The chromatic equivalence class of graph $\psi_n^3(n-3,1)$ *

Yaping Mao[1] † Chengfu Ye[1] Hengzhe Li[2] Shumin Zhang[1]

[1] Department of Mathematics, Qinghai Normal University, Xining, Qinghai 810008, P. R. China

[2] College of Mathematics and Information Science, Henan Normal University, Xingxiang 453007 China

Abstract

Two graphs are defined to be adjointly equivalent if their complements are chromatically equivalent. Recently, we introduced a new invariant of a graph G, which is called the fifth character $R_5(G)$. Using this invariant and the properties of the adjoint polynomials, we completely determine the adjoint equivalence class of $\psi_n^3(n-3,1)$. According to the relations between adjoint polynomial and chromatic polynomial, we also simultaneously determine the chromatic equivalence class of $\psi_n^3(n-3,1)$.

Keywords: chromatic equivalence class; adjoint polynomial; the smallest real root; the fifth character.

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1 Introduction

The graphs considered in this paper are finite undirected and simple graphs. We follow the notation of Bondy and Murty [1], unless otherwise stated. For a graph G, let $V(G)$, $E(G)$, $p(G)$, $q(G)$ and $\overline{G}$ be the set of vertices, the set of edges, the order, the size and the complement of G, respectively. For a graph G, we denote by $P(G,\lambda)$ the chromatic polynomial of G. A partition $\{A_1, A_2, \cdots, A_r\}$ of $V(G)$, where r is a positive integer, is called an *r-independent partition* of graph G if every A_i is nonempty independent set of G. Denote by $\alpha(G,r)$ the number of r-independent partitions of G. Thus the chromatic polynomial G is $P(G,\lambda) = \sum_{r\geq 1}\alpha(G,r)(\lambda)_r$, where $(\lambda)_r = \lambda(\lambda-1)\cdots(\lambda-r+1)$ for all $r \geq 1$. The readers can turn to [15] for details on chromatic polynomials. Two graphs G and H are said to be *chromatically equivalent*, denoted by $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$. By $[G]$ we denote the equivalence class determined by G under "$\sim$". It is obvious that "$\sim$" is an equivalence relation on the family of all

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†Corresponding author. Email: maoyaping@ymail.com

graphs. A graph G is called *chromatically unique* (or simply χ-*unique*) if $H \cong G$ whenever $H \sim G$. See [6, 7] for many results on this field.

Definition 1.1. [9] *Let G be a graph with p vertices, the polynomial*

$$h(G,x) = \sum_{i=1}^{p} \alpha(G,i)x^i$$

is called its adjoint polynomial.

Definition 1.2. [9] *Let G be a graph and $h_1(G,x)$ be the polynomial with a nonzero constant term such that $h(G,x) = x^{\rho(G)}h_1(G,x)$. If $h_1(G,x)$ is an irreducible polynomial over the rational number field, then G is called* irreducible graph.

Two graphs G and H are said to be *adjointly equivalent*, denoted by $G \sim^h H$, if $h(G,x) = h(H,x)$. Evidently, "$\sim^h$" is an equivalence relation on the family of all graphs. Let $[G]_h = \{H \mid H \sim^h G\}$. A graph G is said to be *adjointly unique*(or simply h-*unique*) if $G \cong H$ whenever $G \sim^h H$.

Theorem 1.1. [3] (1) $G \sim^h H$ *if and only if* $\overline{G} \sim \overline{H}$.

(2) $[G]_h = \{H \mid \overline{H} \in [\overline{G}]\}$.

(3) G *is χ-unique if and only if $\overline{G}$ is h-unique.*

Now we define some classes of graphs with order n, which will be used throughout the paper.

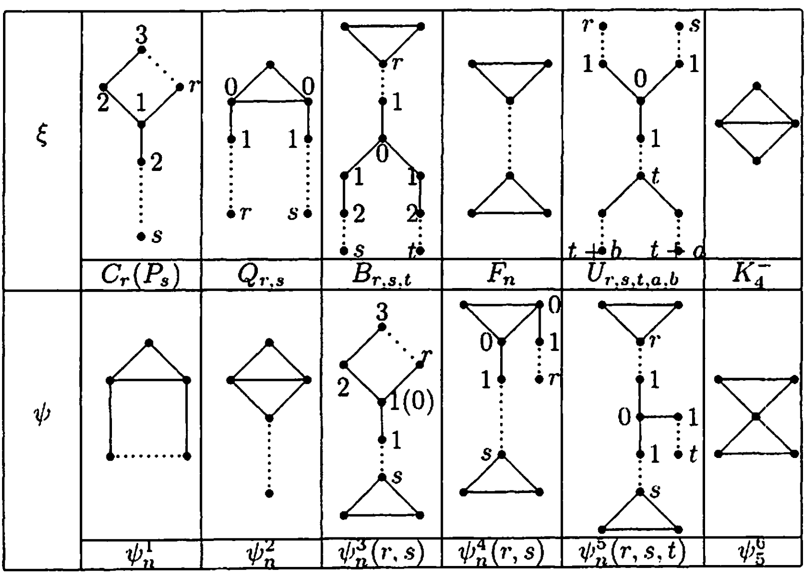

Figure 1 Families ξ and $\psi(n_1 = r+s+t, n_2 = r+s+t+a+b)$

(1) C_n (resp. P_n) denotes the cycle (resp. the path) of order n, and write $\mathscr{C} = \{C_n \,|\, n \geq 3\}$, $\mathscr{P} = \{P_n \,|\, n \geq 2\}$ and $\mathscr{U} = \{U_{1,1,t,1,1} \,|\, t \geq 1\}$.

(2) $D_n (n \geq 4)$ denotes the graph obtained from C_3 and P_{n-2} by identifying a vertex of C_3 with a pendent vertex of P_{n-2}.

(3) T_{l_1,l_2,l_3} is a tree with a vertex v of degree 3 such that $T_{l_1,l_2,l_3} - v = P_{l_1} \cup P_{l_2} \cup P_{l_3}$ and $l_3 \geq l_2 \geq l_1$, write $\mathscr{T}^0 = \{T_{1,1,l_3} \,|\, l_3 \geq 1\}$ and $\mathscr{T} = \{T_{l_1,l_2,l_3} \,|\, (l_1, l_2, l_3) \neq (1, 1, 1)\}$.

(4) $\vartheta = \{C_n, D_n, K_1, T_{l_1,l_2,l_3} \,|\, n \geq 4\}$.

(5) $\xi = \{C_r(P_s), Q(r, s), B_{r,s,t}, F_n, U_{r,s,t,a,b}, K_4^-\}$.

(6) $\psi = \{\psi_n^1, \psi_n^2, \psi_n^3(r, s), \psi_n^4(r, s), \psi_n^5(r, s, t), \psi_8^6\}$.

For convenience, we simply denote $h(G, x)$ by $h(G)$ and $h_1(G, x)$ by $h_1(G)$. By $\beta(G)$ and $\beta_{min}(G)$ we denote the smallest real root and the minimal extremes of the smallest real root of $h(G)$, respectively. Let $d_G(v)$, simply denoted by $d(v)$, be the degree of vertex v. For two graphs G and H, denote by $G \cup H$ the disjoint union of G and H, and mH stands for the disjoint union of m copies. By K_n we denote the complete graph with order n, let $n_G(K_3)$ and $n_G(K_4)$ denote the number of subgraphs isomorphic to K_3 and K_4, respectively. Let $g(x) \,|\, f(x)$(resp. $g(x) \nmid f(x)$) denote $g(x)$ divides $f(x)$(resp. $g(x)$ does not divide $f(x)$) and $\partial(f(x))$ denote the degree of $f(x)$. By $(f(x), g(x))$ we denote the largest common factor of $f(x)$ and $g(x)$ on the real field. Let $N_G(v)$ be the neighborhood set of a vertex v.

It is an important problem to determine $[G]$ for a given graph G. From Theorem 1.1, it is obvious that the goal of determining $[G]$ can be realized by determining $[G]_h$. Thus, if $q(G)$ is large, it may be easier to study $[\overline{G}]_h$ rather than $[G]$. The determination of $[G]$ for a given graph G has received much attention in [12, 13, 14, 21, 22, 23] recently. In this paper, using the properties of adjoint polynomials, we determine the $[\psi_n^3(n-3, 1)]_h$ of graph $\psi_n^3(n-3, 1)$, simultaneously, $[\overline{\psi_n^3(n-3, 1)}]$ is also determined, where $n \geq 7$.

2 Preliminaries

For a polynomial $f(x) = x^n + b_1 x^{n-1} + b_2 x^{n-2} + \cdots + b_n$, we define

$$R_1(f(x)) = \begin{cases} -\binom{b_1}{2} + 1, & if\ n = 1; \\ b_2 - \binom{b_1 - 1}{2} + 1, & if\ n \geq 2. \end{cases}$$

For a graph G, we write $R_1(G)$ instead of $R_1(h(G))$.

Definition 2.1. [2, 9] *Let G be a graph with q edges.*

(1) *The first character of a graph G is defined as*

$$R_1(G) = \begin{cases} 0, & if\ q = 0; \\ b_2 - \binom{b_1 - 1}{2} + 1, & if\ q > 0. \end{cases}$$

(2) *The second character of a graph G is defined as*

$$R_2(G) = b_3(G) - \binom{b_1(G)}{3} - (b_1(G) - 2)\left(b_2(G) - \binom{b_1(G)}{2}\right) - b_1(G),$$

where $b_i(G)(0 \le i \le 3)$ is the first four coefficients of $h(G)$.

Lemma 2.1. [2, 9] *Let G be a graph with k components of $G_1, G_2, \cdots, G_k$. Then $h(G) = \prod_{i=1}^{k} h(G_i)$ and $R_j(G) = \sum_{i=1}^{k} R_j(G_i)$ for $j = 1, 2$.*

It is obvious that $R_j(G)$ is an invariant of graphs. So, for any two graphs G and H, we have $R_j(G) = R_j(H)$ for $j = 1, 2$ if $h(G) = h(H)$ or $h_1(G) = h_1(H)$.

Lemma 2.2. [9, 10] *Let G be a graph with p vertices and q edges. Denote M the set of the triangles in G and by $M(i)$ the number of triangles which cover the vertex i in G. If the degree sequence of G is $(d_1, d_2, \cdots, d_p)$, then the first four coefficients of $h(G)$ are, respectively,*

(1) $b_0(G) = 1, b_1(G) = q$.

(2) $b_2(G) = \binom{q+1}{2} - \frac{1}{2}\sum_{i=1}^{p} d_i^2 + n_G(K_3)$.

(3) $b_3(G) = \frac{q}{6}(q^2+3q+4) - \frac{q+2}{2}\sum_{i=1}^{p} d_i^2 + \frac{1}{3}\sum_{i=1}^{p} d_i^3 - \sum_{ij \in E(G)} d_i d_j - \sum_{i \in M} M(i) d_i + (q+2) n_G(K_3) + n_G(K_4)$, *where* $b_i(G) = \alpha(\overline{G}, p-i)$ $(i = 0, 1, 2, 3)$.

For an edge $e = v_1 v_2$ of a graph G, the graph $G * e$ is defined as follow: the vertex set of $G * e$ is $(V(G) - \{v_1, v_2\}) \bigcup v(v \notin G)$, and the edge set of $G * e$ is $\{e' | e' \in E(G), e'$ is not incident with v_1 or $v_2\} \cup \{uv | u \in N_G(v_1) \cap N_G(v_2)\}$, where $N_G(v)$ is the set of vertices of G which are adjacent to v.

Lemma 2.3. [9] *Let G be a graph with $e \in E(G)$. Then*

$$h(G, x) = h(G - e, x) + h(G * e, x),$$

where $G - e$ denotes the graph obtained by deleting the edge e from G.

Lemma 2.4. [9] (1) *For $n \ge 2$, $h(P_n) = \sum_{k \le n} \binom{k}{n-k} x^k$.*

(2) *For $n \ge 4$, $h(D_n) = \sum_{k \le n} (\frac{n}{k}\binom{k}{n-k} + \binom{k-2}{n-k-3}) x^k$.*

(3) *For $n \ge 4$, $m \ge 6$, $h(P_n) = x(h(P_{n-1}) + h(P_{n-2}))$, $h(D_m) = x(h(D_{m-1}) + h(D_{m-2}))$.*

Lemma 2.5. [25] *Let $\{g_i(x)\}$, simply denoted by $\{g_i\}$, be a polynomial sequence with integer coefficients and $g_n(x) = x(g_{n-1}(x) + g_{n-2}(x))$. Then*

(1) $g_n(x) = h(P_k) g_{n-k}(x) + x h(P_{k-1}) g_{n-k-1}(x)$.

(2) $h_1(P_n) \mid g_{k(n+1)+i}(x)$ *if and only if* $h_1(P_n) | g_i(x)$, *where* $0 \le i \le n$, $n \ge 2$ *and* $k \ge 1$.

Lemma 2.6. [4, 8] *Let G be a nontrivial connected graph with n vertices. Then*

(1) $R_1(G) \le 1$, *and the equality holds if and only if* $G \cong P_n (n \ge 2)$ *or* $G \cong K_3$.

(2) $R_1(G) = 0$ *if and only if* $G \in \vartheta$.

(3) $R_1(G) = -1$ *if and only if* $G \in \xi$, *especially*, $q(G) = p(G) + 1$ *if and only if* $G \in \{F_n | n \ge 6\} \cup \{K_4^-\}$.

(4) $R_1(G) = -2$ *if and only if* $G \in \varphi$ *for* $q(G) = p(G)$, $G \in \psi$ *for* $q(G) = p(G) + 1$ *and* $G \cong K_4$ *for* $q(G) = p(G) + 2$.

(5) $R_1(G) = -3$ *if and only if* $G \in \phi$ *(see Figure 3) for* $q(G) = p(G) + 1$ *and* $G \in \zeta$ *(see Figure 2) for* $q(G) = p(G) + 2$.

(6) $R_1(G) = -4$ *if and only if* $G \in \theta$ *(see Figure 4) for* $q(G) = p(G) + 2$.

Lemma 2.7. [5] *Let G be a connected graph.*

(1) *If* $R_1(G) = 0, -1, -2$, *then* $q(G) - p(G) \leq |R_1(G)|$.

(2) *If* $R_1(G) = -3$, *then* $q(G) - p(G) \leq |R_1(G) + 1|$.

(3) *If* $R_1(G) \leq -4$, *then* $q(G) - p(G) < |R_1(G) + 1|$.

Lemma 2.8. [25] *Let G be a connected graph and H a proper subgraph of G. Then* $\beta(G) < \beta(H)$.

Lemma 2.9. [25] *Let G be a connected graph. Then*

(1) $\beta(G) = -4$ *if and only if* $G \in \{T(1,2,5), T(2,2,2), T(1,3,3), K_{1,4}, C_4(P_2), Q(1,1), K_4^-, D_8\} \cup \mathcal{U}$.

(2) $\beta(G) > -4$ *if and only if* $G \in \{K_1, T(1,2,i)(2 \leq i \leq 4), D_i(4 \leq i \leq 7)\} \cup \mathcal{P} \cup \mathscr{C} \cup \mathscr{T}^0$.

Lemma 2.10. [25] *Let G be a connected graph. Then* $-(2+\sqrt{5}) \leq \beta(G) < -4$ *if and only if G is one of the following graphs:*

(1) T_{l_1,l_2,l_3} *for* $l_1 = 1, l_2 = 2, l_3 > 5$ *or* $l_1 = 1, l_2 > 2, l_3 > 3$ *or* $l_1 = l_2 = 2, l_3 > 2$ *or* $l_1 = 2, l_2 = l_3 = 3$.

(2) $U_{r,s,t,a,b}$ *for* $r = a = 1$, $(r,s,t) \in \{(1,1,2), (2,4,2), (2,5,3), (3,7,3), (3,8,4)\}$, *or* $r = a = 1$, $s \geq 1$, $t \geq t^*(s,b)$, $b \geq 1$, *where* $(s,b) \neq (1,1)$ *and*

$$t^* = \begin{cases} s+b+2, & if\ s \geq 3. \\ b+3, & if\ s = 2. \\ b, & if\ s = 1. \end{cases}$$

(3) D_n *for* $n \geq 9$.

(4) $C_n(P_2)$ *for* $n \geq 5$.

(5) F_n *for* $n \geq 9$.

(6) $B_{r,s,t}$ *for* $r = 5, s = 1$ *and* $t = 3$, *or* $r \geq 1, s = 1$ *if* $t = 1$, *or* $r \geq 4, s = 1$ *if* $t = 2$, *or* $b \geq c + 3, s = 1$ *if* $t \geq 3$.

(7) $G \cong C_4(P_3)$ *or* $G \cong Q(1,2)$.

Corollary 2.1. [21] *If graph G such that* $R_1(G) \leq -2$, *then* $\beta(G) < -2 - \sqrt{5}$.

3 The algebraic properties of adjoint polynomials

3.1 The divisibility of adjoint polynomials and the fifth characters of graphs

Lemma 3.1. [25] *For* $n, m \geq 2$, $h(P_n) \mid h(P_m)$ *if and only if* $(n+1) | (m+1)$.

Theorem 3.1. (1) *For* $n \geq 7$, $\rho(\psi_n^3(n-3,1)) = \begin{cases} \frac{n}{2}, & if\ n\ is\ even; \\ \frac{n-1}{2}, & otherwise. \end{cases}$

(2) *For* $n \geq 7$, $\partial(\psi_n^3(n-3,1)) = \begin{cases} \frac{n}{2}, & if\ n\ is\ even; \\ \frac{n+1}{2}, & otherwise. \end{cases}$

(3) *For* $n \geq 7$, $h(\psi_n^3(n-3,1)) = x(h(\psi_{n-1}^3(n-4,1)) + h(\psi_{n-2}^3(n-5,1)))$.

Proof. (1) Choosing an edge $e = uv \in E(\psi_n^3(n-3,1))$ whose deletion brings about a proper subgraph D_n of $\psi_n^3(n-3,1)$. By Lemma 2.3, we have $h(\psi_n^3(n-3,1)) = h(D_n) + xh(K_3)h(P_{n-5})$. We have, from Lemma 2.4, that

$$\rho(D_n) = \left\lfloor \frac{n}{2} \right\rfloor \ and\ \rho(K_1 \cup K_3 \cup P_{n-5}) = 2 + \left\lfloor \frac{n-4}{2} \right\rfloor.$$

If n is even, then $\rho(D_n) = \rho(K_1 \cup K_3 \cup P_{n-5}) = \frac{n}{2}$ and hence $\rho(\psi_n^3(n-3,1)) = \frac{n}{2}$. If n is odd, then we arrive at $\rho(D_n) = \rho(K_1 \cup K_3 \cup P_{n-5}) = \frac{n-1}{2}$, which implies $\rho(\psi_n^3(n-3,1)) = \frac{n-1}{2}$, as desired.

(2) It obviously follows from (1).

(3) Choosing an edge $e = uv \in E(\psi_n^3(n-3,1))$ whose deletion brings about a proper subgraph D_n of $\psi_n^3(n-3,1)$. From Lemma 2.4, we have

$$\begin{aligned}
& h(\psi_n^3(n-3,1)) \\
= \ & h(D_n) + xh(K_3)h(P_{n-5}) \\
= \ & (xh(D_{n-1}) + xh(D_{n-2})) + xh(K_3)(xh(P_{n-6}) + xh(P_{n-7})) \\
= \ & x\big(h(\psi_{n-1}^3(n-4,1)) + h(\psi_{n-2}^3(n-5,1))\big)
\end{aligned}$$

□

Theorem 3.2. *For $n \geq 2$, $m \geq 7$, $h(P_n) \mid h(\psi_m^3(m-3,1))$ if and only if $n = 4$ and $m = 5k+3$ for $k \geq 1$, or $n = 3$ and $m = 4k+2$ for $k \geq 2$.*

Proof. Let $g_0(x) = -x^3 - 5x^2 - 8x - 2$, $g_1(x) = x^3 + 4x^2 + 6x + 2$ and $g_m(x) = x(g_{m-1}(x) + g_{m-2}(x))$. We can deduce that

$$\begin{aligned}
g_0(x) &= -x^3 - 5x^2 - 8x - 2, \\
g_1(x) &= x^3 + 4x^2 + 6x + 2, \\
g_2(x) &= -x^3 - 2x^2, \\
g_3(x) &= 2x^3 + 6x^2 + 2x, \\
g_4(x) &= x^4 + 4x^3 + 2x^2, \\
g_5(x) &= x^5 + 6x^4 + 8x^3 + 2x^2, \\
g_6(x) &= x^6 + 7x^5 + 12x^4 + 4x^3, \\
g_m(x) &= h(\psi_m^3(m-3,1)),\ if\ m \geq 7.
\end{aligned} \tag{3.1}$$

Let $m = (n+1)k + i$, where $0 \leq i \leq n$. It is obvious that $h_1(P_n) \mid h(\psi_m^3(m-3,1))$ if and only if $h_1(P_n) | g_m(x)$. From Lemma 2.5, it follows that $h_1(P_n) | g_m(x)$ if and only if $h_1(P_n) \mid g_i(x)$, where $0 \leq i \leq n$. We distinguish the following two cases:

Case 1: $n \geq 7$.

If $0 \leq i \leq 6$, it follows from (3.1) that $h_1(P_n) \nmid g_i(x)$. If $i \geq 7$, then it follows from $i \leq n$, Lemma 2.4 and Theorem 3.1 that

$$\partial(h_1(P_n)) = \lfloor n/2 \rfloor \ and\ \partial(h_1(\psi_i^3(i-3,1))) = \lfloor (i+1)/2 \rfloor. \tag{3.2}$$

The following cases are taken into account.

Subcase 1.1: $i = n$.

It follows from (3.2) that $\partial(h_1(\psi_i^3(i-3,1))) = \partial(h_1(P_n)) = \frac{n}{2}$ if n is even and $\partial(h_1(\psi_i^3(i-3,1))) = \partial(h_1(P_n)) + 1 = \frac{n+1}{2}$ if n is odd. First, we consider the case $\partial(h_1(\psi_i^3(i-3,1))) = \partial(h_1(P_n))$. Suppose $h_1(P_n) \mid h_1(\psi_i^3(i-3,1))$. Then $h_1(P_n) = h_1(\psi_i^3(i-3,1))$, which implies that $R_1(P_n) = R_1(\psi_i^3(i-3,1))$. By Lemma 2.6, we know it is impossible. So $h_1(P_n) \nmid h_1(\psi_i^3(i-3,1)))$. Combining this with $(h_1(P_n), x^{\alpha(\psi_i^3(i-3,1))}) = 1$, we have $h_1(P_n) \nmid h(\psi_i^3(i-3,1))$. Next, we consider the case $\partial(h_1(\psi_i^3(i-3,1))) = \partial(h_1(P_n))+1$. Suppose $h_1(P_n) \mid h_1(\psi_i^3(i-3,1))$. Then $h_1(\psi_i^3(i-3,1)) = (x+a)h_1(P_n)$. Note that $R_1(\psi_i^3(i-3,1)) = -2$ and $R_1(P_n) = 1$. Therefore, $R_1(x+a) = -3$ and hence $a = \frac{3\pm\sqrt{33}}{2}$, which contradicts to a is an integer number. Hence $h_1(P_n) \nmid h_1(\psi_i^3(i-3,1))$. Since $(h_1(P_n), x^{\alpha(\psi_i^3(i-3,1))}) = 1, h_1(P_n) \nmid h(\psi_i^3(i-3,1))$.

Subcase 1.2: $i \leq n-1$.

It follows by (3.2) that $\partial(h_1(\psi_i^3(i-3,1))) \leq \partial(h_1(P_n))$. Assume that $h_1(P_n)|h_1(\psi_i^3(i-3,1))$. Then $\partial(h_1(\psi_i^3(i-3,1))) = \partial(h_1(P_n))$ and $h_1(\psi_i^3(i-3,1)) = h_1(P_n)$. So we can turn to Subcase 1.1 for the same contradiction.

Case 2: $2 \leq n \leq 6$.

From (1) of Lemma 2.4 and (3.1) , we can verify that $h_1(P_n) = g_i(x)$ if and only if $n = 3$ and $i = 2$, or $n = 4$ and $i = 3$ for $0 \leq i \leq n \leq 7$. From Lemma 2.5, we have that $h_1(P_n)|h(\psi_m^3(m-3,1))$ if and only if $n = 3$ and $m = 4k+2$, or $n = 4$ and $m = 5k+3$. From $\rho(P_3) = 2, \rho(P_4) = 2$ and $\rho(\psi_i^3(i-3,1)) \geq 3$ for $m \geq 7$, we know that the result holds.

Theorem 3.3. *For* $m \geq 7, h^2(P_4) \nmid h(\psi_m^3(m-3,1)), h^2(P_3) \nmid h(\psi_m^3(m-3,1))$.

Proof. Suppose $h^2(P_4) \mid h(\psi_m^3(m-3,1))$. From Theorem 3.2, we have $m = 5k+3$, where $k \geq 1$. Let $g_m(x) = h(\psi_m^3(m-3,1))$ for $m \geq 7$. By (3) of Theorem 3.1 and (1) of Lemma 2.5, we have

$$\begin{aligned}
g_m(x) &= h(P_4)g_{m-4}(x) + xh(P_3)g_{m-5}(x) \\
&= h^2(P_4)g_{m-8}(x) + 2xh(P_3)h(P_4)g_{m-9}(x) + (xh(P_3))^2 g_{m-10}(x) \\
&= h^2(P_4)(g_{m-8}(x) + 2xh(P_3)g_{m-13}(x)) + 3(xh(P_3))^2 h(P_4)g_{m-14}(x) \\
&\quad +(xh(P_3))^3 g_{m-15}(x) \\
&= h^2(P_4)(g_{m-8}(x) + 2xh(P_3)g_{m-13}(x) + 3(xh(P_3))^2 g_{m-18}(x)) \\
&\quad +4(xh(P_3))^3 h(P_4)g_{m-19}(x) + (xh(P_3))^4 g_{m-20}(x) \\
&= \cdots \\
&= h^2(P_4)\sum_{s=1}^{k-2} s(xh(P_3))^{s-1} g_{m-5s-3}(x) + (k-1)(xh(P_3))^{k-2}h(P_4) \\
&\quad g_{m+1-(5k-1)}(x) + (xh(P_3))^{k-1} g_{m-(5k-1)}(x).
\end{aligned}$$

According to the assumption and $m = 5k+3$, we arrive at, by (3.1), that

$$h^2(P_4) \mid ((k-1)x^{3k-6}(x+2)^{k-2}h(P_4)g_5(x) + x^{3k-3}(x+2)^{k-1}g_4(x))$$

By calculation, we have $k = -1$, which contradicts to $k \geq 1$.
Using the similar method, we can also prove $h^2(P_3) \nmid h(\psi_m^3(m-3,1))$. □

In [13], we introduced a new character.

Definition 3.1. [13] *Let G be a graph with q edges. Then the fifth character of a graph G is defined as follow:*

$$R_5(G) = R_2(G) - R_1(G) + p - q.$$

It is obvious that $R_5(G)$ is an invariant of graph G. So, for any two graphs G and H, we have $R_5(G) = R_5(H)$ if $h(G) = h(H)$ or $h_1(G) = h_1(H)$.

Theorem 3.4. [13] *Let graph G with k components $G_1, G_2, \cdots, G_k$. Then $R_5(G) = \sum_{i=1}^{k} R_5(G_k)$.*

It is obvious that $R_5(G)$ is an invariant of graphs. So, for any two graphs G and H, we have $R_5(G) = R_5(H)$ if $h(G) = h(H)$ or $h_1(G) = h_1(H)$.

Theorem 3.5. [13] (1) $R_5(C_n) = 0$ *for* $n \geq 4$; $R_5(C_3) = -3$; $R_5(K_1) = 1$.
(2) $R_5(B_{r,1,1}) = 4$ *for* $r \geq 1$; $R_5(B_{r,1,t}) = 5$ *for* $r, t > 1$.
(3) $R_5(F_6) = 5$; $R_5(F_n) = 4$ *for* $n \geq 7$; $R_5(K_4^-) = 3$.
(4) $R_5(D_4) = 0$; $R_5(D_n) = 1$ *for* $n \geq 5$; $R_5(T_{1,1,1}) = 0$.
(5) $R_5(T_{1,1,l_3}) = 1$; $R_5(T_{1,l_2,l_3}) = 2$; $R_5(T_{l_1,l_2,l_3}) = 3$ *for* $l_3 \geq l_2 \geq l_1 \geq 2$.
(6) $R_5(C_r(P_2)) = 4$ *for* $r \geq 4$; $R_5(C_4(P_3)) = R_5(Q_{1,2}) = 5$.
(7) $R_5(P_2) = -1$; $R_5(P_n) = -2$ *for* $n \geq 3$.
(8) $R_5(K_4) = 7$; $R_5(\psi_n^3(n-3,1)) = 9$ *for* $n \geq 7$.

Lemma 3.2. [13] *Let graph $G \in \varphi$. Then* $9 \leq R_5(G) \leq 14$.

From the definition of $R_5(G)$, we have the following results.

Lemma 3.3. [16] *Let graph $G \in \xi \backslash \{F_n, U_{r,s,t,a,b}, K_4^-\}$. Then*
(1) $R_5(G) = 4$ *if and only if* $G \in \{C_{n-1}(P_2) \mid n \geq 5\} \cup \{Q_{1,1}\} \cup \{B_{n-5,1,1} \mid n \geq 7\}$.
(2) $R_5(G) = 5$ *if and only if* $G \in \{C_r(P_s) \mid r \geq 4, s \geq 3\} \cup \{Q_{1,n-4} \mid n \geq 6\} \cup \{B_{r,1,t}, B_{1,1,1} \mid r, t \geq 2\}$.
(3) $R_5(G) = 6$ *if and only if* $G \in \{Q_{r,s} \mid r, s \geq 2\} \cup \{B_{1,1,t}, B_{r,s,t} \mid r, s, t \geq 2\}$.
(4) $R_5(G) = 7$ *if and only if* $G \in \{B_{1,s,t} \mid s, t \geq 2\}$.

Corollary 3.1. *Let graph $G \in \xi \backslash \{F_n, U_{r,s,t,a,b}, K_4^-\}$. Then* $R_5(G) \geq 4$.

Lemma 3.4. [16] *Let graph $G \in \psi$. Then*
(1) $R_5(G) = 8$ *if and only if* $G \in \{\psi_n^1\} \cup \{\psi_5^2\} \cup \{\psi_n^3(r,s) \mid r \geq 4, s \geq 2\} \cup \{\psi_n^4(n-6,1) \mid n \geq 8\} \cup \{\psi_n^5(1,s,t) \mid s, t \geq 2\}$.
(2) $R_5(G) = 9$ *if and only if* $G \in \{\psi_n^2\} \cup \{\psi_n^3(n-3,2) \mid n \geq 6\} \cup \{\psi_n^4(r,s) \mid r, s \geq 2\} \cup \{\psi_7^4(1,1)\} \cup \{\psi_n^5(1,1,t), \psi_n^5(r,s,t) \mid r, s, t \geq 2\} \cup \{\psi_5^6\}$.
(3) $R_5(G) = 10$ *if and only if* $G \in \{\psi_n^4(1, n-6) \mid n \geq 8\} \cup \{\psi_n^5(r,1,t) \mid r, t \geq 2\} \cup \{\psi_8^5(1,1,1)\}$.
(4) $R_5(G) = 11$ *if and only if* $G \in \{\psi_n^5(n-7,1,1) \mid n \geq 9\}$.

Corollary 3.2. *Let graph $G \in \psi$. Then* $R_5(G) \geq 8$.

Lemma 3.5. [16] *Let graph $G \in \zeta$. Then*
(1) $R_5(G) = 12$ *if and only if* $G \in \{\zeta_n^1 \mid n \geq 8\} \cup \{\zeta_n^2(r,s) \mid r,s \geq 2\} \cup \{\zeta_n^3(r,s,t) \mid r,s,t \geq 2\}$.
(2) $R_5(G) = 13$ *if and only if* $G \in \{\zeta_7^1\} \cup \{\zeta_n^2(1,n-8) \mid n \geq 10\} \cup \{\zeta_n^3(1,s,t) \mid s,t \geq 2\}$.
(3) $R_5(G) = 14$ *if and only if* $G \in \{\zeta_9^2(1,1)\} \cup \{\zeta_n^3(1,1,n-9) \mid n \geq 11\}$.
(4) $R_5(G) = 15$ *if and only if* $G \in \{\zeta_n^3(1,1,1) \mid n \geq 9\}$.

Corollary 3.3. *Let graph $G \in \zeta$. Then* $R_5(G) \geq 12$.

Lemma 3.6. [17] *Let graph $G \in \theta$. Then* $16 \leq R_5(G) \leq 22$.

Lemma 3.7. [16] *Let graph $G \in \phi$. Then* $12 \leq R_5(G) \leq 17$.

3.2 The smallest real roots of adjoint polynomials of a graph

In [18, 19, 20], Ren and Liu obtained the following results.

Lemma 3.8. [18, 19, 20] (1) *For* $n \geq 4$, $m \geq 6$, $\beta(K_4) < \beta(F_m) < \beta(D_n) < \beta(C_n) < \beta(P_n)$.
(2) $\beta_{min}(B_{r,s,t}) \leq \beta_{min}(Q(r,s)) \leq \beta_{min}(C_r(P_s)) \leq \beta_{min}(T_n)$ *for* $n \geq 6$.
(3) $\beta_{min}(\psi_n^5(r,s,t)) \leq \beta_{min}(\psi_n^4(r,s)) \leq \beta_{min}(\psi_n^3(r,s)) \leq \beta_{min}(\psi_n^2) \leq \beta_{min}(\psi_n^1)$ *for* $n \geq 8$.
(4) $\beta_{min}(B_{r,s,t}) = \beta(B_{1,1,n-5})$; $\beta_{min}(Q(r,s)) = \beta(Q(1,n-4))$.
(5) $\beta_{min}(\zeta_n^3) \leq \beta_{min}(\zeta_n^2) \leq \beta_{min}(\zeta_n^1)$.
(6) $\beta_{min}(\psi_n^3(r,s)) = \beta(\psi_n^3(n-3,1))$; $\beta_{min}(\psi_n^4(r,s)) = \beta(\psi_n^4(1,n-6))$; $\beta_{min}(\psi_n^5(r,s,t)) = \beta(\psi_n^5(n-7,1,1))$.
(7) $\beta_{min}(\zeta_n^2(r,s)) = \beta(\zeta_n^2(1,n-8))$; $\beta_{min}(\zeta_n^3(r,s,t)) = \beta(\zeta_n^3(1,1,n-9))$.
(8) $\beta_{min}(\psi_n^1) < \beta(\psi_n^5(1,s,t))$.

Lemma 3.9. (1) *For* $n \geq 7$, $\beta(\psi_n^3(n-3,1)) < \beta(\psi_{n-1}^3(n-4,1))$.
(2) *For* $n \geq 7, r \geq 5, m \geq 6$, $\beta(\psi_n^3(n-3,1)) < \beta(K_4^-)$; $\beta(\psi_n^3(n-3,1)) < \beta(C_{n-1}(P_2))$; $\beta(\psi_n^3(n-3,1)) < \beta(B_{m-5,1,1})$; $\beta(\psi_n^3(n-3,1)) < \beta(F_m)$; $\beta(\psi_n^3(n-3,1)) < \beta(Q_{1,1})$.
(3) *For* $n \geq 7$, $m \geq 6$, $\beta(\psi_n^3(n-3,1)) < \beta(K_4) = \beta(\psi_5^2)$; $\beta(\psi_n^3(n-3,1)) < \beta(B_{1,1,m-5}) < \beta(C_r(P_s))$.
(4) *For* $n \geq 7$, $m \geq 6$, $\beta(\psi_n^3(n-3,1)) < \beta(Q_{1,m-4})$; $\beta(\psi_n^3(n-3,1)) < \beta(B_{1,1,1})$.

Proof. (1) Using Software Mathematica, for $n_1 \geq 18$, we have $\beta(\psi_7^3(4,1)) = -4.68554 > \beta(\psi_8^3(5,1)) = -4.73205 > \beta(\psi_9^3(6,1)) = -4.75047 > \beta(\psi_{10}^3(7,1)) = -4.75802 > \beta(\psi_{11}^3(8,1)) = -4.76118 > \beta(\psi_{12}^3(9,1)) = -4.76251 > \beta(\psi_{13}^3(10,1)) = -4.76308 > \beta(\psi_{14}^3(11,1)) = -4.76332 > \beta(\psi_{15}^3(12,1)) = -4.76343 > \beta(\psi_{16}^3(13,1)) = -4.76347 > \beta(\psi_{17}^3(14,1)) = -4.76349 > \beta(\psi_{n_1}^3(n_1-3,1))$.
(2) From Lemmas 2.9, 2.10 and Corollary 2.1, it is easy to see that the result holds.

(3) For $n_1 \geq 8, \beta(\psi^3_{n_1}(n_1-3,1)) < \beta(\psi^3_7(4,1)) = -4.68554 < \beta(K_4) = -4.49086$; From $n_1 \geq 8$, $m_1 \geq 14$, $\beta(\psi^3_{n_1}(n_1-3,1)) < \beta(\psi^3_7(4,1)) = -4.68554 < \beta(B_{1,1,m_1-5}) < \beta(B_{1,1,15}) = -4.51729 < \beta(B_{1,1,14}) = -4.51728 < \beta(B_{1,1,13}) = -4.51726 < \beta(B_{1,1,12}) = -4.51721 < \beta(B_{1,1,11}) = -4.51713 < \beta(B_{1,1,10}) = -4.51695 < \beta(B_{1,1,9}) = -4.51658 < \beta(B_{1,1,8}) = -4.51584 < \beta(B_{1,1,7}) = -4.51432 < \beta(B_{1,1,6}) = -4.51119 < \beta(B_{1,1,5}) = -4.50469 < \beta(B_{1,1,4}) = -4.49086 < \beta(B_{1,1,3}) = -4.4605 < \beta(B_{1,1,2}) = -4.39026 < \beta(B_{1,1,1}) = -4.21432$.

(4) For $n_1 \geq 8, m_1 \geq 16, \beta(\psi^3_{n_1}(n_1-3,1)) < \beta(\psi^3_7(4,1)) = -4.68554 < \beta(Q_{1,m_1-4}) < \beta(Q_{1,11}) = -4.38249 < \beta(Q_{1,10}) = -4.38207 < \beta(Q_{1,9}) = -4.38131 < \beta(Q_{1,8}) = -4.37988 < \beta(Q_{1,7}) = -4.3772 < \beta(Q_{1,6}) = -4.37213 < \beta(Q_{1,5}) = -4.36232 < \beta(Q_{1,4}) = -4.334292 < \beta(Q_{1,3}) = -4.30278 < \beta(Q_{1,2}) = -4.21342$. Since $B_{1,1,1}$ is a subgraph of $\psi^3_n(n-3,1)$, it follows from Lemma 2.8 that $\beta(\psi^3_n(n-3,1)) < \beta(B_{1,1,1})$. □

Lemma 3.10. (1) *For* $n \geq 7, m \geq 5, \beta(\psi^3_n(n-3,1)) < \beta(\psi^1_m) < \beta(\psi^5_n(1,s,t))$.

(2) *For* $n \geq 7, m \geq 5, \beta(\psi^3_n(n-3,1)) = \beta(\psi^2_m)$ *if and only if* $m = 8$ *and* $n = 8$.

(3) *For* $n \geq 7, m \geq 8, \beta(\psi^4_m(1,m-6)) \leq \beta(\psi^3_n(n-3,1))$ *the equality holds if and only if* $m = n = 7$; $\beta(\psi^3_n(n-3,1)) < \beta(\psi^4_m(m-6,1))$.

(4) *For* $n \geq 7, m \geq 10, \beta(\psi^5_n(m-7,1,1)) < \beta(\psi^3_n(n-3,1))$.

(5) *For* $n \geq 7, m \geq 10, \beta(\psi^3_n(n-3,1)) < \beta(\psi^6_5)$.

Proof. (1) For $n_1 \geq 8, m_1 \geq 6, \beta(\psi^3_{n_1}(n_1-3,2)) < \beta(\psi^3_7(4,2)) = -4.68554 < \beta(\psi^1_{m_1}) < \beta(\psi^1_{18}) = -4.61347 < \beta(\psi^1_{17}) = -4.61346 < \beta(\psi^1_{16}) = -4.61345 < \beta(\psi^1_{15}) = -4.61342 < \beta(\psi^1_{14}) = -4.61337 < \beta(\psi^1_{13}) = -4.61325 < \beta(\psi^1_{12}) = -4.613 < \beta(\psi^1_{11}) = -4.61246 < \beta(\psi^1_{10}) = -4.61128 < \beta(\psi^1_9) = -4.60873 < \beta(\psi^1_8) = -4.60212 < \beta(\psi^1_7) = -4.59056 < \beta(\psi^1_6) = -4.56155 < \beta(\psi^1_5) = -4.49086$. From (8) of Lemma 3.8, the result holds.

(2) For $n_1 \geq 10, m_1 \geq 9, \beta(\psi^3_{n_1}(n_1-3,1)) < \beta(\psi^3_9(6,1)) = -4.75047 < \beta(\psi^2_{m_1}) < \beta(\psi^2_{17}) = -4.74819 < \beta(\psi^2_{16}) = -4.74818 < \beta(\psi^2_{15}) = -4.74815 < \beta(\psi^2_{14}) = -4.7481 < \beta(\psi^2_{13}) = -4.74796 < \beta(\psi^2_{12}) = -4.74766 < \beta(\psi^2_{11}) = -4.74694 < \beta(\psi^2_{10}) = -4.74528 < \beta(\psi^2_9) = -4.74137 < \beta(\psi^2_8) = \beta(\psi^3_8(5,1)) = -4.73205 < \beta(\psi^2_7) = -4.70928 < \beta(\psi^3_7(4,1)) = -4.68554 < \beta(\psi^2_6) = -4.65109 < \beta(\psi^2_5) = -4.49086$.

(3) For $n_1 \geq 8, m_1 \geq 16, m_2 \geq 12, \beta(\psi^4_{m_1}(1,m_1-6)) < \beta(\psi^4_{16}(1,10)) = -4.85505 < \beta(\psi^4_{15}(1,9)) = -4.85498 < \beta(\psi^4_{14}(1,8)) = -4.85482 < \beta(\psi^4_{13}(1,7)) = -4.85443 < \beta(\psi^4_{12}(1,6)) = -4.85347 < \beta(\psi^4_{11}(1,5)) = -4.85109 < \beta(\psi^4_{10}(1,4)) = -4.84517 < \beta(\psi^4_9(1,3)) = -4.83021 < \beta(\psi^4_8(1,2)) = -4.79129 < \beta(\psi^3_{n_1}(n_1-3,2)) < \beta(\psi^3_7(4,2)) = \beta(\psi^4_7(1,1)) = -4.68554$; $\beta(\psi^3_{n_1}(n_1-3,1)) < \beta(\psi^3_7(4,1)) = -4.68554 < \beta(\psi^4_8(2,1)) = -4.56155 < \beta(\psi^4_9(3,1)) = -4.49086 < \beta(\psi^4_{10}(4,1)) = -4.4887 < \beta(\psi^4_{11}(5,1)) = -4.4217 < \beta(\psi^4_{m_2}(m_2-6,1))$.

(4) For $n_1 \geq 8$, $m_1 \geq 10$, $\beta(\psi^5_{m_1}(m_1-7,1,1)) < \beta(\psi^5_9(1,1,1)) = -5.53103 < \beta(\psi^3_{n_1}(n_1-3,1)) < \beta(\psi^3_7(4,1)) = -4.68554$.

(5) $\beta(\psi_5^6) = -6.17508 < \beta(\psi_{n_1}^3(n-3,1))$. □

Lemma 3.11. *(1) For $n \geq 7$, $m \geq 8$, $\beta(\psi_n^3(n-3,1)) = \beta(\zeta_m^1)$ if and only if $m = 13$ and $n = 9$.*
(2) For $n \geq 7, m \geq 8$, $\beta(\zeta_m^2(1, m-8)) < \beta(\psi_n^3(n-3,1))$.
(3) For $n \geq 10, m \geq 14$, $\beta(\zeta_m^3(1,1,m-9)) < \beta(\psi_n^3(n-3,1))$.

Proof. Using Software Mathematica, we have

(1) For $n_1 \geq 10, m \geq 14, \beta(\zeta_7^1) = -5 < \beta(\zeta_8^1) = -4.86906 < \beta(\zeta_9^1) = -4.80535 < \beta(\zeta_{10}^1) = -4.77448 < \beta(\zeta_{11}^1) = -4.75999 < \beta(\zeta_{12}^1) = -4.7534 < \beta(\psi_{n_1}^3(n_1-3,2)) < \beta(\psi_9^3(6,2)) = \beta(\zeta_{13}^1) = -4.75047 < \beta(\zeta_{m_1}^1) < \beta(\psi_8^3(5,1)) = -4.73205 < \beta(\psi_7^3(4,1)) = -4.68554$.

(2) For $n_1 \geq 9, m_1 \geq 18, \beta(\zeta_9^2(1,1)) = -5.04892 < \beta(\zeta_{10}^2(1,2)) = -4.9418 < \beta(\zeta_{11}^2(1,3)) = -4.89307 < \beta(\zeta_{12}^2(1,4)) = -4.8713 < \beta(\zeta_{13}^2(1,5)) = -4.86188 < \beta(\zeta_{14}^2(1,6)) = -4.8579 < \beta(\zeta_{15}^2(1,7)) = -4.85626 < \beta(\zeta_{16}^2(1,8)) = -4.85557 < \beta(\zeta_{17}^2(1,9)) = -4.85529 < \beta(\zeta_{18}^2(1,10)) = -4.85517 < \beta(\zeta_{m_1}^2(1,m_1-8)) < \beta(\psi_{n_1}^3(n_1-3,1)) < \beta(\psi_8^3(5,1)) = -4.73205 < \beta(\psi_7^3(4,1)) = -4.68554$.

(3) For $n_1 \geq 9, m_1 \geq 20, \beta(\zeta_{10}^3(1,1,1)) = -5.23607 < \beta(\zeta_{11}^3(1,1,2)) = -5.10522 < \beta(\zeta_{12}^3(1,1,3)) = -5.04892 < \beta(\zeta_{13}^3(1,1,4)) = -5.0254 < \beta(\zeta_{14}^3(1,1,5)) = -5.01594 < \beta(\zeta_{15}^3(1,1,6)) = -5.01224 < \beta(\zeta_{16}^3(1,1,7)) = -5.01082 < \beta(\zeta_{17}^3(1,1,8)) = -5.01027 < \beta(\zeta_{18}^3(1,1,9)) = -5.01006 < \beta(\zeta_{19}^3(1,1,10)) = -5.00998 < \beta(\zeta_{m_1}^3(1,1,m_1-9)) < \beta(\psi_{n_1}^3(n_1-3,1)) < \beta(\psi_8^3(5,1)) = -4.73205 < \beta(\psi_7^3(4,1)) = -4.68554$. □

4 The chromaticity of graph $\overline{\psi_n^3(n-3,1)}$

Lemma 4.1. [24] *For $n \geq 4$, D_n is adjointly unique if and only if $n \neq 4, 8$.*

Lemma 4.2. *Let G be a graph such that $G \sim^h \psi_n^3(n-3,1)$, where $n \geq 7$. Then G does not contain K_4^- as one of its components.*

Proof. Suppose $h(K_4^-) \mid h(\psi_n^3(n-3,1))$. From Lemma 2.3, we have $h(K_4^-) = x^2(x+1)(x+4)$ and hence $h_1(P_2) \mid h(\psi_n^3(n-3,1))$, which contradicts to Theorem 3.3. □

Theorem 4.1. *Let G be a graph satisfying $G \sim^h \psi_n^3(n-3,1)$ where $n \geq 7$. Then G contains at most two components whose first characters are 1, furthermore, one of both is P_3 and the other is P_4 or one of both is P_3 and the other is C_3.*

Proof. Let G_1 be one of the components of G such that $R_1(G) = 1$. From Lemma 2.6 and Theorem 3.2, $h(G_1)|h(\psi_n^3(n-3,1))$ if and only if $G_1 \cong P_3$ and $n = 4k+2$, or $G_1 \cong P_4$ and $n = 5k+3$. According to (1) of Lemma 2.5, we obtain the following equality:

$$h(\psi_{20k+18}^3(20(k-1)+15,1)) = h(P_{20})h(\psi_{20(k-1)+18}^3(20(k-1)+15,1))$$

$$+xh(P_{19})h(\psi^3_{20(k-1)+17}(20(k-1)+14,1)) \quad (4.1)$$

Noting that $\{n\,|\,n=4k+2, k\geq 1\}\cap\{n\,|\,n=5k+3, k\geq 1\}=\{n\,|\,n=20k+18, k\geq 0\}$, we have

$$h(P_3)h(P_4)\mid h(\psi^3_{20(k-1)+18}(20(k-1)+15,1)) \quad (4.2)$$

By Lemma 3.1, we get $h(P_3)|h(P_{19})$ and $h(P_4)|h(P_{19})$. Combining this with $(h(P_3),h(P_4))=1$, we have

$$h(P_3)h(P_4)\mid h(P_{19}) \quad (4.3)$$

From (4.1) to (4.3), we obtain $h(P_3)h(P_4)\mid h(\psi^3_{20k+18}(20k+15,2))$. Note that $h(P_4)=h(K_1\cup C_3)$ and hence $h(P_3)h(C_3)\mid h(\psi^3_{20k+18}(20k+15,2))$. From Theorem 3.3, we know that the theorem holds.

Theorem 4.2. *Let G be a graph such that $G\sim^h\psi^3_n(n-3,1)$, where $n\geq 9$.*
(1) *If $n=8$, then $[G]_h=\{\psi^3_8(5,1),\phi^1_5\cup C_3,\psi^2_8\}$.*
(2) *If $n\neq 8$, then $[G]_h=\{\psi^3_n(n-3,1)\}$.*

Proof. **(1)** When $n=8$, let G be a graph satisfying $h(G)=h(\psi^3_8(5,2))$. From Lemmas 2.1, 2.2 and 2.6, we obtain that $q(G)-p(G)=1$ and $R_1(G)=-2$. If G is a connected graph, then $G\in\mathscr{G}=\{\psi^2_8,\psi^3_8(5,1),\psi^4_8(2,2),\psi^4_8(1,3),\psi^4_8(3,1),\psi^5_8(1,1,1)\}$ by $R_5(G)=R_5(\psi^3_8(5,1))=9$ and (2) of Lemma 3.4. By calculation, we have $\{\psi^2_8,\psi^3_8(5,1)\}\in[G]_h$. We now assume that G is not a connected graph. By calculation, we have $h(G)=h(\psi^3_8(5,2))=x^4(x^2+3x+1)(x^2+6x+6)$. Let $h(G)=h(\psi^3_8(5,2))=x^6f_1(x)f_2(x)$, where $f_1(x)=x^2+3x+1$, $f_2(x)=x^2+6x+6$. Note that $R_1(f_1(x))=1$ and $b_1(f_1(x))=3$. Then Lemma 2.6 implies that $f_1(x)=h_1(P_4)=h_1(C_3)$ if $f_1(x)$ is a factor of adjoint polynomial of some graph. Then P_4 or C_3 is a component of G. If P_4 is a component of G, then $G=P_4\cup G_1$ and hence $h_1(f_2(x))=x^2+6x+6$, which implies that $R_1(G_1)=R_1(f_2(x))=-3$ and $q(G_1)-p(G_1)=2$. From (5) of Lemma 2.6, we have $G_1\in\zeta$, which contradicts to $p(G_1)=4$. Suppose that C_3 is a component of G. Then $G=C_3\cup G_1$ and so $h_1(f_2(x))=x^2+6x+6$, which implies that $R_1(G_1)=R_1(f_2(x))=-3$ and $q(G_1)-p(G_1)=1$. From Lemma 2.6, we have $G_1\in\phi$. Since $p(G)=8$, we can only find one graph $G_1\in\phi$ such that $p(G_1)=5$. Then $G_1=\phi^1_5$. So $G=C_3\cup\phi^1_5$. By calculation, $C_3\cup\phi^1_5\in[G]_h$.

(2) When $n\geq 7$ and $n\neq 8$, let $G=\bigcup_{i=1}^t G_i$. From Lemma 2.1, we have

$$h(G)=\prod_{i=1}^t h(G_i)=h(\psi^3_n(n-3,1)), \quad (4.4)$$

which results in $\beta(G)=\beta(\psi^3_n(n-3,1))\in(-\infty,-2-\sqrt{5})$ by Corollary 2.1. Let s_i denote the number of components G_i such that $R(G_i)=-i$, where $i\geq -1$. From Theorem 4.1, Lemmas 4.1, 2.1 and 2.2, it follows that $0\leq s_{-1}\leq 2$ and

$$R_1(G)=\sum_{i=1}^t R_1(G_i)=-2 \text{ and } q(G)=p(G)+1, \quad (4.5)$$

which implies $$-4 \le R_1(G_i) \le 1,$$

$$s_{-1} = s_1 + 2s_2 + 3s_3 + 4s_4 - 2, \tag{4.6}$$

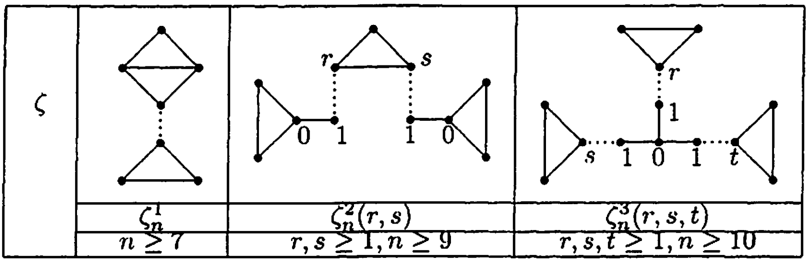

Figure 2 Family of ζ

Let $\cup_{T\in\mathcal{T}_0} T_{l_1,l_2,l_3} = (\cup_{T\in\mathcal{T}_1} T_{1,1,l_3}) \cup (\cup_{T\in\mathcal{T}_2} T_{1,l_2,l_3}) \cup (\cup_{T\in\mathcal{T}_3} T_{l_1,l_2,l_3})$, $\mathcal{T}_1 = \{T_{1,1,l_3} \mid l_3 \ge 2\}$, $\mathcal{T}_2 = \{T_{1,l_2,l_3} \mid l_3 \ge l_2 \ge 2\}$, $\mathcal{T}_3 = \{T_{l_1,l_2,l_3} \mid l_3 \ge l_2 \ge l_1 \ge 2\}$, $\mathcal{T}_0 = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, the tree T_{l_1,l_2,l_3} is denoted by T for short, $A = \{i \mid i \ge 4\}$ and $B = \{j \mid j \ge 5\}$.

We distinguish the following cases by $0 \le s_{-1} \le 2$:

Case 1: $s_{-1} = 0$.

It follows from (4.6) that$s_4 = s_3 = 0$ and $s_1 + 2s_2 = 2$. We distinguish the following subcases:

Subcase 1.1: $s_2 = 1$ and $s_1 = 0$.

From Lemma 2.6, we set

$$G = G_1 \cup (\cup_{i\in A} C_i) \cup (\cup_{j\in B} D_j) \cup f D_4 \cup a K_1 \cup b T_{1,1,1} \cup (\cup_{T\in\mathcal{T}_0} T_{l_1,l_2,l_3}), \tag{4.7}$$

where $R_1(G_1) = -2$.

By Theorems 3.4 and 3.5, we arrive at

$$R_5(G) = R_5(\psi_n^3(n-3,1)) = 9 = R_5(G_1) + |B| + a + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3| \tag{4.8}$$

Recall that $q(G) = p(G) + 1$. Then $q(G_1) - p(G_1) \ge 1$. By (1) of Lemma 2.7, it follows that $q(G_1) - p(G_1) \le 2$. Thus $1 \le q(G_1) - p(G_1) \le 2$. So we have the following subcases to consider.

Subcase 1.1.1: $q(G_1) - p(G_1) = 2$.

From (4) of Lemma 2.6 and $R_1(G_1) = -2$, we have $G_1 \cong K_4$. Since $q(G) = p(G) + 1$, we can obtain $a + b + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3| = 1$ from (4.7), which implies that $|\mathcal{T}_2| = |\mathcal{T}_3| = 0$ and $0 \le b \le 1$. From this together with (4.8), if $b = 0$, then $9 = R_5(K_4) + |B| + 1$. Since $R_5(K_4) = 7$, we have $|B| = 1$ and $G = K_4 \cup (\cup_{i\in A} C_i) \cup D_j \cup f D_4$. If $b = 1$, then it follows from (4.8) that $9 = R_5(K_4) + |B|$, which leads to $|B| = 2$ and $G = K_4 \cup (\cup_{i\in A} C_i) \cup 2D_j \cup f D_4 \cup T_{1,1,1}$. As stated above, we conclude, from Lemma 2.9 and (1) of Lemma 3.8, that $\beta(\psi_n^3(n-3,1)) = \beta(G) = \beta(K_4)$, which contradicts to $\beta(\psi_n^3(n-3,1)) < \beta(K_4)$ by (3) of Lemma 3.9.

Subcase 1.1.2: $q(G_1) - p(G_1) = 1$.

Since $q(G) = p(G) + 1$, it follows that $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$ and $G_1 \in \psi$ by (4) of Lemma 2.6 and (4.7). From (4.8), $9 = R_5(G_1) + |B|$ and hence $|B| = 0$ and $R_5(G_1) = 9$ or $|B| = 1$ and $R_5(G_1) = 8$ by Lemma 3.4.

If $|B| = 1$, then $G = G_1 \cup (\cup_{i\in A} C_i) \cup D_j \cup fD_4$, where $G_1 \in \{\psi_n^1\} \cup \{\psi_5^2\} \cup \{\psi_n^3(r,s)\} \cup \{\psi_n^4(n-6,1)\} \cup \{\psi_n^5(1,s,t)\}$ by (1) of Lemma 3.4. By Lemma 2.9 and Corollary 2.1, it follows that $\beta(\psi_n^3(n-3,1)) = \beta(G) = \beta(G_1)$. From (1), (2), (3) of Lemma 3.10, $\beta(\psi_n^3(n-3,1)) < \beta(\psi_n^1) < \beta(\psi_n^5(1,s,t))$ and $\beta(\psi_n^3(n-3,1)) < \beta(\psi_n^4(n-6,1))$. Therefore, $\beta(\psi_n^3(n-3,1)) = \beta(G_1) = \beta_{min}(\psi_n^3(r,s))$. From this together with (6) of Lemma 3.8, $G_1 \cong \psi_m^3(m-3,1)$ and $m < n$, which contradicts to $p(G) = q(G)$ by (1) of Lemma 3.9.

If $|B| = 0$, then $G = G_1 \cup (\cup_{i\in A} C_i) \cup fD_4$, where $G_1 \in \{\psi_n^2\} \cup \{\psi_n^3(n-3,1)\} \cup \{\psi_n^4(r,s)\} \cup \{\psi_7^4(1,1)\} \cup \{\psi_n^5(1,1,t), \psi_n^5(r,s,t)\} \cup \{\psi_5^6\}$ by (2) of Lemma 3.4. If $G_1 \cong \psi_n^2$, then $p(G_1) = p(G) = 8$ by (2) of Lemma 3.10. It is impossible. If $G_1 \cong \psi_n^4(r,s)$, then $p(G_1) = p(G) = 7$ by (6) of Lemma 3.8 and (3) of Lemma 3.10. One can see that it is impossible. From (4), (5) of Lemma 3.10 and (6) of Lemma 3.8, $G_1 \not\cong \psi_5^6, \psi_n^5(r,s,t), \psi_n^5(1,1,t)$. So $G_1 \cong \psi_m^3(m-3,1)$. From (1) of Lemma 3.9, we have $m = n$. It is impossible.

Subcase 1.2: $s_1 = 2$ and $s_2 = 0$.

From Lemma 2.6 and (4.5), let

$$G = G_1 \cup G_2 \cup (\cup_{i\in A} C_i) \cup (\cup_{j\in B} D_j) \cup fD_4 \cup aK_1 \cup bT_{1,1,1} \cup (\cup_{T\in \mathcal{T}_0} T_{l_1,l_2,l_3}) \tag{4.9}$$

where $R_1(G_1) = R_1(G_2) = -1$.

By Theorems 3.4 and 3.5, we have

$$R_5(G) = R_5(\psi_n^3(n-3,1)) = 9 = R_5(G_1) + R_5(G_2) + |B| + a + |\mathcal{T}_1| + 2|\mathcal{T}_2| - 3|\mathcal{T}_3| \tag{4.10}$$

Recall that $q(G) = p(G) + 1$. Then $\sum_{i=1}^{2}(q(G_i) - p(G_i)) \geq 1$. Using (1) of Lemma 2.7, it follows that $\sum_{i=1}^{2}(q(G_i) - p(G_i)) \leq 2$. Thus $1 \leq \sum_{i=1}^{2}(q(G_i) - p(G_i)) \leq 2$, which brings about the following two subcases to be considered.

Subcase 1.2.1: $\sum_{i=1}^{2}(q(G_i) - p(G_i)) = 2$.

From (3) of Lemma 2.6 and Lemma 4.2 and (4.5), we have $G_i \cong F_m (i = 1, 2)$ and $a + b + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3| = 1$, which implies that $|\mathcal{T}_2| = |\mathcal{T}_3| = 0$ and $0 \leq b \leq 1$. If $b = 0$, then it follows from (4.10) that $9 = 2R_5(F_m) + |B| + 1$. Then $|B| = 0$ and $G = F_m \cup F_m \cup (\cup_{i\in A} C_i) \cup fD_4$. If $b = 1$, then it follows from (4.10) that $9 = 2R_5(F_m) + |B|$. Then $|B| = 1$ and $G = F_m \cup F_m \cup (\cup_{i\in A} C_i) \cup D_j \cup fD_4$. Using (1) of Lemma 3.8, we have $\beta(\psi_n^3(n-3,1)) = \beta(G) = \beta(F_m)$, which contradicts to $\beta(\psi_n^3(n-3,1)) < \beta(F_m)$ by (2) of Lemma 3.9.

Subcase 1.2.2: $\sum_{i=1}^{2} q(G_i) - p(G_i) = 1$.

It is obvious that $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$, $G_1 \cong F_m$ and $G_2 \in \xi$ by Lemmas 2.6 and 4.2 and (4.5). Then $9 = R_5(F_m) + R_5(G_2) + |B|$, that is $R_5(G_2) = 5 + |B|$. Since $G_2 \in \xi$, it follows that $R_5(G_2) \geq 4$ by Corollary 3.1. Then $4 \leq R_5(G_2) \leq 5$ since $|B|$ is an integer. If $R_5(G_2) = 4$, then $|B| = 1$ and $G = F_m \cup G_2 \cup (\cup_{i\in A} C_i) \cup D_j \cup fD_4$ by (4.9) and (1) of Lemma 3.3, where $\{C_{n-1}(P_2)|n \geq 5\} \cup \{Q_{1,1}\} \cup \{B_{n-5,1,1}|n \geq 7\}$. By Lemma 2.9, 2.10 and Corollary 2.1, we know that $\beta(\psi_n^3(n-3,1)) = \beta(G) = \beta(G_2)$ or $\beta(\psi_n^3(n-3,1)) = \beta(G) = \beta(F_m)$, which contradicts to (2) of Lemma 3.9. If $R_5(G_2) = 5$, then $|B| = 0$ and $G = F_m \cup G_2 \cup (\cup_{i\in A} C_i) \cup fD_4$, where

$\{C_r(P_s)|r \geq 4, s \geq 3\} \cup \{Q_{1,n-4}|n \geq 6\} \cup \{B_{r,1,t}, B_{1,1,1}|r, t \geq 2\}$ by (2) of Lemma 3.3. From (1) of Lemma 3.8, $\beta(G) = \beta(F_m)$ or $\beta(G) = \beta(G_2)$. From (2) of Lemma 3.9, $\beta(\psi_n^3(n-3,1)) = \beta(G_2) < \beta(F_m)$. So $\beta(G) = \beta(G_2)$, which contradicts to (4) of Lemma 3.8 and (3), (4) of Lemma 3.9.

Case 2: $s_{-1} = 1$.

It follows from (4.6) that $s_4 = 0$ and $s_1 + 2s_2 + 3s_3 = 3$. Thus we have the following subcases to consider.

Subcase 2.1: $s_3 = 1, s_2 = s_1 = 0$.

Without loss of generality, let

$$G = G_1 \cup G_2 \cup (\cup_{i\in A} C_i) \cup (\cup_{j\in B} D_j) \cup fD_4 \cup aK_1 \cup bT_{1,1,1} \cup (\cup_{T\in \mathcal{T}_0} T_{l_1,l_2,l_3}), \tag{4.11}$$

where $G_1 \in \{P_3, P_4, C_3\}, R_1(G_2) = -3$.

By Theorems 3.4 and 3.5, we arrive at

$$R_5(G) = R_5(\psi_n^3(n-3,1)) = 9 = R_5(G_1) + R_5(G_2) + |B| + a + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3| \tag{4.12}$$

Subcase 2.1.1: $G_1 \cong P_3$ or $G_1 \cong P_4$.

Recall that $q(G) = p(G) + 1$. Then $q(G_2) - p(G_2) \geq 2$. From (2) of Lemma 2.7, it follows that $q(G_2) - p(G_2) \leq 2$. Then $q(G_2) - p(G_2) = 2$, which implies $G_2 \in \zeta$ and $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$. Hence we have, from (4.12), that $9 = -2 + R_5(G_2) + |B|$, which results in $R_5(G_2) = 11 - |B| \leq 11$. It contradicts to Corollary 3.3.

Subcase 2.1.2: $G_1 \cong C_3$.

Applying (4.5) and Lemma 2.7, we have $1 \leq q(G_2) - p(G_2) \leq 2$. If $q(G_2) - p(G_2) = 1$, then $G_2 \in \phi$ and $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$ by (5) of Lemma 2.6 and (4.5). From (4.12), it follows that $9 = -3 + R_5(G_2) + |B| + 1$, which leads to $R_5(G_2) = 11 - |B| \leq 11$. It contradicts to Lemma 3.7.

Suppose $q(G_2) - p(G_2) = 2$. It is easy to see that $G_2 \in \zeta$ and $a + b + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3| = 1$ by (5) of Lemma 2.6 and (4.5). If $b = 0$, then we obtain, from (4.12), that $9 = -3 + R_5(G_2) + |B| + 1$, which leads to $R_5(G_2) = 11 - |B| \leq 11$. It contradicts to Corollary 3.3. If $b = 1$, then we have, from (4.12), that $9 = -3 + R_5(G_2) + |B|$, which results in $G = C_3 \cup G_2 \cup (\cup_{i\in A} C_i) \cup fD_4 \cup T_{1,1,1}$, where $R_5(G_2) = 12$. It implies that $G_2 \in \{\zeta_n^1\} \cup \{\zeta_n^2(r,s)\} \cup \{\zeta_n^3(r,s,t)\}$ by (1) of Lemma 3.5. From Lemma 2.1 and (1) of Lemma 3.8, $\beta(\psi_n^3(n-3,1)) = \beta(G) = \beta(G_2)$. By (7) of Lemma 3.8 and Lemma 3.11, we know that $G_2 \cong \zeta_n^1$ if and only if $p(G) = 13$ and $p(G_2) = 9$. One can see that it is impossible.

Subcase 2.2: $s_2 = s_1 = 1$.

Without loss of generality, let

$$G = G_1 \cup G_2 \cup G_3 \cup (\cup_{i\in A} C_i) \cup (\cup_{j\in B} D_j) \cup fD_4 \cup aK_1 \cup bT_{1,1,1} \cup (\cup_{T\in \mathcal{T}_0} T_{l_1,l_2,l_3}), \tag{4.13}$$

where $G_1 \in \{P_3, P_4, C_3\}, R_1(G_1) = -1, R_1(G_2) = -2$.

From Theorems 3.4 and 3.5, we arrive at

$$R_5(G) = R_5(\psi_n^3(n-3,1)) = 9 = \sum_{i=1}^{3} R_5(G_i) + |B| + a + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3| \tag{4.14}$$

Subcase 2.2.1: $G_1 \cong P_3$ or $G_1 \cong P_4$.

Using (4.5) and Lemma 2.7, we get that $2 \le \sum_{i=2}^{3}(q(G_2) - p(G_2)) \le 3$. We have the following cases to consider.

First, we consider the case that $q(G_2) - p(G_2) = 1$ and $q(G_3) - p(G_3) = 2$. From (3) and (4) of Lemmas 2.6 and Lemma 4.2, we have that $G_2 \cong F_m$, $G_3 \cong K_4$ and $a + b + |T_1| + 2|T_2| + 3|T_3| = 1$. If $b = 0$, then $9 = -2 + R_5(F_m) + R_5(K_4) + |B| + 1$, which results in $|B| = -1$. It contradicts to that $|B|$ is an positive integer. If $b = 1$, then $9 = -2 + R_5(F_m) + R_5(K_4) + |B|$, which implies $|B| = 0$ and $G = G_1 \cup F_m \cup K_4 \cup (\cup_{i\in A} C_i) \cup (\cup_{j\in B} D_j) \cup fD_4 \cup T_{1,1,1}$. From Lemma 2.9 and (1) of Lemma 3.8, we have $\beta(\psi_n^3(n-3,1)) = \beta(G) = \beta(K_4)$, which contradicts to $\beta(\psi_n^3(n-3,1)) < \beta(K_4)$ by (3) of Lemma 3.9.

Next, we consider the case that $q(G_2) - p(G_2) = 1$ and $q(G_3) - p(G_3) = 1$. It is obvious that $G_2 \cong F_m$, $G_3 \in \psi$ and $a = b = |T_1| = |T_2| = |T_3| = 0$ by Lemma 2.6, Lemma 4.2. By (4.14), we have $9 = -2 + R_5(F_m) + R_5(G_3) + |B|$ and hence $R_5(G_3) = 7 - |B| \le 7$, which contradicts to Corollary 3.2.

In this end, we consider the case that $q(G_2) = p(G_2)$ and $q(G_3) - p(G_3) = 2$. Applying Lemma 2.6 and (4.5), it follows that $G_2 \in \xi$, $G_3 \cong K_4$ and $a = b = |T_1| = |T_2| = |T_3| = 0$. Then $9 = -2 + R_5(G_2) + R_5(K_4) + |B|$ and hence $|B| = 0$ and $R_5(G_2) = 4$. By (1) of Lemma 3.3, we know that $G = C_3 \cup G_2 \cup K_4 \cup (\cup_{i\in A} C_i) \cup fD_4$, where $G_2 \in \{C_{n-1}(P_1)\} \cup \{Q_{1,1}\} \cup \{B_{n-5,1,1}\}$. We can get the same contradiction as Subcase 1.2.2.

Subcase 2.2.2: $G_1 \cong C_3$.

From (4.5) and Lemma 2.7, we have $1 \le \sum_{i=2}^{3}(q(G_2) - p(G_2)) \le 3$. Thus we distinguish the following subcases.

If $q(G_2) - p(G_2) = 1$ and $q(G_3) - p(G_3) = 2$, then $G_2 \cong F_m$, $G_3 \cong K_4$ and $a + b + |T_1| + 2|T_2| + 3|T_3| = 2$ by Lemmas 2.6, 4.2, (4.5) and (4.13), which implies that $|T_3| = |T_2| = 0$ and $0 \le b \le 2$. If $b = 0$, then we have, from (4.14), that $9 = -3 + R_5(F_m) + R_5(K_4) + |B| + 2$ and hence $|B| = -1$, a contradiction. If $b = 1$, then $9 = -3 + R_5(F_m) + R_5(K_4) + |B| + 1$ and hence $G = C_3 \cup F_m \cup K_4 \cup (\cup_{i\in A} C_i) \cup fD_4 \cup T_{1,1,1}$. If $b = 2$, then $9 = -3 + R_5(F_m) + R_5(K_4) + |B|$, which results in $G = C_3 \cup F_m \cup K_4 \cup (\cup_{i\in A} C_i) \cup D_j \cup fD_4 \cup 2T_{1,1,1}$. As stated above, from (1) of Lemma 3.8, we have $\beta(\psi_n^3(n-3,1)) = \beta(G) = \beta(K_4)$, which contradicts to $\beta(\psi_n^3(n-3,1)) < \beta(K_4)$ by (3) of Lemma 3.9.

If $q(G_2) - p(G_2) = 1$ and $q(G_3) - p(G_3) = 1$, then $G_2 \cong F_m$, $G_3 \in \phi$ and $a + b + |T_1| + 2|T_2| + 3|T_3| = 1$ by Lemmas 2.6. From this together with (4.14), if $b = 0$, then $9 = -3 + R_5(F_m) + R_5(G_3) + |B| + 1$ and hence $R_5(G_3) = 7 - |B| \le 7$, which contradicts to $G_3 \in \phi$ by Lemma 3.7. If $b = 1$, then $9 = -3 + R_5(F_m) + R_5(G_3) + |B|$ and hence $R_5(G_3) = 8 - |B| \le 8$, which contradicts to $G_3 \in \phi$ by Lemma 3.7.

If $q(G_2) = p(G_2)$ and $q(G_3) - p(G_3) = 1$, then it follows from Lemmas 2.6 and (4.15) that $G_2 \in \xi$, $G_3 \in \psi$ and $a = b = |T_1| = |T_2| = |T_3| = 0$. By (4.16), we have $R_5(G_3) = 12 - R_5(G_2) - |B|$, which results in $|B| = 0$, $R_5(G_2) = 4$ and $R_5(G_3) = 8$. From this together with (1) of Lemma 3.3 and (1) of Lemma 3.4, we know that $G = C_3 \cup G_2 \cup G_3 \cup (\cup_{i\in A} C_i) \cup fD_4$, where $G_2 \in \{C_{n-1}(P_1) | n \ge 5\} \cup \{Q_{1,1}\} \cup \{B_{n-5,1,1} | n \ge 7\}$, $G_3 \in \{\psi_n^1\} \cup \{\psi_5^2\} \cup \{\psi_n^3(r,s) | r \ge 4, s \ge 2\} \cup \{\psi_n^4(n-6,1) | n \ge 8\} \cup \{\psi_n^5(1,s,t) | s,t \ge 2\}$. Using the similar discussing method as Subcase 1.2.2, we can get a contradiction.

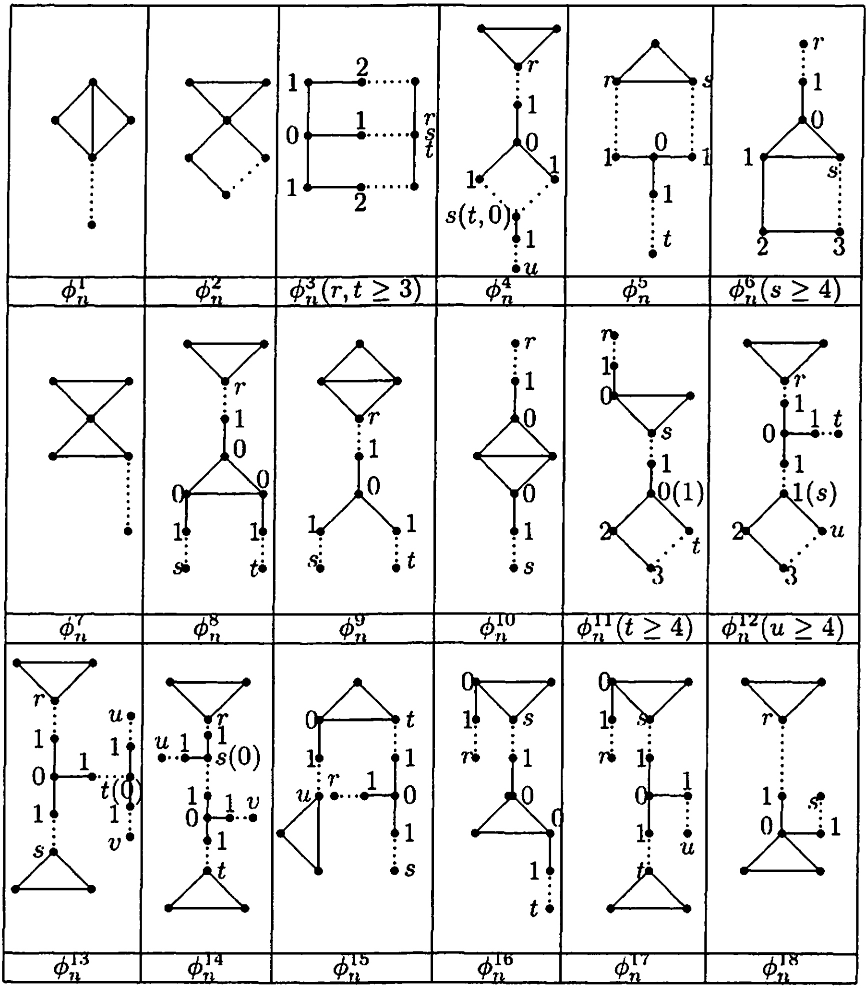

Figure 3 Family of ϕ

Suppose that $q(G_2) - p(G_2) = 1$ and $q(G_3) = p(G_3)$. Applying Lemmas 2.6, 4.2 and (4.15), we have that $G_2 \cong F_m$, $G_3 \in \varphi$ and $a = b = |T_1| = |T_2| = |T_3| = 0$. Hence $R_5(G_3) = 8 - |B| \leq 8$, which contradicts to $G_3 \in \varphi$ by Lemma 3.2.

Subcase 2.3: $s_1 = 3$.

Without loss of generality, let

$$G = \cup_{i=1}^{4} G_i \cup (\cup_{i\in A} C_i) \cup (\cup_{j\in B} D_j) \cup f D_4 \cup aK_1 \cup bT_{1,1,1} \cup (\cup_{T\in \mathcal{T}_0} T_{l_1,l_2,l_3}), \quad (4.15)$$

where $G_1 \in \{P_3, P_4, C_3\}, R_1(G_i) = -1 (i = 2, 3, 4)$.

Using Theorems 3.4 and 3.5, it follows that

$$R_5(G) = R_5(\psi_n^3(n-3,1)) = 9 = \sum_{i=1}^{4} R_5(G_i) + |B| + a + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3| \tag{4.16}$$

Subcase 2.3.1: $G_1 \cong P_3$ or $G_1 \cong P_4$.

Using Lemma 2.7 and (4.5), we know that $2 \le \sum_{i=2}^{4}(q(G_i) - p(G_i)) \le 3$. If $\sum_{i=2}^{4}(q(G_i) - p(G_i)) = 3$, then $G_i \cong F_m (i = 2, 3, 4)$ and $a + b + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3| = 1$ by Lemmas 2.6 and 4.2, which implies that $|\mathcal{T}_3| = |\mathcal{T}_2| = 0$ and $0 \le b \le 1$. If $b = 0$, then we obtain, from (4.16), that $9 = -2 + 3R_5(F_m) + |B| + 1$, which contradicts to $R_5(F_m) = 4$. If $b = 1$, then we have $9 = -2 + 3R_5(F_m) + |B|$, which also contradicts to $R_5(F_m) = 4$. Suppose $\sum_{i=2}^{4}(q(G_i) - p(G_i)) = 2$. Applying Lemmas 2.6 and 4.2, we obtain that $G_i \cong F_m (i = 2, 3)$ and $G_4 \in \xi$ and $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$. Hence $9 = -2 + 2R_5(F_m) + R_5(G_4) + |B|$, which implies $R_5(G_4) = 3 - |B| \le 3$. It contradicts to $G_4 \in \xi$ by Corollary 3.1.

Subcase 2.3.2: $G_1 \cong C_3$.

Using Lemma 2.7 and (4.5), it follows that $1 \le \sum_{i=2}^{3}(q(G_i) - p(G_i)) \le 3$. If $\sum_{i=2}^{3}(q(G_i) - p(G_i)) = 3$, then $G_i \cong F_m (i = 2, 3, 4)$ and $a + b + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3| = 2$ by Lemmas 2.6 and 4.2. If $b = 0$, then $9 = -3 + 3R_5(F_m) + |B| + 2$, which contradicts to $R_5(F_m) = 4$. If $b = 1$, then $9 = -3 + 3R_5(F_m) + |B| + 1$, which also contradicts to $R_5(F_m) = 4$. If $b = 2$, then we arrive, from (4.16), at $9 = -3 + 3R_5(F_m) + |B|$, which implies $G = C_3 \cup F_m \cup F_m \cup F_m \cup (\cup_{i \in A} C_i) \cup fD_4 \cup \cup 2T_{1,1,1}$. From (1) of Lemma 3.8 and Lemma 2.9, it follows that $\beta(\psi_n^3(n-3,1)) = \beta(G) = \beta(F_m)$, which contradicts to $\beta(\psi_n^3(n-3,1)) < \beta(F_m)$ by (2) of Lemma 3.9.

If $\sum_{i=2}^{4}(q(G_i) - p(G_i)) = 2$, then $G_i \cong F_m (i = 2, 3)$ and $G_4 \in \xi$ and $a + b + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3| = 1$ by (3) of Lemmas 2.6 and 4.2. From this together with (4.16), if $b = 0$, then $9 = -3 + 2R_5(F_m) + R_5(G_4) + |B| + 1$, which results in $R_5(G_4) = 3 - |B| \le 3$. It contradicts to $G_4 \in \xi$ by Corollary 3.1. If $b = 1$, then $9 = -3 + 2R_5(F_m) + R_5(G_4) + |B|$, which implies $|B| = 0$ and $G = C_3 \cup F_m \cup F_m \cup G_4 \cup (\cup_{i \in A} C_i) \cup fD_4 \cup \cup T_{1,1,1}$, where $R_5(G_4) = 4$. From (1) of Lemma 3.3, $G_4 \in \{C_{n-1}(P_1)\} \cup \{Q_{1,1}\} \cup \{B_{n-5,1,1}\}$. Combining this with (1) of Lemma 3.8, it follows that $\beta(\psi_n^3(n-3,1)) = \beta(G) = \beta(G_4)$, which contradicts to (2) of Lemma 3.9.

Suppose $\sum_{i=2}^{4} q(G_i) - p(G_i) = 1$. Clearly, $G_2 \cong F_m$ and $G_4 \in \xi$ $(i = 3, 4)$ and $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$ by (3) of Lemma 2.6, Lemma 4.2 and (4.5). Hence $9 = -3 + R_5(F_m) + R_5(G_3) + R_5(G_4) + |B|$, which implies that $G = C_3 \cup F_m \cup G_3 \cup G_4 \cup (\cup_{i \in A} C_i) \cup fD_4$, where $R_5(G_i) = 4 (i = 3, 4)$. We can also get the same contradiction as the above case.

Case 3: $s_{-1} = 2$.

It follows, from (4.6), that $s_1 + 2s_2 + 3s_3 + 4s_4 = 4$, which brings about the following subcases to consider.

Subcase 3.1: $s_4 = 1, s_3 = s_2 = s_1 = 0$.

Without loss of generality, let

$$G = P_3 \cup G_1 \cup G_2 \cup (\cup_{i\in A} C_i) \cup (\cup_{j\in B} D_j) \cup f D_4 \cup a K_1 \cup b T_{1,1,1} \cup (\cup_{T\in \mathcal{T}_0} T_{l_1,l_2,l_3}), \tag{4.17}$$

where $G_1 \in \{P_4, C_3\}, R_1(G_2) = -4$.

From Theorems 3.4 and 3.5, we arrive at

$$R_5(G) = R_5(\psi_n^3(n-3,1)) = 9 = -2 + \sum_{i=1}^{2} R_5(G_i) + |B| + a + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3| \tag{4.18}$$

Recall that $q(G) = p(G) + 1$. If $G_1 \cong P_4$, then $q(G_2) - p(G_2) \geq 3$. By (3) of Lemma 2.7, we have $q(G_2) - p(G_2) < 3$, a contradiction. We now assume $G_1 \cong C_3$. It is obvious that $q(G_2) - p(G_2) \geq 2$ by (4.5) and (4.17). By (3) of Lemma 2.7, we arrive at $q(G_2) - p(G_2) < 3$. Then $q(G_2) - p(G_2) = 2$ and $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$, which implies $G_2 \in \theta$ by (6) of Lemma 2.6. From (4.18), we have $R_5(G_2) = 13 - |B| \leq 13$, which contradicts to Lemma 3.6.

Subcase 3.2: $s_4 = s_2 = 0, s_3 = s_1 = 1$.

Without loss of generality, let

$$G = P_3 \cup (\cup_{i=1}^{3} G_i) \cup (\cup_{i\in A} C_i) \cup (\cup_{j\in B} D_j) \cup f D_4 \cup a K_1 \cup b T_{1,1,1} \cup (\cup_{T\in \mathcal{T}_0} T_{l_1,l_2,l_3}), \tag{4.19}$$

where $G_1 \in \{P_4, C_3\}, R_1(G_2) = -1, R_1(G_3) = -3$.

From Theorems 3.4 and 3.5, we arrive at

$$R_5(G) = R_5(\psi_n^3(n-3,1)) = 9 = -2 + \sum_{i=1}^{3} R_5(G_i) + |B| + a + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3| \tag{4.20}$$

If $G_1 \cong P_4$, then $\sum_{i=2}^{3}(q(G_i) - p(G_i)) \geq 3$ by (4.5) and (4.19). From Lemmas 2.6 and 2.7, we have $\sum_{i=2}^{3}(q(G_i) - p(G_i)) \leq 3$. Then $\sum_{i=2}^{3}(q(G_i) - p(G_i)) = 3$, which implies $G_2 \cong F_m, G_3 \in \zeta$ and $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$. By (4.20), $9 = -2 - 2 + R_5(F_m) + R_5(G_3) + |B|$ and hence $R_5(G_3) = 9 - |B| \leq 9$, which contradicts to $G_3 \in \zeta$ by Corollary 3.3.

Suppose $G_1 \cong C_3$. Applying Lemma 2.7 and (4.5), we have $2 \leq \sum_{i=2}^{3}(q(G_i) - p(G_i)) \leq 3$. Consider the case $\sum_{i=2}^{3}(q(G_i) - p(G_i)) = 3$. From Lemmas 2.6, 4.2 and (4.5), we have $G_2 \cong F_m$ and $G_3 \in \zeta$ and $a + b + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3| = 1$. If $b = 0$, then $9 = -2 - 3 + R_5(F_m) + R_5(G_3) + |B| + 1$, which results in $R_5(G_3) = 9 - |B| \leq 9$, which contradicts to $G_3 \in \zeta$. If $b = 1$, then $9 = -2 - 3 + R_5(F_m) + R_5(G_3) + |B|$ and hence $R_5(G_3) = 10 - |B| \leq 10$, which also contradicts to $G_3 \in \zeta$.

Consider the case $\sum_{i=2}^{3}(q(G_i) - p(G_i)) = 2$. If $q(G_2) = p(G_2)$ and $q(G_3) - p(G_3) = 2$, then $G_2 \in \xi$ and $G_3 \in \zeta$ and $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$ by Lemma 2.6 and (4.5). Then $9 = -2 - 3 + R_5(G_2) + R_5(G_3) + |B|$ and hence $R_5(G_2) + R_5(G_3) = 14 - |B| \leq 14$, which contradicts to $G_3 \in \zeta$ by Corollary 3.1 and Corollary 3.3. If $q(G_2) - p(G_2) = 1$ and $q(G_3) - p(G_3) = 1$, then

$G_2 \cong F_m$ and $G_3 \in \phi$ and $a = b = |T_1| = |T_2| = |T_3| = 0$ by 2.6, 4.2 and (4.5). From this together with (4.20), we get that $9 = -2-3+R_5(F_m)+R_5(G_3)+|B|$. Hence $R_5(G_3) = 10 - |B| \le 10$, which contradicts to Lemma 3.7.

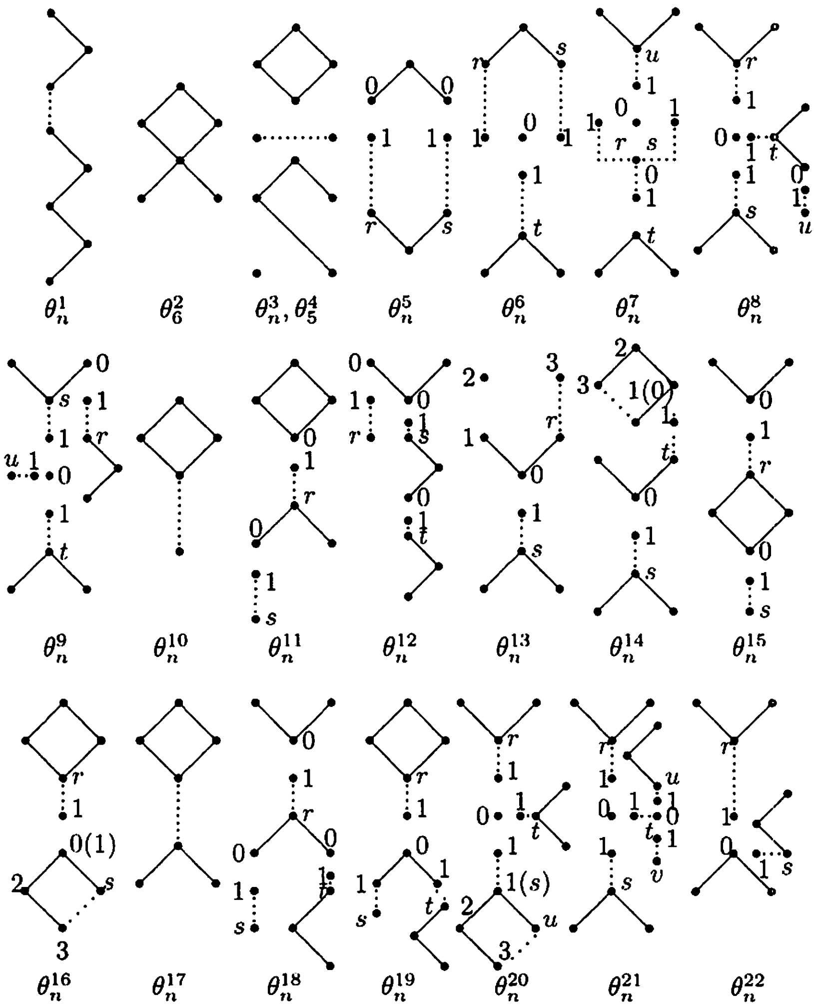

Figure 4 Family of θ

Subcase 3.3: $s_4 = s_3 = s_1 = 0, s_2 = 2$.
Without loss of generality, let

$$G = P_3 \cup (\cup_{i=1}^4 G_i) \cup (\cup_{i\in A} C_i) \cup (\cup_{j\in B} D_j) \cup f D_4 \cup a K_1 \cup b T_{1,1,1} \cup (\cup_{T\in \mathcal{T}_0} T_{l_1,l_2,l_3}), \tag{4.21}$$

where $G_1 \in \{P_4, C_3\}, R_1(G_i) = -2 (i = 3, 4)$.

By Theorems 3.4 and 3.5, we have

$$R_5(G) = R_5(\psi_n^3(n-3,1)) = 9 = -2+\sum_{i=1}^{3} R_5(G_i)+|B|+a+|T_1|+2|T_2|+3|T_3| \quad (4.22)$$

Suppose $G_1 \cong P_4$. Recall that $q(G) = p(G) + 1$. Then $\sum_{i=2}^{3}(q(G_i) - p(G_i)) \geq 3$. From Lemma 2.7, $\sum_{i=2}^{3}(q(G_i) - p(G_i)) \leq 4$. Therefore, $3 \leq \sum_{i=2}^{3}(q(G_i) - p(G_i)) \leq 4$. If $\sum_{i=2}^{3}(q(G_i) - p(G_i)) = 4$, then $G_i \cong K_4(i = 2,3)$ and $a+b+|T_1|+2|T_2|+3|T_3| = 1$ by (4) of Lemma 2.6 and (4.5). If $b = 0$, then $9 = -2-2+2R_5(K_4)+|B|+1$, which contradicts to $R_5(K_4) = 7$ by (8) of Theorem 3.5. If $b = 1$, then $9 = -2-2+2R_5(K_4)+|B|$, which also contradicts to $R_5(K_4) = 7$. If $\sum_{i=2}^{3}(q(G_i) - p(G_i)) = 3$, then it follows (4) of Lemma 2.6 and (4.5) that $G_2 \cong K_4$, $G_3 \in \psi$ and $a = b = |T_1| = |T_2| = |T_3| = 0$. Combining this with (4.22), we have $9 = -2-2+R_5(K_4)+R_5(G_3)+|B|$ and hence $R_5(G_3) = 6-|B| \leq 6$, which contradicts to $G_3 \in \psi$ by Corollary 3.2.

Suppose $G_1 \cong C_3$. From (4.5) and Lemma 2.7, we have $2 \leq \sum_{i=2}^{3}(q(G_i) - p(G_i)) \leq 4$. If $\sum_{i=2}^{3}(q(G_i) - p(G_i)) = 4$, then $G_i \cong K_4(i = 2,3)$ and $a+b+|T_1|+2|T_2|+3|T_3| = 2$ by Lemma 2.6 and (4.5) and hence $|T_3| = |T_2| = 0$ and $0 \leq b \leq 1$. Combining this with (4.22), if $b = 0$, then $9 = -2-3+2R_5(K_4)+|B|+2$, which contradicts to $R_5(K_4) = 7$. If $b = 1$, then $9 = -2-3+2R_5(K_4)+|B|+1$, which also contradicts to $R_5(K_4) = 7$. If $b = 2$, then $9 = -2-3+2R_5(K_4)+|B|$ and hence $|B| = 0$ and $G = P_3 \cup C_3 \cup K_4 \cup K_4 \cup (\cup_{i\in A}C_i) \cup fD_4 \cup 2T_{1,1,1}$. From Lemma 2.9 and (1) of Lemma 3.8, $\beta(\psi_n^3(n-3,1)) = \beta(G) = \beta(K_4)$, which contradicts to $\beta(\psi_n^3(n-3,1)) < \beta(K_4)$ by (3) of Lemma 3.9.

If $\sum_{i=2}^{3}(q(G_i) - p(G_i)) = 3$, then $G_2 \cong K_4$, $G_3 \in \psi$ and $a+b+|T_1|+2|T_2|+3|T_3| = 1$ by (4) of Lemma 2.6 and (4.5). If $b = 0$, then $9 = -2-3+R_5(K_4)+R_5(G_3)+|B|+1$ and hence $R_5(G_3) = 6-|B| \leq 6$, which contradicts to Corollary 3.2. If $b = 1$, then $9 = -2-3+R_5(K_4)+R_5(G_3)+|B|$ and hence $R_5(G_3) = 7-|B| \leq 7$, which also contradicts to Corollary 3.2.

Suppose $\sum_{i=2}^{3}(q(G_i) - p(G_i)) = 2$. From Lemma 2.6 and (4.5), we know that $G_i \in \psi(i = 2,3)$ and $a = b = |T_1| = |T_2| = |T_3| = 0$. Combining this with (4.22), $9 = -2-3+R_5(G_2)+R_5(G_3)+|B|$ and hence $R_5(G_2)+R_5(G_3) = 14-|B| \leq 14$, which contradicts to $G_3 \in \psi$ by Corollary 3.2.

Subcase 3.4: $s_4 = s_3 = s_2 = 0, s_1 = 4$.

Without loss of generality, let

$$G = P_3 \cup(\cup_{i=1}^{5} G_i)\cup(\cup_{i\in A}C_i)\cup(\cup_{j\in B}D_j)\cup fD_4 \cup aK_1 \cup bT_{1,1,1}\cup(\cup_{T\in \mathcal{T}_0}T_{l_1,l_2,l_3}), \quad (4.23)$$

where $G_1 \in \{P_4, C_3\}$, $R_1(G_i) = -1(i = 2,3,4,5)$.

Applying Theorems 3.4 and 3.5, we get that

$$R_5(G) = R_5(\psi_n^3(n-3,1)) = 9 = -2+\sum_{i=1}^{5} R_5(G_i)+|B|+a+|T_1|+2|T_2|+3|T_3| \quad (4.24)$$

Suppose $G_1 \cong P_4$. Using (1) of Lemma 2.7 and (4.5), we have $3 \leq \sum_{i=2}^{5}(q(G_i) - p(G_i)) \leq 4$. If $\sum_{i=2}^{5}(q(G_i) - p(G_i)) = 4$, then $G_i \cong F_m (i = 2,3,4,5)$ and $a + b + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3| = 1$ by Lemmas 2.6 and 4.2. If $b = 0$, then $9 = -2 - 2 + 4R_5(F_m) + |B| + 1$, which contradicts to $R_5(F_m) = 4$. If $b = 1$, then $9 = -2 - 2 + 4R_5(F_m) + |B|$, which also contradicts to $R_5(F_m) = 4$. If $\sum_{i=2}^{5}(q(G_i) - p(G_i)) = 3$, then $G_i \cong F_m$ $(i = 2,3,4)$, $G_5 \in \xi$ and $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$ by Lemmas 2.6 and 4.2. Hence $9 = -2 - 2 + 3R_5(F_m) + R_5(G_5) + |B|$ and hence $R_5(G_5) = 1 - |B| \leq 1$, which contradicts to $G_5 \in \xi$ by Corollary 3.1.

Suppose $G_1 \cong C_3$. Recall that $q(G) = p(G) + 1$. Then $2 \leq \sum_{i=2}^{5}(q(G_i) - p(G_i)) \leq 4$ by Lemma 2.7. If $\sum_{i=2}^{5}(q(G_i) - p(G_i)) = 4$, then $G_i \cong F_m$ $(i = 2,3,4,5)$ and $a + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3| = 2$. Combining this with (4.24), if $b = 0$, then $9 = -2 - 3 + 4R_5(F_m) + |B| + 2$, which contradicts to $R_5(F_m) = 4$. We can get a contradiction for $b = 1$ and $b = 2$. If $\sum_{i=2}^{3}(q(G_i) - p(G_i)) = 3$, then $G_i \cong F_m$ $(i = 2,3,4)$, $G_5 \in \xi$ and $a + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3| = 1$ by Lemmas 2.6 and 4.2. If $b = 0$, then it follows from (4.24) that $9 = -2 - 3 + 3R_5(F_m) + R_5(G_5) + |B| + 1$ and hence $R_5(G_5) = 1 - |B| \leq 1$, which contradicts to $G_5 \in \xi$. If $b = 1$, then $9 = -2 - 3 + 3R_5(F_m) + R_5(G_5) + |B|$ and hence $R_5(G_5) = 2 - |B| \leq 2$, which also contradicts to $G_5 \in \xi$. If $\sum_{i=2}^{3}(q(G_i) - p(G_i)) = 2$, then $G_i \cong F_m$ $(i = 2,3)$, $G_i \in \xi$ $(i = 4,5)$ and $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$ by Lemmas 2.6, 4.2 and (4.23). From this together with (4.24), $9 = -2 - 3 + |B| + 2R_5(F_m) + R_5(G_4) + R_5(G_5)$. By Corollary 3.1, $R_5(G_3) \geq 4$. Hence $R_5(G_4) + R_5(G_5) = 6 - |B| \leq 6$, which contradicts to $G_4, G_5 \in \xi$ by Corollary 3.1. ❏

Corollary 4.1. *If $n \geq 7$, graph $\psi_n^3(n-3,1)$ is adjoint uniqueness if and only if $n \neq 8$.*

Corollary 4.2. *If $n \geq 7$, the chromatic equivalence class of $\psi_n^3(n-3,1)$ only contains the complements of graphs that are in Theorem 4.2.*

Corollary 4.3. *If $n \geq 7$, graph $\psi_n^3(n-3,1)$ is chromatic uniqueness if and only if $n \neq 8$.*

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