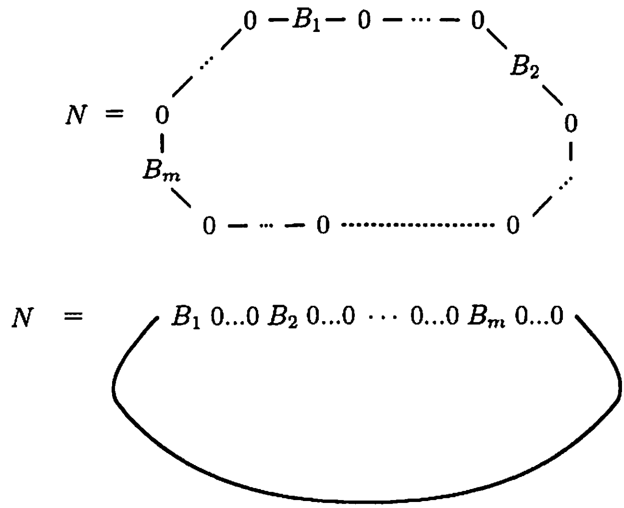


Figure 2: The necklace N depicted in two different, yet equivalent forms

Note that

- Sum of all pearls $= |B_1| + ... + |B_m| = n$,
- Number of pearls $= \ell(B_1) + ... + \ell(B_m) + \underbrace{m}_{\text{for } m \text{ zeros}} = n + 1$.

Therefore,

$$n + 1 = \ell(B_1) + ... + \ell(B_m) + m.$$

Because blocks $B_1, \dots, B_m$ are maximal,

$$n + 1 = |B_1| + ... + |B_m| + m = n + m.$$

If follows that $m = 1$, i.e. the necklace N contains a unique maximal block B, where $|B| = \ell(B) = n$. □

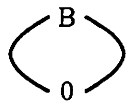
Notice that the maximal block of a necklace is an element of the set

$$V(n) = \left\{ (v_i) \in \mathbb{N}^n : \sum_{i=1}^{n} v_i = n \text{ and } \sum_{i=1}^{j} v_i \geq j \text{ for all } j = 1, ..., n \right\}.$$

A direct consequence of the previous lemma is

Theorem 7. *The following map is a bijection:*

$$\phi : V(n) \longrightarrow \underbrace{U(n)/\langle s \rangle}_{Necklaces}$$

B ⟶ The necklace 

A direct corollary of the bijection is

Theorem 8 (Conjecture 1.1 [1]). *For a positive integer n and an integer r, the generating functions $u(n,r)$ and $v(n)$ coincide.*

Proof. For v in $V(n)$, let $u = \phi(n)$. Then, $\binom{n}{u} = \binom{n}{v}$. By Theorem 7 and Corollary 3,

$$v(n) = \sum_{v \in V(n)} q^{\binom{n}{v}} = \sum_{u \in U(n)/\langle s \rangle} q^{\binom{n}{u}} = \sum_{u \in U(n,r)} q^{\binom{n}{u}} = u(n,r).$$

□

References

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