On Some Variants of Gracefulness of Cycle Graphs

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Abstract

A graph $G = (V(G), E(G))$ is even graceful and equivalently graceful, if there exists an injection f from the set of vertices $V(G)$ *to* $\{0,1,2,3,4,\ldots,2|E(G)|\}$ such that when each edge uv is assigned the label $|f(u) - f(v)|$, the resulting edge labels are $2,4,6,\ldots,2|E(G)|$. In this work, we use even graceful labeling to give a new proof for necessary and sufficient conditions for the gracefulness of the cycle graph. We extend this technique to odd graceful and super Fibonacci graceful labelings of cycle graphs via some number theoretic concept, called a balanced set of natural numbers

Keywords: Graph labeling; Graceful; Odd graceful; Even graceful; Super Fibonacci graceful.

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1. Introduction

Labeling of graphs appeared in the late sixties of the last century with the paper of Rosa [1]. Now, there is a huge number of graph labeling techniques which are interesting from a mathematical point of view. Moreover, some of these labelings proved to have real life applications including crystallography, cryptography and experimental design [11].

A graph $G = (V, E)$ consists of $V = V(G)$, its set of vertices and $E = E(G)$, its set of edges. The order of G is $|V|$ and the size of G is $|E|$, where $|X|$ denotes the cardinality of the set X. Here we deal with simple finite non directed graphs. One can refer to [10] for definitions of these graph theoretic terms.

The path graph P_n has vertices $v_1, v_2, \cdots, v_n$ and edges $v_1v_2, v_2v_3, \cdots, v_{n-1}v_n$; $n \geq 2$.

The cycle graph C_n consists of P_n and the additional edge v_1v_n; $n \geq 3$.

The study of graceful labeling of a graph is an active research area in graph theory. The graceful labeling problem is to determine which graphs are graceful. The graceful graphs and graceful labeling methods originated with the study in [1], the author called such a function a β −valuation of the graph. Golomb [13], subsequently called such labeling graceful. A function f is called graceful labeling of graph G of size q if $f: V \rightarrow \{0,1,2,\dots,q\}$ is injective and the induced function $F: E \rightarrow \{1,2,3,\dots,q\}$ defined by $f^*(uv) = |f(u) - f(v)|$ is bijective, a graph which admits graceful labeling is called graceful graph, see [14]. There are few general results on graceful labeling including some cycle related graphs e.g., wheels [9], gears [4], helms [3], webs [7] and double wheels [11].

There is a number of variants of graceful labeling of graphs including odd, even, Fibonacci and super Fibonacci graceful labeling of graphs.

Definition 1.1 [15,6] A graph G with p vertices and q edges is odd graceful, if there exists an injection f from $V(G)$ to $\{0,1,2,3,\dots,2q-1\}$ such that, when each edge uv is assigned the label $/f(u) - f(v)|$, the resulting edge labels are $1,3,5,\dots,2q-1$.

In [8], the author proved that the cycle graph C_n is odd graceful if and only if n is even. In this work, we study the even gracefulness of cycle graphs using balanced sets of natural numbers and apply a similar method to odd graceful and super Fibonacci graceful labeling of cycle graphs.

Definition 1.2 [14] A graph G of size n is even graceful, if there exists an injection f from $V(G)$ to $\{0,2,4,\dots,2n\}$ such that, when each edge uv is assigned the label $/f(u) - f(v)|$, the resulting edge labels are $2,4,6,\dots,2n$.

Definition 1.3 [5,2] Let $F_1 = 1$, $F_2 = 1$, $F_i = F_{i-1} + F_{i-2}; i \geq 3$. A graph G of size F_n is Fibonacci graceful, if there exists an injection f from $V(G)$ to $\{0,1,2,\cdots,F_n\}$ such that, when each edge uv is assigned the label $/f(u) - f(v)|$, the resulting edge labels are $F_1, F_2, \dots, F_n$.

Definition 1.4 [12] Let $F_0 = 0$, $F_1 = 1$, $F_2 = 2$, $F_i = F_{i-1} + F_{i-2}; i \geq 3$. A graph G of size F_n is super Fibonacci graceful, if there exists an injection f from $V(G)$ to $\{F_0, F_1, F_2, \cdots, F_n\}$ such that, when each edge uv is assigned the label $/f(u) - f(v)|$, the resulting edge labels are $F_1, F_2,, \cdots, F_n$.

Definition 1.5 Let H be a finite subset of the set of natural numbers $N = \{1,2,3,\cdots\}$. A graph G of size $|H|$ is H-graceful, if there exists an injection f from $V(G)$ to $\{0,1,2,\cdots, max\,(H)\}$ such that, when each edge uv is assigned the label $/f(u) - f(v)|$, the resulting edge labels are the elements of H.

Definition 1.6 Let H be a finite subset of the set of natural numbers $N = \{1,2,3,\cdots\}$. H is a balanced set if there exist two subsets H_1, H_2 of H satisfying $H = H_1 \cup H_2$, $H_1 \cap H_2 = \emptyset$, and $\sum_{y \in H_1} y = \sum_{y \in H_2} y$.

2. Main results

Lemma 2.1: If C_n is H-graceful graph, then H is a balanced set.

Proof. Let $b_{i,j}$ be the label of the edge $v_i v_j$, $i = 1,2,\cdots,n; j \equiv i+1 \ (mod\ n)$. It is clear that

$H = \{b_{i,j} \ ; \ i = 1,2,\cdots,n, j \equiv i+1 \ (mod\ n)\}$ and the following equations hold

$f(v_i) - f(v_{i-1}) = \delta_{i-1,i}\, b_{i-1,i}, \ \ i = 2,\cdots,n$

$f(v_1) - f(v_n) = \delta_{1,n} b_{1,n}$, where $\delta_{i,j} = \delta_{j,i} \in \{-1,1\}$.

Summing up these equations we get

$\delta_{1,n}\ b_{1,n} + \sum_{i=1}^{n-1} \delta_{i,i+1}\ b_{i,i+1} = 0.$

Let $H_1 = \{b_{i,j} ; \delta_{i,j} = 1\}$ and $H_2 = \{b_{i,j} ; \delta_{i,j} = -1\}$.

Therefore, $H = H_1 \cup H_2$, $H_1 \cap H_2 = \emptyset$, and $\sum_{y \in H_1} y = \sum_{y \in H_2} y$ and hence the lemma.

Theorem 2.2 The following conditions are equivalent.

1. C_n is odd graceful.
2. The set $\{1,3,\cdots,2n-1\}$ is a balanced set.
3. n is an even number.

Proof. If C_n is odd graceful, then by Lemma 2.1 C_n is H-graceful, where $H = \{1,3,\cdots,2n-1\}$ is a balanced set. This proves that $1. \Longrightarrow 2.$ It is well known that $\sum_{x \in H} x = n^2$ and this sum is even if and only if n is even. Therefore, if H is a balanced set then n is even. This proves that $2. \Longrightarrow 3.$

The following two cases proves that $3. \Longrightarrow 2.$

Case1: $n \equiv 0 (mod\ 4) \ \ i.e., n = 4m.$

$H = \{1,3,5,\cdots,8m-3,8m-1\}$

Let $H_1 = \{1,5,\cdots,4m-3; 4m+3,4m+7,\cdots,8m-1\}$,

$H_2 = \{3,7,\cdots,4m-1; 4m+1,4m+5,\cdots,8m-3\}$.

Clearly, $H = H_1 \cup H_2$, $H_1 \cap H_2 = \emptyset$ and $\sum_{y \in H_1} y = \sum_{y \in H_2} y$.

Case2: $n \equiv 2 (mod\ 4) \ \ i.e., n = 4m+2, m \geq 1.$

Claim: $\exists\ H_1, H_2 \subset H$, satisfying $|H_1| = 2m, |H_2| = 2m+2$, for

$H = \{1,3,5,7,9,11,\cdots,8m+3\}, m \geq 1$ satisfying

$H = H_1 \cup H_2$, $H_1 \cap H_2 = \emptyset$ and $\sum_{x \in H_1} x = \sum_{x \in H_2} x$.

Proof of claim . By induction, the initial step is clear when $m = 1$, as

$H = \{1,3,5,7,9,11\}$ and the sets

$H_1 = \{11,7\}$, $H_2 = \{9,5,3,1\}$ satisfy the claim. For the inductive step, assume the claim is true for some $m \geq 1$. i.e.,

$H = \{1,3,\cdots,8m+3\}$ is balanced set for some partition $H = H_1 \cup H_2$.

For $m+1$ the underlined sets will be

$H^{\sim} = H \cup \{8m+5, 8m+7, 8m+9, 8m+11\}$

$H_1^{\sim} = H_1 \cup \{8m+5, 8m+11\}$ and

$H_2^{\sim} = H_2 \cup \{8m+7, 8m+9\}$.

It is obvious that the assertion of the claim is true for $m+1$ and hence by the principle of mathematical induction the claim follows.

The last part of the proof is to show an odd graceful labeling of C_n when n is even, see [8] for such a labeling.

Lemma 2.3: The set $\{2,4,\cdots,2n\}$ is a balanced set if and only if n is congruent to 0 or $3(mod\ 4)$.

Proof. Assume $H = \{2,4,\cdots,2n\}$ is a balanced set.

It is well known that $\sum_{x\in H} x = n(n+1)$.

Let $S = \frac{1}{2}n(n+1)$. We will study the following cases.

1. $n \equiv 0\ (mod\ 4)$ i.e., $n = 4m, m \in N$.

$S = 2m(4m+1)$ which is an even number. In this case

$H = \{2,4,\cdots,8m\} = H_1 \cup H_2$ where

$H_1 = \{2,6,\cdots,4m-2,4m+4,\cdots,8m-4,8m\}$

and $H_2 = \{4,8,\cdots,4m,4m+2,\cdots,8m-6,8m-2\}$.

we see that $|H_1| = |H_2| = 2m$.

These sets satisfy: $H_1 \cap H_2 = \emptyset$ and $\sum_{y\in H_1} y = \sum_{y\in H_2} y$. This proves that H is a balanced set.

2. $n \equiv 1\ (mod\ 4)$ i.e., $n = 4m+1, m \in N$.

$S = (4m+1)(2m+1)$ which is an odd number. This is impossible as all elements in H are even, and hence H is not a balanced set in this case.

3. $n \equiv 2\ (mod\ 4)$ i.e., $n = 4m+2, m \in N$.

$S = (2m+1)(4m+3)$ which is impossible as in the previous case and hence H is not a balanced set in this case as well.

4. $n \equiv 3$ (mod 4) i.e., $n = 4m+3, m \in N$.

$S = (4m+3)(2m+2)$ which is an even number.

$H = \{2,4,\cdots,8m+6\} = H_1 \cup H_2$ where

$H_1 = \{8m+6,8m+2,\cdots,4m+6;\ 4m,4m-4,\cdots,4\}$

and $H_2 = \{8m+4,8m,\cdots,4m+4;\ 4m+2,4m-2,\cdots,2\}$.

we see that $|H_1| = 2m+1$ and $|H_2| = 2m+2$.

These sets satisfy the conditions for H being a balanced set.

Theorem 2.4 The cycle graph C_n is even graceful for every $n \equiv 0(mod4)$.

Proof. Let $n \equiv 0(mod\ 4)$ and $V(C_n) = \{v_1, v_2, v_3, \cdots, v_n\}$, where $V(C_n)$ is the vertex set of the cycle graph C_n see Fig(1). The labels of the vertices: $v_1, v_2, v_3, \dots, v_n$ are defined as follows:

$f(v_i) = i-1$ when i is odd

$f(v_i) = 2n-(i-2)$ when $i \le \frac{n}{2}$ is even

$f(v_i) = 2n - i$ when $i > \frac{n}{2}$ is even.

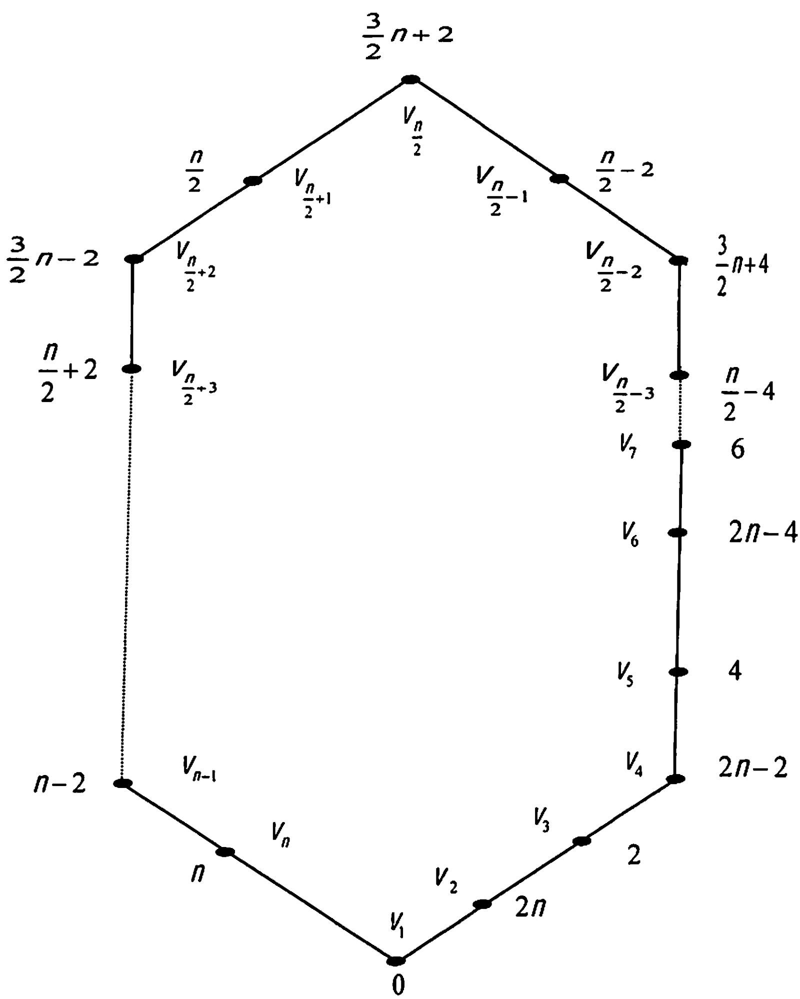
Applying Definition 1.2, we get the edge labeling function l as follows:

$l(v_1v_2) = 2n,\quad l(v_2v_3) = 2n - 2,\quad l(v_3v_4) = 2n - 4,$

$l(v_4v_5) = 2n - 6,\quad l(v_5v_6) = 2n - 8,\ l(v_6v_7) = 2n - 10, \ldots,$

$l\left(v_{\frac{n}{2}-1}\ v_{\frac{n}{2}}\right) = n + 4, l\left(v_{\frac{n}{2}}\ v_{\frac{n}{2}+1}\right) = n + 2,$

$l\left(v_{\frac{n}{2}+1}\ v_{\frac{n}{2}+2}\right) = n - 2,\quad l(v_{n-1}v_n) = 2., l(v_n\, v_1) = n.$



Then the edge labels using this rule are as follows:

$2n, 2n-2, 2n-4, 2n-6, \dots, n+4, n+2, n-2, n-4, n-6, \dots, 2, n.$

Hence C_n is even graceful if $n \equiv 0(mod4)$ and the proof is complete.

For example, C_8, see Fig(2) admits even graceful labeling where the vertex labels are given by

$f(v_1) = 0,\quad f(v_2) = 16,\quad f(v_3) = 2,\quad f(v_4) = 14,$

$f(v_5) = 4$, $f(v_6) = 10$, $f(v_7) = 6$, $f(v_8) = 8$ and the corresponding edge labels are

$l(v_1v_2) = 16$, $l(v_2v_3) = 14$, $l(v_3v_4) = 12$, $l(v_4v_5) = 10$,
$l(v_5v_6) = 6$, $l(v_6v_7) = 4$, $l(v_7v_8) = 2$, $l(v_8v_1) = 8$.

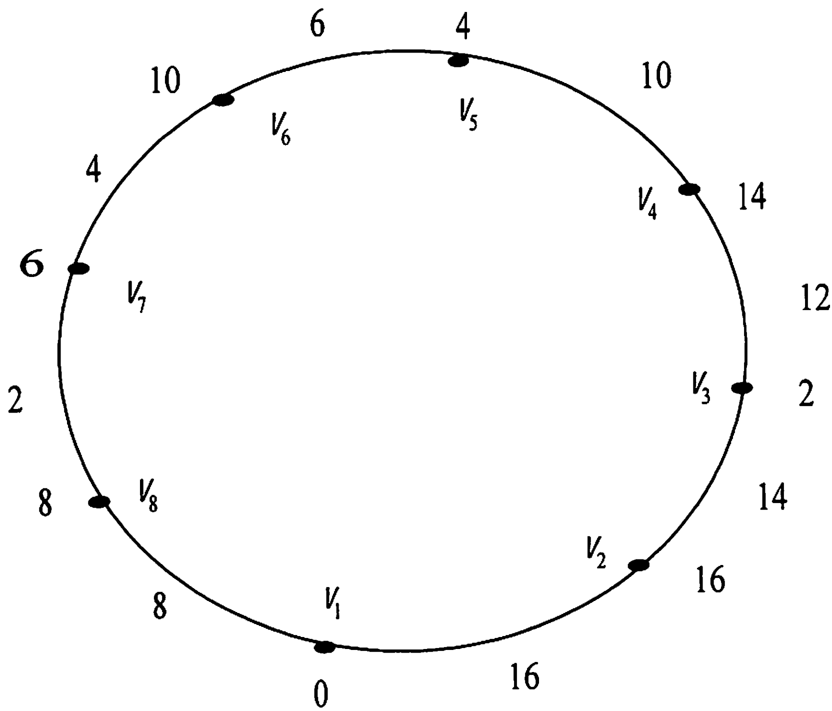


Fig (2): even graceful labeling of c_8

Theorem 2.5 The cycle graph C_n is even graceful for every $n \equiv 3(mod 4)$.

Proof. Let $n \equiv 3 \; mod \; 4$ and $V(C_n) = \{v_1, v_2, v_3, \cdots, v_n\}$, where $V(C_n)$ is the vertex set of the cycle graph C_n, see Fig(3). The even labels of the vertices $v_1, v_2, v_3, \cdots, v_n$ are given by

$f(v_i) = i - 1$ when i is odd

$f(v_i) = 2n - (i - 2)$ when $i \leq \frac{n+1}{2}$ is even

(of course $i = \frac{n+1}{2}$ is even as $n \equiv 3(mod 4)$)

$f(v_i) = 2n - i$ when $i > \frac{n+1}{2}$ is even.

Applying the Definition 1.2, we get the edge labeling function l as follows:

$l(v_1 v_2) = 2n, l(v_2 v_3) = 2n - 2$,

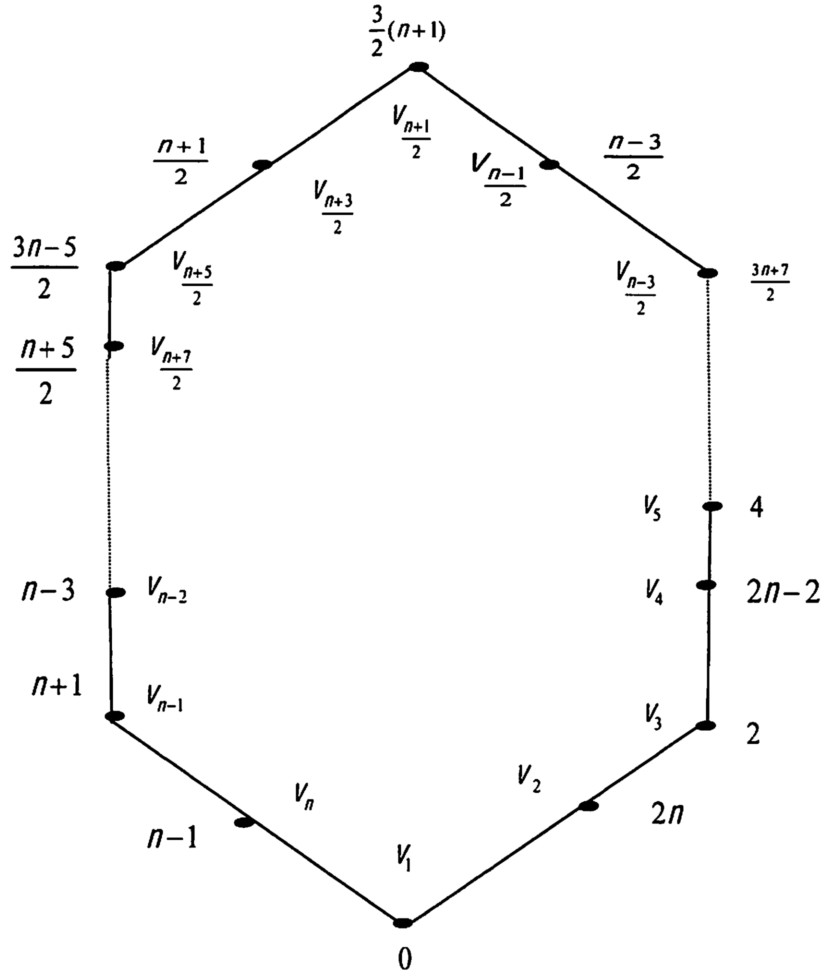
$l(v_3 v_4) = 2n - 4, \quad l(v_4 v_5) = 2n - 6,$

$l(v_5 v_6) = 2n - 8, \; l(v_6 v_7) = 2n - 10, \cdots,$

$l\left(v_{\frac{n-3}{2}} v_{\frac{n-1}{2}}\right) = n + 5, l\left(v_{\frac{n+1}{2}} v_{\frac{n+3}{2}}\right) = n + 1,$

$l\left(v_{\frac{n+3}{2}} v_{\frac{n+5}{2}}\right) = n - 3, l\left(v_{\frac{n+5}{2}} v_{\frac{n+7}{2}}\right) = q - 5, \cdots,$

$l(v_{n-2} v_{n-1}) = 4, \; l(v_{n-1} v_n) = 2,. \, l(v_n v_1) = n - 1.$



Fig(3):even graceful labeling of c_n, $n \equiv 3(mod4)$

Then the edge labels using this rule are as follows:

$2n, 2n-2, 2n-4, \cdots, n+5, n+3,$

$n+1, n-3, n-5, \cdots, 4, 2, n-1$. This completes the proof.

As an illustration, C_7, see Fig(4), has the following vertex labels:

$f(v_1) = 0, \quad f(v_2) = 14, \quad f(v_3) = 2, \quad f(v_4) = 12,$

$f(v_5) = 4,\ f(v_6) = 8,\ f(v_7) = 6$ and the corresponding edge labels are

$l(v_1v_2) = 14,\quad l(v_2v_3) = 12,\quad l(v_3v_4) = 10,\quad l(v_4v_5) = 8,$

$l(v_5v_6) = 4,\ l(v_6v_7) = 2, l(v_7v_1) = 6$

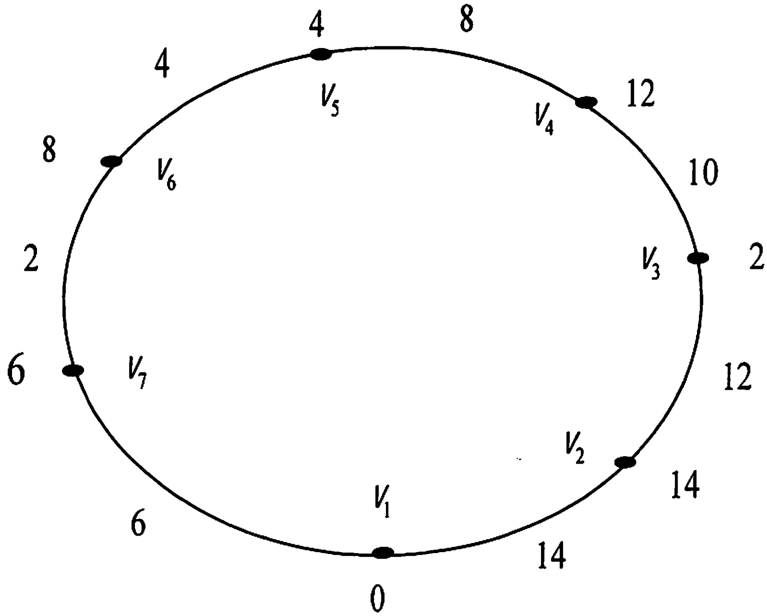


Fig (4): even graceful labeling of c_7

Combining Lemma 2.3, Theorem 2.4 and Theorem 2.5, we conclude the following characterization.

Theorem 2.6 The following conditions are equivalent.

1. C_n is graceful .
2. C_n is even graceful.
3. The set $\{2,4, \ldots ,2n\}$ is a balanced set.
4. $n \equiv 0$ or $3(mod 4)$.

Applying the technique adopted previously in odd graceful cycle graphs and even graceful cycle graphs for super Fibonacci cycle graphs we get the following results.

Lemma 2.7 The set $H = \{F_1, F_2, \cdots, F_n\}$ where $F_1 = 1,\ F_2 = 2,\ F_i = F_{i-1} + F_{i-2};\ i \geq 3$ is balanced if and only if $n \equiv 0 \pmod 3$.

Proof. By straight forward induction, one can easily verify that

1. F_n is an even number if and only if $n \equiv 2 \pmod 3$.
2. $\sum_{i=1}^{n} F_i = F_{n+2} - 2;\ n \geq 1$.
3. $\sum_{x \in H} x$ is an even number if and only if $n \equiv 0 \pmod 3$.

The rest of the proof is to show a partition of H which satisfy requirements in Definition 1.6, in the case $n \equiv 0 \pmod 3$. Therefore, $n = 3m$ for some natural number $m \geq 1$, it is clear that $H = H_1 \cup H_2$ where $H_1 = \{F_1, F_2, F_4, F_5, \cdots, F_{3m-2}, F_{3m-1}\}, H_2 = \{F_3, F_6, \cdots, F_{3m}\}$.

These sets satisfy: $H_1 \cap H_2 = \emptyset$ and $\sum_{y \in H_1} y = \sum_{y \in H_2} y$ and hence the proof is complete.

Theorem 2.8 The following conditions are equivalent.

1. C_n is super Fibonacci graceful.
2. The set $\{F_1, F_2, \cdots, F_n\}$ is a balanced set.
3. $n \equiv 0 \pmod 3$.

Proof. From Lemma 2.1, if C_n is super Fibonacci graceful, then it is H-graceful, where $H = \{F_1, F_2, \cdots, F_n\}$ and hence 1. ⟹2.

From the previous Lemma 2.7, we have equivalence between conditions 2. and 3.

The rest of the proof is to show that 3. ⟹1.

If $n \equiv 0 \pmod 3$, then we can label the vertices of C_n by the sequence

$F_0, F_n, F_{n-2}, F_{n-1}, \cdots, F_{n-3j}, F_{n-3j-2}, F_{n-3j-1}, \cdots, F_6, F_4, F_5, F_3, F_1$.

$j = 0,1, \cdots m-1$,where $n = 3m$ for some natural number $m \geq 1$. The corresponding edge labels are $F_1, \cdots, F_n$. which completes the proof.

3. Conclusion

We have introduced the concept of balanced finite set of natural numbers with respect to addition. This concept was used in relation to different types of graceful labelings of graphs. Necessary and sufficient conditions for odd graceful, even graceful and super Fibonacci graceful labelings of graphs are deduced. This concept may be studied for other sets and applied to different types of graceful labeling and. It may be used to study special systems of equations over finite sets of natural numbers.

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