On the path edge-connectivity of graphs*

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Abstract

Dirac showed that in a $(k-1)$-connected graph there is a path through each k vertices. The path k-connectivity $\pi_k(G)$ of a graph G, which is a generalization of Dirac's notion, was introduced by Hager in 1986. Recently, Mao introduced the concept of path k-edge-connectivity $\omega_k(G)$ of a graph G. Denote by $G \circ H$ the lexicographic product of two graphs G and H. In this paper, we prove that $\omega_4(G \circ H) \geq \omega_4(G)\lfloor \frac{3|V(H)|}{5} \rfloor$ for any two graphs G and H. Moreover, the bound is sharp.

Keywords: Edge-connectivity; Steiner tree; packing; path edge-connectivity; lexicographic product.

AMS subject classification 2010: 05C05, 05C40, 05C70, 05C76.

1 Introduction

All graphs considered in this paper are undirected, finite and simple. We refer to [2] for graph theoretical notation and terminology not described here. For a graph G, let $V(G)$, $E(G)$ and $\delta(G)$ denote the set of vertices, the set of edges and the minimum degree of G, respectively. For $S \subseteq V(G)$, we denote by $G - S$ the subgraph obtained by deleting from G the vertices of S together with the edges incident with them.

In [8], Dirac showed that in a $(k-1)$-connected graph there is a path through each k vertices; see [34]. In [16], Hager revised this statement to the question of how many internally disjoint paths P_i with the exception of a given set S of k vertices exist such that $S \subseteq V(P_i)$. The path connectivity of a graph G, introduced by Hager [16], is a natural specialization of the generalized connectivity and is also a natural generalization of the 'path' version definition of connectivity. For a graph $G = (V, E)$ and a set $S \subseteq V(G)$ of at least two vertices, *a path connecting* S (or simply, *an* S*-path*) is a subgraph $P = (V', E')$ of G that is a path with $S \subseteq V'$. Note that a path connecting S is also a tree connecting S. Two paths P and P' connecting S are said to be *internally disjoint* if

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$E(P) \cap E(P') = \varnothing$ and $V(P) \cap V(P') = S$. For $S \subseteq V(G)$ and $|S| \geq 2$, the *local path connectivity* $\pi_G(S)$ is the maximum number of internally disjoint paths connecting S in G, that is, we search for the maximum cardinality of edge-disjoint paths which contain S and are vertex-disjoint with the exception of the vertices in S. For an integer k with $2 \leq k \leq n$, the *path k-connectivity* is defined as $\pi_k(G) = \min\{\pi_G(S) \mid S \subseteq V(G), |S| = k\}$, that is, $\pi_k(G)$ is the minimum value of $\pi_G(S)$ when S runs over all k-subsets of $V(G)$. Clearly, $\pi_1(G) = \delta(G)$ and $\pi_2(G) = \kappa(G)$. For $k \geq 3$, $\pi_k(G) \leq \kappa_k(G)$ holds because each path is also a tree. Another tree-connectivity parameter, called generalized connectivity, are studied in [4, 25, 26, 28, 31].

As a natural counterpart of path k-connectivity, Mao [30] recently introduced the concept of path k-edge-connectivity. Two paths P and P' connecting S are said to be *edge-disjoint* if $E(P) \cap E(P') = \varnothing$. For $S \subseteq V(G)$ and $|S| \geq 2$, the *local path edge-connectivity* $\omega_G(S)$ is the maximum number of edge-disjoint paths connecting S in G. For an integer k with $2 \leq k \leq n$, the *path k-edge-connectivity* is defined as $\omega_k(G) = \min\{\omega_G(S) \mid S \subseteq V(G), |S| = k\}$, that is, $\omega_k(G)$ is the minimum value of $\omega_G(S)$ when S runs over all k-subsets of $V(G)$. Clearly, we have

$$\begin{cases} \omega_k(G) = \delta(G), & \text{for } k = 1; \\ \omega_k(G) = \lambda(G), & \text{for } k = 2; \\ \omega_k(G) \leq \lambda_k(G), & \text{for } k \geq 3. \end{cases} \quad (1)$$

The path k-(edge-)connectivity and generalized k-(edge-)connectivity can be motivated by their interesting interpretation in practice. For example, suppose that G represents a network. If one considers to connect a pair of vertices of G, then a path is used to connect them. However, if one wants to connect a set S of vertices of G with $|S| \geq 3$, then a tree has to be used to connect them. This kind of tree for connecting a set of vertices is usually called a *Steiner tree*, and popularly used in the physical design of VLSI circuits (see [10, 11, 32]). In this application, a Steiner tree is needed to share an electric signal by a set of terminal nodes. Usually, one wants to consider how tough a network can be, for the connection of a set of vertices. Then, the number of totally independent ways to connect them is a measure for this purpose. The k-path-connectivity and generalized k-connectivity can serve for measuring the capability of a network G to connect any k vertices in G.

Product networks were proposed based upon the idea of using the cross product as a tool for "combining" two known graphs with established properties to obtain a new one that inherits properties from both [7]. Recently, there has been an increasing interest in a class of interconnection networks called Cartesian product networks; see [7, 22]. Lexicographic product is also studied extensively; see [17]. Some applications in networks of the lexicographic product were studied; see [1, 9, 23, 27].

Recently, Li and Mao [27] investigated the sharp upper and lower bounds of $\kappa_3(G \circ H)$, i.e., the lexicographic product of G and H. For generalized 3-edge-connectivity, Sun [28] got a sharp lower bound of $\lambda_3(G \circ H)$. Mao [29] obtained upper and lower bounds of $\omega_3(G \circ H)$. Here we will study upper and lower bounds of $\omega_4(G \circ H)$.

The lexicographic product of two graphs G and H, written as $G \circ H$, is defined as follows: $V(G \circ H) = V(G) \times V(H)$, and two distinct vertices (u, v) and (u', v') of $G \circ H$ are adjacent if and only if either $(u, u') \in E(G)$ or $u = u'$ and $(v, v') \in E(H)$. Note that unlike the Cartesian product, the lexicographic product is a non-commutative product since $G \circ H$ is usually not isomorphic to $H \circ G$.

Observation 1 (1) *Let G be a connected graph. Then $\pi_4(G) \leq \omega_4(G) \leq \delta(G)$.*

(2) *Let G be a connected graph with minimum degree δ. If G has two adjacent vertices of degree δ, then $\omega_k(G) \leq \delta - 1$.*

In this paper, we obtain the following lower bound of $\omega_4(G \circ H)$.

Theorem 2 *Let G and H be two graphs. Then*

$$\omega_4(G \circ H) \geq \omega_4(G) \left\lfloor \frac{3|V(H)|}{5} \right\rfloor.$$

Moreover, the bound is sharp.

The following observation is immediate.

Observation 3 *For any connected graph G, if $\omega_4(G) \geq \ell$, then $\delta(G) \geq \ell$ and there are at most two vertices with degree ℓ.*

Example 1: Set $G = P_n$ and $H = 2K_1$. Clearly, $\omega_4(G) = 1$ and $|V(H)| = 2$. From Theorem 2, we obtain that $\omega_4(P_n \circ 2K_1) \geq 1$. Note that there are at least 4 vertices with minimum degree 2. From Observation 3, we have $\omega_4(P_n \circ 2K_1) \leq 1$. So $\omega_4(P_n \circ 2K_1) = 1$. So the bound in Theorem 2 is sharp.

2 Proof of Theorem 2

In this section, let G and H be two connected graphs with $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(H) = \{v_1, v_2, \ldots, v_m\}$, respectively. Then $V(G \circ H) = \{(u_i, v_j) \mid 1 \leq i \leq n,\ 1 \leq j \leq m\}$. For $v \in V(H)$, we use $G(v)$ to denote the subgraph of $G \circ H$ induced by the vertex set $\{(u_i, v) \mid 1 \leq i \leq n\}$. Similarly, for $u \in V(G)$, we use $H(u)$ to denote the subgraph of $G \circ H$ induced by the vertex set $\{(u, v_j) \mid 1 \leq j \leq m\}$. In the sequel, let K_n and P_n denote the complete graph of order n and path of order n, respectively. If G is a connected graph and $x, y \in V(G)$, then the *distance* $d_G(x, y)$ between x and y is the length of a shortest path connecting x and y in G. The degree of a vertex v in G is denoted by $d_G(v)$.

We now introduce the general idea of the proof of Theorem 2. In Section 2.1, we first study the path 4-edge-connectivity of the lexicographic product of a path P and a graph H and show $\omega_4(P \circ H) \geq \lfloor \frac{3|V(H)|}{5} \rfloor$. After this preparation, we consider the graph $G \circ H$ and prove $\omega_4(G \circ H) \geq \omega_4(G)\lfloor \frac{3|V(H)|}{5} \rfloor$ in Subsection 2.2.

Before realizing the above two steps, we introduce the following two well-known lemmas, which will be used later.

Given a vertex x and a set U of vertices, an *(x, U)-fan* is a set of paths from x to U such that any two of them share only the vertex x. The size of a (x, U)-fan is the number of internally disjoint paths from x to U.

Lemma 1 (Fan Lemma, [33], p-170) *A graph is k-connected if and only if it has at least $k+1$ vertices and, for every choice of x, U with $|U| \geq k$, it has an (x, U)-fan of size k.*

Lemma 2 (Expansion Lemma, [33], p-162) *If G is a k-connected graph, and G' is obtained from G by adding a new vertex y with at least k neighbors in G, then G' is k-connected.*

Let G be a k-connected graph. Choose $U \subseteq V(G)$ with $|U| = k$. Then the graph G' is obtained from G by adding a new vertex y and joining each vertex of U and the vertex y. We call this operation *an expansion operation at y and U*. Denote the resulting graph G' by $G' = G \vee \{y, U\}$.

2.1 Lexicographic product of a path and a connected graph

To start with, we show the following proposition, which is a preparation of the next subsection.

Proposition 1 *Let H be a connected graph and P_n be a path with n vertices. Then $\omega_4(P_n \circ H) \geq \lfloor \frac{3|V(H)|}{5} \rfloor$. Moreover, the bound is sharp.*

Let $V(H) = \{v_1, v_2, \ldots, v_m\}$ and $V(P_n) = \{u_1, u_2, \ldots, u_n\}$. Without loss of generality, let u_i and u_j be adjacent if and only if $|i - j| = 1$, where $1 \leq i \neq j \leq n$. It suffices to show that $\omega_4(P_n \circ H)(S) \geq \lfloor \frac{3m}{5} \rfloor$ for any $S = \{x, y, z, t\} \subseteq V(P_n \circ H)$, that is, there exist $\lfloor \frac{3m}{5} \rfloor$ edge-disjoint S-Steiner paths in $P_n \circ H$. We proceed our proof by the following four lemmas.

Lemma 3 *If x, y, z, t belong to the same $V(H(u_i))$ $(1 \leq i \leq n)$, then there exist $\lfloor \frac{3m}{5} \rfloor$ edge-disjoint S-Steiner paths.*

Proof. Without loss of generality, we assume $x, y, z, t \in V(H(u_1))$. For any five vertices in $H(u_2)$, we say $(u_2, v_{j_1}), (u_2, v_{j_2}), (u_2, v_{j_3}), (u_2, v_{j_4}), (u_2, v_{j_5})$, where $j_i \in \{1, 2, \ldots, m\}$ and $1 \leq i \leq 5$. For any vertex in $H(u_1) - \{x, y, z, t\}$, we say (u_1, v_{i_1}). Then the path induced by the edges in $\{x(u_2, v_{j_1}),$ $(u_2, v_{j_1})(u_1, v_{i_1}), (u_1, v_{i_1})(u_2, v_{j_5}), (u_2, v_{j_5})y, y(u_2, v_{j_3}), (u_2, v_{j_3})t, t(u_2, v_{j_2}),$ $(u_2, v_{j_2})z\}$ and the path induced by the edges in $\{z(u_2, v_{j_4}), (u_2, v_{j_4})x,$

$x(u_2, v_{j_5}), (u_2, v_{j_5})t, t(u_2, v_{j_1}), (u_2, v_{j_1})y\}$and the path induced by the edges in $\{z(u_2, v_{j_3}), (u_2, v_{j_3})x, x(u_2, v_{j_2}), (u_2, v_{j_2})y, y(u_2, v_{j_4}), (u_2, v_{j_4})t\}$ are 3 edge-disjoint S-Steiner paths. For the arbitrariness of the five vertices in $H(u_2)$ and the vertex (u_1, v_{i_1}) in $H(u_1) - \{x, y, z, t\}$, we can obtain $\lfloor \frac{3m}{5} \rfloor$ edge-disjoint S-Steiner paths; see Figure 2.1.

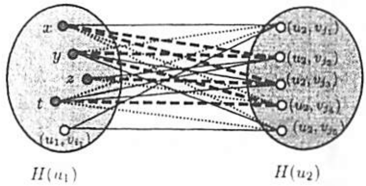

Figure 2.1 The graph for Lemma 3.

Lemma 4 *If three vertices of* $\{x, y, z, t\}$ *belong to some copy* $H(u_i)$ $(1 \leq i \leq n)$, *then there exist* $\lfloor \frac{3m}{5} \rfloor$ *edge-disjoint* S*-Steiner paths.*

Proof. Without loss of generality, we may assume $x, y, z \in V(H(u_1))$ and $t \in V(H(u_i))$ $(2 \leq i \leq n)$. In the following argument, we can see that this assumption has no impact on the correctness of our proof. We distinguish the following two cases to show this lemma.

Case 1. $i = 2$.

Without loss of generality, we assume $t \in V(H(u_2))$. Let x', y', z' be the vertices corresponding to x, y, z in $H(u_2)$ and t' be the vertex corresponding to t in $H(u_1)$.

Suppose $t' \notin \{x, y, z\}$. Without loss of generality, let

$$\{x, y, z, t'\} = \{(u_1, v_j) \,|\, 1 \leq j \leq 4\}$$

and $\{x', y', z', t\} = \{(u_2, v_j) \,|\, 1 \leq j \leq 4\}$. Then the path Q_1 induced by the edges in $\{xt, t(u_1, v_5), (u_1, v_5)x', x'y, yy', y'z\}$, the path Q_2 induced by the edges in $\{zt, tt', t'z', z'x, x(u_2, v_5), (u_2, v_5)y\}$ and the path Q_3 induced by the edges in $\{ty, yz', z'z, zx', x'x\}$ are 3 edge-disjoint S-Steiner paths; see Figure 2.2 (a).

For any five vertices in $H(u_1) - \{x, y, z, t', (u_1, v_5)\}$, we say (u_1, v_{i_1}), $(u_1, v_{i_2}), (u_1, v_{i_3}), (u_1, v_{i_4}), (u_1, v_{i_5})$, where $i_r \in \{6, 7, \ldots, m\}$ and $1 \leq r \leq 5$. For any five vertices in $H(u_2) - \{x', y', z', t, (u_2, v_5)\}$, we say $(u_2, v_{j_1}), (u_2, v_{j_2})$, $(u_2, v_{j_3}), (u_2, v_{j_4}), (u_2, v_{j_5})$, where $j_r \in \{6, 7, \ldots, m\}$ and $1 \leq r \leq 5$. Then we can get the path induced by the edges in $\{x(u_2, v_{j_1}), (u_2, v_{j_1})y, y(u_2, v_{j_3})$, $(u_2, v_{j_3})(u_1, v_{i_2}), (u_1, v_{i_2})t, t(u_1, v_{i_1}), (u_1, v_{i_1})(u_2, v_{j_2}), (u_2, v_{j_2})z\}$ and the path induced by the edges in $\{x(u_2, v_{j_3}), (u_2, v_{j_3})(u_1, v_{i_4}), (u_1, v_{i_4})(u_2, v_{j_5})$, $(u_2, v_{j_5})z, z(u_2, v_{j_1}), (u_2, v_{j_1})(u_1, v_{i_3}), (u_1, v_{i_3})t, t(u_1, v_{i_5}), (u_1, v_{i_5})(u_2, v_{j_4})$,

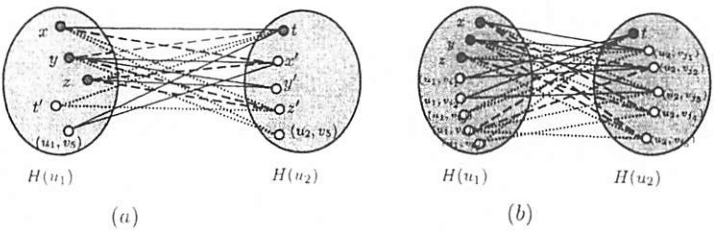

Figure 2.2 The graphs for Lemma 4.

$(u_2, v_{j_4})y\}$ and the the path induced by the edges in $\{t(u_1, v_{i_4}), (u_1, v_{i_4})(u_2, v_{j_2}), (u_2, v_{j_2})y, y(u_2, v_{j_5}), (u_2, v_{j_5})x, x(u_2, v_{j_4}), (u_2, v_{j_4})z\}$ are 3 edge-disjoint S-Steiner paths; see Figure 2.2 (b).

Note that the arbitrariness of the five vertices in $H(u_1)-\{x, y, z, t', (u_1, v_5)\}$ and the five vertices in $H(u_2) - \{x', y', z', t, (u_2, v_5)\}$, we can obtain $\lfloor \frac{3(m-5)}{5} \rfloor$ edge-disjoint S-Steiner paths. These paths together with Q_1, Q_2, Q_3 are $\lfloor \frac{3m}{5} \rfloor$ edge-disjoint S-Steiner paths, as desired.

Suppose $t' \in \{x, y, z\}$. Without loss of generality, let $t' = z$ and $\{x, y, z\} = \{(u_1, v_1), (u_1, v_2), (u_1, v_3)\}$ and $\{x', y', t\} = \{(u_2, v_1), (u_2, v_2), (u_2, v_3)\}$. Then the path Q_1 induced by the edges in $\{xt, ty, yx', x'z\}$, the path Q_2 induced by the edges in $\{xy', y'z, z(u_2, v_4), (u_2, v_4)y, y(u_2, v_5), (u_2, v_5)(u_1, v_4), (u_1, v_4)t\}$ and the path Q_3 induced by the edges in $\{yy', y'(u_1, v_5), (u_1, v_5)t, tz, z(u_2, v_5), (u_2, v_5)x\}$ are 3 edge-disjoint S-Steiner paths; see Figure 2.3 (a).

For any five vertices in $H(u_1)-\{x, y, z, (u_1, v_4), (u_1, v_5)\}$, we say (u_1, v_{i_1}), $(u_1, v_{i_2}), (u_1, v_{i_3}), (u_1, v_{i_4}), (u_1, v_{i_5})$, where $i_r \in \{6, 7, \ldots, m\}$ and $1 \leq r \leq 5$. For any five vertices in $H(u_2) - \{x', y', t, (u_2, v_4), (u_2, v_5)\}$, we say (u_2, v_{j_1}), $(u_2, v_{j_2}), (u_2, v_{j_3}), (u_2, v_{j_4}), (u_1, v_{j_5})$, where $j_r \in \{6, 7, \ldots, m\}$ and $1 \leq r \leq 5$. Similarly to the proof of the above case, we can get 3 edge-disjoint S-Steiner paths; see Figure 2.3(b).

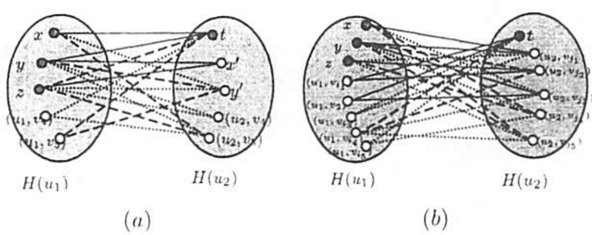

Figure 2.3 The graphs for Lemma 4.

Note that the arbitrariness of the five vertices in $H(u_1) - \{x, y, z, (u_1, v_4), (u_1, v_5)\}$ and five vertices in $H(u_2) - \{x', y', t, (u_2, v_4), (u_2, v_5)\}$, we can obtain $\lfloor \frac{3(m-5)}{5} \rfloor$ edge-disjoint S-Steiner paths. These path together with Q_1, Q_2, Q_3 are $\lfloor \frac{3m}{5} \rfloor$ edge-disjoint S-Steiner paths, as desired.

Case 2. $i \geq 3$.

Let $P' = u_2 u_3 \cdots u_n$. Clearly, $\kappa(P' \circ H) \geq m$. From Lemma 1, there is a t, U-fan in $P' \circ H$, where $U = V(H(u_2)) = \{(u_2, v_j) \,|\, 1 \leq j \leq m\}$. Thus, there exist m internally disjoint paths $P_1, P_2, \cdots, P_m$ such that P_j $(1 \leq j \leq m)$ is a path connecting t and (u_2, v_j). Without loss of generality, let $\{x, y, z\} = \{(u_1, v_j) \,|\, 1 \leq j \leq 3\}$ and any five vertices in $H(u_2)$, we say $(u_2, v_1), (u_2, v_2), (u_2, v_3), (u_2, v_4), (u_2, v_5)$. Then the path Q_1 induced by the edges in $\{x(u_2, v_1), (u_2, v_1)y, y(u_2, v_2), (u_2, v_2)z, z(u_2, v_3)\} \cup E(P_3)$ the path Q_2 induced by the edges in $\{y(u_2, v_4), (u_2, v_4)x, x(u_2, v_2)\} \cup E(P_2) \cup E(P_1) \cup \{(u_2, v_1)z\}$ and the path Q_3 induced by the edges in $\{z(u_2, v_4)\} \cup E(P_4) \cup E(P_5) \cup \{(u_2, v_5)x, x(u_2, v_3), (u_2, v_3)y\}$ are 3 edge-disjoint S-Steiner paths; see Figure 2.4(a).

For any five vertices in $H(u_1) - \{x, y, z, (u_1, v_4), (u_1, v_5)\}$, we say $(u_1, v_{i_1}), (u_1, v_{i_2}), (u_1, v_{i_3}), (u_1, v_{i_4}), (u_1, v_{i_5})$, where $i_r \in \{6, 7, \ldots, m\}$ and $1 \leq r \leq 5$. For any five vertices in $H(u_2) - \{(u_2, v_1), (u_2, v_2), (u_2, v_3), (u_2, v_4), (u_2, v_5)\}$, we say $(u_2, v_{j_1}), (u_2, v_{j_2}), (u_2, v_{j_3}), (u_2, v_{j_4}), (u_1, v_{j_5})$,where $j_r \in \{6, 7, \ldots, m\}$ and $1 \leq r \leq 5$. Similarly to the proof of the above case, we can get 3 edge-disjoint S-Steiner paths. They are the path induced by the edges in $\{x(u_2, v_{j_2}), (u_2, v_{j_2})y, y(u_2, v_{j_1}), (u_2, v_{j_1})z, z(u_2, v_{j_3})\} \cup E(P_{j_3})$ and the path induced by the edges in $\{x(u_2, v_{j_1}), (u_2, v_{j_1})(u_1, v_{i_1}), (u_1, v_{i_1})(u_2, v_{j_4}), (u_2, v_{j_4})y, y(u_2, v_{j_5}), (u_2, v_{j_5})z, z(u_2, v_{j_2})\} \cup E(P_{j_2})$ and the path induced by the edges in $\{z(u_2, v_{j_4})\} \cup E(P_{j_4}) \cup E(P_{j_5})\} \cup \{ (u_2, v_{j_5})x, x(u_2, v_{j_3}), (u_2, v_{j_3})y\}$; see Figure 2.4 (b).

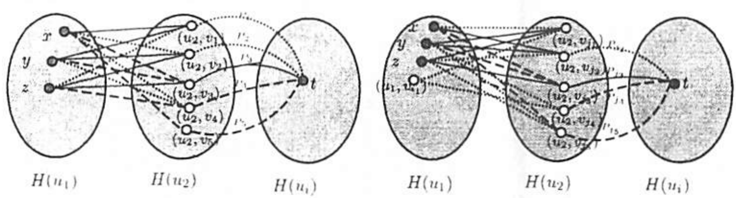

Figure 2.4 The graph for Lemma 4

Note that the arbitrariness of the five vertices in $H(u_1) - \{x, y, z, (u_1, v_4), (u_1, v_5)\}$ and the five vertices in $H(u_2) - \{(u_2, v_1), (u_2, v_2), (u_2, v_3), (u_2, v_4), (u_2, v_5)\}$, we can obtain $\lfloor \frac{3(m-5)}{5} \rfloor$ edge-disjoint S-Steiner paths. These path together with Q_1, Q_2, Q_3 are $\lfloor \frac{3m}{5} \rfloor$ edge-disjoint S-Steiner paths, as desired. ∎

Lemma 5 *If two vertices of* $\{x, y, z, t\}$ *belong to some copy* $H(u_i)$ $(1 \le i \le n)$, *then there exist* $\lfloor \frac{2m}{3} \rfloor$ *edge-disjoint* S*-Steiner paths.*

Proof. We have the following cases to be considered.

Case 1. $x, y \in V(H(u_i))$, $z \in V(H(u_j))$ and $t \in V(H(u_k))$, where $i < j < k$, $1 \le i \le n-2, 2 \le j \le n-1, 3 \le k \le n$.

Without loss of generality, we may assume that $x, y \in V(H(u_1))$ and $z \in V(H(u_j))$ $(2 \le j \le n-1)$.

Subcase 1.1 $z \in V(H(u_2))$ and $t \in V(H(u_k))$, where $3 \le k \le n$.

Consider the case $k \ge 4$. Let $P' = u_3u_4 \cdots u_k$. Clearly, $\kappa(P' \circ H) \ge m$. From Lemma 1, there is a t, U-fan in $P' \circ H$, where $U = V(H(u_3)) = \{(u_3, v_r) \mid 1 \le r \le m\}$. Thus, there exist m internally disjoint paths $P_1, P_2, \cdots, P_m$ such that P_r $(1 \le r \le m)$ is a path connecting t and (u_3, v_j).

Without loss of generality, we may assume that $x = (u_1, v_1), y = (u_1, v_2)$ and $z = (u_2, v_1)$. Then we can get the path Q_1 induced by the edges in $\{xz, zy, y(u_2, v_2), (u_2, v_2)\ (u_3, v_1)\} \cup E(P_1)$ and the path Q_2 induced by the edges in $E(P_2) \cup \{(u_3, v_2)z, z(u_1, v_3), (u_1, v_3)(u_2, v_2), (u_2, v_2)x, x(u_2, v_3), (u_2, v_3)y\}$. For any three vertices in $H(u_1) - \{x, y, (u_1, v_3)\}$, we say $(u_1, v_{i_1}), (u_1, v_{i_2}), (u_1, v_{i_3})$, where $i_r \in \{4, 5, \ldots, m\}$ and $1 \le r \le 3$. For any three vertices in $H(u_2) - \{z, (u_2, v_2), (u_2, v_3)\}$, we say $(u_2, v_{j_1}), (u_2, v_{j_2}), (u_2, v_{j_3})$, where $j_r \in \{4, 5, \ldots, m\}$ and $1 \le r \le 3$. For any three vertices in $H(u_3) - \{(u_3, v_1), (u_3, v_2), (u_3, v_3)\}$, we say $(u_3, v_{k_1}), (u_3, v_{k_2}), (u_3, v_{k_3})$, where $k_r \in \{4, 5, \ldots, m\}$ and $1 \le r \le 3$. Similarly to the proof of the above case, we can get 2 edge-disjoint S-Steiner paths. They are the path induced by the edges in $\{x(u_2, v_{j_1}), (u_2, v_{j_1})y, y(u_2, v_{j_2}), (u_2, v_{j_2})(u_1, v_{i_1}), (u_1, v_{i_1})z, z(u_3, v_{k_1})\} \cup E(P_{k_1})$ and the path induced by the edges in $E(P_{k_2}) \cup \{(u_3, v_{k_2})z, z(u_1, v_{i_2}), (u_1, v_{i_2})(u_2, v_{j_2}), (u_2, v_{j_2})x, x(u_2, v_{j_3}), (u_2, v_{j_3})y\}$.

Note that the arbitrariness of the three vertices in $H(u_1) - \{x, y, (u_1, v_3)\}$, the three vertices in $H(u_2) - \{z, (u_2, v_2), (u_2, v_3)\}$ and the three vertices in $H(u_3) - \{(u_3, v_1), (u_3, v_2), (u_3, v_3)\}$, we can obtain $\lfloor \frac{2(m-3)}{3} \rfloor$ edge-disjoint S-Steiner paths. These path together with Q_1, Q_2 are $\lfloor \frac{2m}{3} \rfloor$ edge-disjoint S-Steiner paths, as desired.

Consider the case $k = 3$. We may assume that $t \in V(H(u_3))$ and $x = (u_1, v_1), y = (u_1, v_2)$ and $z = (u_2, v_1)$ and $t = (u_3, v_1)$. Then we can get the path Q_1 induced by the edges in $\{y(u_2, v_2), (u_2, v_2)x, xz, zt\}$ and the path Q_2 induced by the edges in $\{x(u_2, v_3), (u_2, v_3)y, yz, z(u_3, v_2), (u_3, v_2)(u_2, v_2), (u_2, v_2)t\}$.

For any three vertices in $H(u_1) - \{x, y, (u_1, v_3)\}$, we say $(u_1, v_{i_1}), (u_1, v_{i_2}), (u_1, v_{i_3})$, where $i_r \in \{4, 5, \ldots, m\}$ and $1 \le r \le 3$. For any three vertices in $H(u_2) - \{z, (u_2, v_2), (u_2, v_3\}$, we say $(u_2, v_{j_1}), (u_2, v_{j_2}), (u_2, v_{j_3})$, where $j_r \in \{4, 5, \ldots, m\}$ and $1 \le r \le 3$. For any three vertices in $H(u_3) - \{(u_3, v_1), (u_3, v_2),$

$(u_3, v_3)\}$, we say $(u_3, v_{k_1}), (u_3, v_{k_2}), (u_3, v_{k_3})$, where $k_r \in \{4, 5, \ldots, m\}$ and $1 \le r \le 3$. Similarly to the proof of the above case, we can get 2 edge-disjoint S-Steiner paths. They are the path induced by the edges in $\{t(u_2, v_{j_1}), (u_2, v_{j_1})(u_3, v_{k_1}), (u_3, v_{k_1})z, z(u_1, v_{i_1}), (u_1, v_{i_1})(u_2, v_{j_2}), (u_2, v_{j_2})x, x(u_2, v_{j_3}), (u_2, v_{j_3})y\}$ and the path induced by the edges in $\{t(u_2, v_{j_3}), (u_2, v_{j_3})(u_3, v_{k_2}), (u_3, v_{k_2})z, z(u_1, v_{i_2}), (u_1, v_{i_2})(u_2, v_{j_2}), (u_2, v_{j_2})y, y(u_2, v_{j_1}), (u_2, v_{j_1})x\}$.

Note that the arbitrariness of the three vertices in $H(u_1) - \{x, y, (u_1, v_3)\}$, the three vertices in $H(u_2) - \{z, (u_2, v_2), (u_2, v_3)\}$ and the three vertices in $H(u_3) - \{(u_3, v_1), (u_3, v_2), (u_3, v_3)\}$, we can obtain $\lfloor \frac{2(m-3)}{3} \rfloor$ edge-disjoint S-Steiner paths. These path together with Q_1, Q_2 are $\lfloor \frac{2m}{3} \rfloor$ edge-disjoint S-Steiner paths, as desired.

Subcase 1.2 $z \in V(H(u_j))$ and $t \in V(H(u_k))$, where $3 \le j \le n-1, 4 \le k \le n$.

Consider the case $|j - k| \ge 2$ and $j \ge 4$. Let $P' = u_2 u_4 \cdots u_j$. Clearly, $\kappa(P' \circ H) \ge m$. From Lemma 1, there is a z, U'-fan in $P' \circ H$, where $U' = V(H(u_2)) = \{(u_2, v_r) \mid 1 \le r \le m\}$. Thus there exist m pairwise internally disjoint paths $P'_1, P'_2, \cdots, P'_m$ such that each P'_r $(1 \le r \le m)$ is a path connecting z and (u_2, v_r). Let $P'' = u_{j+1} u_{j+2} \cdots u_k$. Clearly, $\kappa(P'' \circ H) \ge m$. From Lemma 1, there is a t, U''-fan in $P'' \circ H$, where $U'' = V(H(u_{j+1})) = \{(u_{j+1}, v_r) \mid 1 \le r \le m\}$. Thus there exist m pairwise internally disjoint paths $P''_1, P''_2, \cdots, P''_m$ such that each P''_r $(1 \le r \le m)$ is a path connecting t and (u_{j+1}, v_r). Without loss of generality, let $x = (u_1, v_1), y = (u_1, v_2)$ and $z = (u_j, v_1)$. Then we can get the path Q_1 induced by the edges in $\{y(u_2, v_2), (u_2, v_2)x, x(u_2, v_1)\} \cup E(P'_1) \cup \{z(u_{j+1}, v_1)\} \cup E(P''_1)$ and the path Q_2 induced by the edges in $\{x(u_2, v_3), (u_2, v_3)y, y(u_2, v_1), (u_2, v_1)(u_1, v_3), (u_1, v_3)(u_2, v_2)\} \cup E(P'_2) \cup \{z(u_{j+1}, v_2)\} \cup E(P''_2)$.

For any three vertices in $H(u_1) - \{x, y, (u_1, v_3)\}$, we say $(u_1, v_{i_1}), (u_1, v_{i_2}), (u_1, v_{i_3})$, any three vertices in $H(u_2) - \{(u_2, v_1), (u_2, v_2), (u_2, v_3)\}$, we say $(u_2, v_{j_1}), (u_2, v_{j_2}), (u_2, v_{j_3})$, any three vertices in $H(u_j) - \{(u_j, v_1), (u_j, v_2), (u_j, v_3)\}$, we say $(u_j, v_{k_1}), (u_j, v_{k_2}), (u_j, v_{k_3})$, any three vertices in $H(u_{j+1}) - \{(u_{j+1}, v_1), (u_{j+1}, v_2), (u_{j+1}, v_3)\}$, we say $(u_{j+1}, v_{s_1}), (u_{j+1}, v_{s_2}), (u_{j+1}, v_{s_3})$, where $i_r, j_r, k_r, s_r \in \{4, 5, \ldots, m\}$ and $1 \le r \le 3$. Then we can get 2 edge-disjoint S-Steiner paths. They are the path induced by the edges in $\{y(u_2, v_{j_2}), (u_2, v_{j_2})x, x(u_2, v_{j_1})\} \cup E(P'_{j_1}) \cup \{z(u_{j+1}, v_{s_1})\} \cup E(P''_{s_1})$ and the path induced by the edges in $\{x(u_2, v_{j_3}), (u_2, v_{j_3})y, y(u_2, v_{j_1}), (u_2, v_{j_1})(u_1, v_{i_3}), (u_1, v_{i_3})(u_2, v_{j_2})\} \cup E(P'_{j_2}) \cup \{z(u_{j+1}, v_{s_2})\} \cup E(P''_{s_2})$.

Note that the arbitrariness of the three vertices in $H(u_1) - \{x, y, (u_1, v_3)\}$, the three vertices in $H(u_2) - \{z, (u_2, v_2), (u_2, v_3)\}$, the three vertices in $H(u_j) - \{(u_j, v_1), (u_j, v_2), (u_j, v_3)\}$ and the three vertices in $H(u_{j+1}) - \{(u_{j+1}, v_1), (u_{j+1}, v_2), (u_{j+1}, v_3)\}$, we can obtain $\lfloor \frac{2(m-3)}{3} \rfloor$ edge-disjoint S-Steiner paths. These path together with Q_1, Q_2 are $\lfloor \frac{2m}{3} \rfloor$ edge-disjoint S-Steiner paths, as desired.

Consider the case $|j-k| \geq 2$ and $j=3$. Let $P = u_4u_5\cdots u_k$. Clearly, $\kappa(P' \circ H) \geq m$. From Lemma 1, there is a t,U-fan in $P' \circ H$, where $U = V(H(u_4)) = \{(u_4, v_r) \mid 1 \leq r \leq m\}$. Thus there exist m pairwise internally disjoint paths $P_1, P_2, \cdots, P_m$ such that each P_r $(1 \leq r \leq m)$ is a path connecting t and (u_4, v_r). Without loss of generality, let $x = (u_1, v_1), y = (u_1, v_2)$ and $z = (u_3, v_1)$. Then we can get 2 edge-disjoint S-Steiner paths. They are the path Q_1 induced by the edges in $\{x(u_2, v_1), (u_2, v_1)y, y(u_2, v_2), (u_2, v_2)z, z(u_4, v_1)\} \cup E(P_1)$ and the path Q_2 induced by the edges in $\{y(u_2, v_3), (u_2, v_3)x, x(u_2, v_2), (u_2, v_2)(u_3, v_2), (u_3, v_2)(u_2, v_1), (u_2, v_1)z, z(u_4, v_2)\} \cup E(P_2)$.

For any three vertices in $H(u_1) - \{x, y, (u_1, v_3)\}$, we say $(u_1, v_{i_1}), (u_1, v_{i_2}), (u_1, v_{i_3})$, where $i_r \in \{4, 5, \ldots, m\}$ and $1 \leq r \leq 3$, any three vertices in $H(u_2) - \{(u_2, v_1), (u_2, v_2), (u_2, v_3)\}$, we say $(u_2, v_{j_1}), (u_2, v_{j_2}), (u_2, v_{j_3})$, where $j_r \in \{4, 5, \ldots, m\}$ and $1 \leq r \leq 3$, any three vertices in $H(u_3) - \{z, (u_3, v_2), (u_3, v_3\}$, we say $(u_3, v_{k_1}), (u_3, v_{k_2}), (u_3, v_{k_3})$, where $k_r \in \{4, 5, \ldots, m\}$ and $1 \leq r \leq 3$ and any three vertices in $H(u_4) - \{(u_4, v_1), (u_4, v_2), (u_4, v_3)\}$, we say $(u_4, v_{s_1}), (u_4, v_{s_2}), (u_4, v_{s_3})$, where $s_r \in \{4, 5, \ldots, m\}$ and $1 \leq r \leq 3$. Then we can get 2 edge-disjoint S-Steiner paths. They are the path induced by the edges in $\{x(u_2, v_{j_1}), (u_2, v_{j_1})y, y(u_2, v_{j_2}), (u_2, v_{j_2})z, z(u_4, v_{s_1})\} \cup E(P_{s_1})$ and the path induced by the edges in $\{y(u_2, v_{j_3}), (u_2, v_{j_3}x, x(u_2, v_{j_2}), (u_2, v_{j_2})(u_3, v_{k_2}), (u_3, v_{k_2})(u_2, v_{j_1}), (u_2, v_{j_1})z, z(u_4, v_{s_2})\} \cup E(P_{s_2})$.

Note that the arbitrariness of the three vertices in $H(u_1) - \{x, y, (u_1, v_3)\}$, the three vertices in $H(u_2) - \{z, (u_2, v_2), (u_2, v_3)\}$, the three vertices in $H(u_3) - \{(u_3, v_1), (u_3, v_2), (u_3, v_3\}$ and the three vertices in $H(u_4) - \{(u_4, v_1), (u_4, v_2), (u_4, v_3)\}$, we can obtain $\lfloor \frac{2(m-3)}{3} \rfloor$ edge-disjoint S-Steiner paths. These paths together with Q_1, Q_2 are $\lfloor \frac{2m}{3} \rfloor$ edge-disjoint S-Steiner paths, as desired.

Consider the case $|j-k| = 1$ and $j \geq 4$. Let $P = u_2u_3\cdots u_j$. Clearly, $\kappa(P' \circ H) \geq m$. From Lemma 1, there is a z,U-fan in $P' \circ H$, where $U = V(H(u_2)) = \{(u_2, v_r) \mid 1 \leq r \leq m\}$. Thus there exist m pairwise internally disjoint paths $P_1, P_2, \cdots, P_m$ such that each P_r $(1 \leq r \leq m)$ is a path connecting z and (u_2, v_r). Without loss of generality, let $x = (u_1, v_1), y = (u_1, v_2)$ and $z = (u_j, v_1), t = (u_{j+1}, v_1)$. Then we can get 2 edge-disjoint S-Steiner paths, the path Q_1 induced by the edges in $\{x(u_2, v_2), (u_2, v_2)y, y(u_2, v_1)\} \cup E(P_1) \cup \{zt\}$ and the path Q_2 induced by the edges in $\{y(u_2, v_3), (u_2, v_3)x, x(u_2, v_1), (u_2, v_1)(u_1, v_3), (u_1, v_3)(u_2, v_2)\} \cup E(P_2) \cup \{z(u_{j+1}, v_2), (u_{j+1}, v_2)(u_j, v_2), (u_j, v_2)t\}$.

For any three vertices in $H(u_1) - \{x, y, (u_1, v_3)\}$, we say $(u_1, v_{i_1}), (u_1, v_{i_2}), (u_1, v_{i_3})$, where $i_r \in \{4, 5, \ldots, m\}$ and $1 \leq r \leq 3$, any three vertices in $H(u_2) - \{(u_2, v_1), (u_2, v_2), (u_2, v_3)\}$, we say $(u_2, v_{j_1}), (u_2, v_{j_2}), (u_2, v_{j_3})$, where $j_r \in \{4, 5, \ldots, m\}$ and $1 \leq r \leq 3$, any three vertices in $H(u_j) - \{z, (u_j, v_2), (u_j, v_3)\}$, we say $(u_j, v_{k_1}), (u_j, v_{k_2}), (u_3, v_{k_3})$, where $k_r \in \{4, 5, \ldots, m\}$ and $1 \leq r \leq 3$ and any three vertices in $H(u_{j+1}) - \{(u_{j+1}, v_1), (u_{j+1}, v_2), (u_{j+1}, v_3)\}$, we say $(u_{j+1}, v_{s_1}), (u_{j+1}, v_{s_2}), (u_{j+1}, v_{s_3})$, where $s_r \in \{4, 5, \ldots, m\}$ and $1 \leq r \leq 3$, we can get 2 edge-disjoint S-Steiner paths, the path induced by the edges in $\{x(u_2, v_{j_2}), (u_2, v_{j_2})y, y(u_2, v_{j_1})\} \cup E(P_{j_1}) \cup \{z(u_{j+1}, v_{s_1}), (u_{j+1}, v_{s_1})(u_j, v_{k_1}),$

$(u_j, v_{k_1})t\}$ and the path induced by the edges in $\{y(u_2, v_{j_3}), (u_2, v_{j_3})x, x(u_2, v_{j_2}), (u_2, v_{j_2})(u_1, v_{i_3}), (u_1, v_{i_3}), (u_2, v_{j_2})\} \cup E(P_{j_2}) \cup \{z(u_{j+1}, v_{s_3}), (u_{j+1}, v_{s_3})(u_j, v_{k_3}), (u_j, v_{k_3})t\} \cup E(P_{s_2})$.

Note that the arbitrariness of the three vertices in $H(u_1) - \{x, y, (u_1, v_3)\}$, the three vertices in $H(u_2) - \{z, (u_2, v_2), (u_2, v_3)\}$, the three vertices in $H(u_j) - \{(u_j, v_1), (u_j, v_2), (u_j, v_3)\}$ and the three vertices in $H(u_{j+1}) - \{(u_{j+1}, v_1), (u_{j+1}, v_2), (u_{j+1}, v_3)\}$, we can obtain $\lfloor\frac{2(m-3)}{3}\rfloor$ edge-disjoint S-Steiner paths. These paths together with Q_1, Q_2 are $\lfloor\frac{2m}{3}\rfloor$ edge-disjoint S-Steiner paths, as desired.

Consider the case $|j-k| = 1$ and $j = 3$. Without loss of generality, we may assume that $z \in V(H(u_3))$ and $t \in V(H(u_4))$ and $x = (u_1, v_1), y = (u_1, v_2), z = (u_3, v_1), t = (u_4, v_1)$. Then we can get 2 edge-disjoint S-Steiner paths, the path Q_1 induced by the edges in $\{x(u_2, v_1), (u_2, v_1)y, y(u_2, v_2), (u_2, v_2)z, zt\}$ and the path Q_2 induced by the edges in $\{y(u_2, v_3), (u_2, v_3)x, x(u_2, v_2), (u_2, v_2)(u_1, v_3), (u_1, v_3)(u_2, v_3), (u_2, v_3)z, z(u_4, v_2), (u_4, v_2)(u_3, v_2), (u_3, v_2)t\}$.

For any three vertices in $H(u_1) - \{x, y, (u_1, v_3)\}$, we say $(u_1, v_{i_1}), (u_1, v_{i_2}), (u_1, v_{i_3})$, where $i_r \in \{4, 5, \ldots, m\}$ and $1 \leq r \leq 3$, any three vertices in $H(u_2) - \{(u_2, v_1), (u_2, v_2), (u_2, v_3)\}$, we say $(u_2, v_{j_1}), (u_2, v_{j_2}), (u_2, v_{j_3})$, where $j_r \in \{4, 5, \ldots, m\}$ and $1 \leq r \leq 3$, any three vertices in $H(u_3) - \{z, (u_3, v_2), (u_3, v_3)\}$, we say $(u_3, v_{k_1}), (u_3, v_{k_2}), (u_3, v_{k_3})$, where $k_r \in \{4, 5, \ldots, m\}$ and $1 \leq r \leq 3$ and any three vertices in $H(u_4) - \{(u_4, v_1), (u_4, v_2), (u_4, v_3)\}$, we say $(u_4, v_{s_1}), (u_4, v_{s_2}), (u_4, v_{s_3})$, where $s_r \in \{4, 5, \ldots, m\}$ and $1 \leq r \leq 3$. Then we can get 2 edge-disjoint S-Steiner paths, the path induced by the edges in $\{x(u_2, v_{j_1}), (u_2, v_{j_1})y, y(u_2, v_{j_2}), (u_2, v_{j_2})z, z(u_4, v_{s_1}), (u_4, v_{s_1})(u_3, v_{k_1}), (u_3, v_{k_1})t\}$ and the path induced by the edges in $\{y(u_2, v_{j_3}), (u_2, v_{j_3})x, x(u_2, v_{j_2}), (u_2, v_{j_2})(u_1, v_{i_3}), (u_1, v_{i_3})(u_2, v_{j_3}), (u_2, v_{j_3})z, z(u_4, v_{s_2}), (u_4, v_{s_2})(u_3, v_{k_2}), (u_3, v_{k_2})t\}$.

Note that the arbitrariness of the three vertices in $H(u_1) - \{x, y, (u_1, v_3)\}$, the three vertices in $H(u_2) - \{z, (u_2, v_2), (u_2, v_3)\}$, the three vertices in $H(u_3) - \{(u_3, v_1), (u_3, v_2), (u_3, v_3)\}$ and the three vertices $H(u_4) - \{(u_4, v_1), (u_4, v_2), (u_4, v_3)\}$, we can obtain $\lfloor\frac{2(m-3)}{3}\rfloor$ edge-disjoint S-Steiner paths. These paths together with Q_1, Q_2 are $\lfloor\frac{2m}{3}\rfloor$ edge-disjoint S-Steiner paths, as desired.

Case 2. $x, y \in V(H(u_i))$, $z, t \in V(H(u_k))$, where $i < k$, $1 \leq i \leq n-1$, $2 \leq k \leq n$.

Without loss of generality, we may assume that $x, y \in V(H(u_1))$, $z, t \in V(H(u_k))$.

At first, we consider the case $k \geq 5$. Let $P' = u_2u_3\cdots u_j$ and $P'' = u_ju_{j+1}\cdots u_{k-1}$, where $i < j < k$ and $|i-j| \geq 2$ and $|k-j| \geq 2$. Clearly, $\kappa(P' \circ H) \geq m$ and $\kappa(P'' \circ H) \geq m$. From Lemma 1, there is a $(u_j, v_1), U'$-fan in $P' \circ H$ and $(u_j, v_1), U''$-fan in $P'' \circ H$, respectively, where $U' = V(H(u_2)) = \{(u_2, v_r)$ and $U'' = V(H(u_{k-1})) = \{(u_{k-1}, v_r) \mid 1 \leq r \leq m\}$. Thus there exist m pairwise internally disjoint paths $P'_1, P'_2, \cdots, P'_m$ such that each P'_r $(1 \leq r \leq m)$ is a path connecting (u_j, v_1) and (u_2, v_r) and there exist m pairwise internally disjoint paths $P_1'', P_2'', \cdots, P_m''$ such that each P_r'' $(1 \le r \le m)$ is a path connecting (u_j, v_1) and (u_{k-1}, v_r). Without loss of generality, let $x = (u_1, v_1), y = (u_1, v_2)$ and $z = (u_k, v_1), t = (u_k, v_2)$. Then we can get 2 edge-disjoint S-Steiner paths, the path Q_1 induced by the edges in $\{x(u_2, v_2), (u_2, v_2)y, y(u_2, v_1)\} \cup E(P_1') \cup E(P_1'') \cup \{(u_{k-1}, v_1)z, z(u_{k-1}, v_2), (u_{k-1}, v_2)t\}$ and the path Q_2 induced by the edges in $\{y(u_2, v_3), (u_2, v_3)x, x(u_2, v_1), (u_2, v_1)(u_1, v_3), (u_1, v_3)(u_2, v_2)\} \cup E(P_2') \cup E(P_2'') \cup \{(u_{k-1}, v_2)(u_k, v_3), (u_k, v_3)(u_{k-1}, v_1), (u_{k-1}, v_1)t, t(u_{k-1}, v_3), (u_{k-1}, v_3)z\}$.

For any three vertices in $H(u_1) - \{x, y, (u_1, v_3)\}$, we say $(u_1, v_{i_1}), (u_1, v_{i_2}), (u_1, v_{i_3})$, where $i_r \in \{4, 5, \ldots, m\}$ and $1 \le r \le 3$, any three vertices in $H(u_2) - \{(u_2, v_1), (u_2, v_2), (u_2, v_3)\}$, we say $(u_2, v_{j_1}), (u_2, v_{j_2}), (u_2, v_{j_3})$, where $j_r \in \{4, 5, \ldots, m\}$ and $1 \le r \le 3$, any three vertices in $H(u_{k-1}) - \{(u_{k-1}, v_1), (u_{k-1}, v_2), (u_{k-1}, v_3)\}$, we say $(u_{k-1}, v_{a_1}), (u_{k-1}, v_{a_2}), (u_{k-1}, v_{a_3})$, where $a_r \in \{4, 5, \ldots, m\}$ and $1 \le r \le 3$ and any three vertices in $H(u_k) - \{z, t, (u_k, v_3)\}$, we say $(u_k, v_{s_1}), (u_k, v_{s_2}), (u_k, v_{s_3})$, where $s_r \in \{4, 5, \ldots, m\}$ and $1 \le r \le 3$. Then we can get 2 edge-disjoint S-Steiner paths, the path induced by the edges in $\{x(u_2, v_{j_2}), (u_2, v_{j_2})y, y(u_2, v_{j_1})\} \cup E(P_{j_1}') \cup E(P_{(k-1)_1}'') \cup \{(u_{k-1}, v_{a_1})z, z(u_{k-1}, v_{a_2}), (u_{k-1}, v_{a_2})t\}$ and the path induced by the edges in $\{y(u_2, v_{j_3}), (u_2, v_{j_3})x, x(u_2, v_{j_1}), (u_2, v_{j_1})(u_1, v_{i_3}), (u_1, v_{i_3})(u_2, v_{j_2})\} \cup E(P_{j_2}') \cup E(P_{(k-1)_2}'') \cup \{(u_{k-1}, v_{a_2})(u_k, v_{s_3}), (u_k, v_{s_3})(u_{k-1}, v_{a_1}), (u_{k-1}, v_{a_1})t, t(u_{k-1}, v_{a_3}), t(u_{k-1}, v_{a_3})z\}$

Note that the arbitrariness of the three vertices in $H(u_1) - \{x, y, (u_1, v_3)\}$, any three vertices $H(u_2) - \{(u_2, v_1), (u_2, v_2), (u_2, v_3)\}$, $H(u_{k-1}) - \{(u_{k-1}, v_1), (u_{k-1}, v_2), (u_{k-1}, v_3)\}$ and any three vertices in $H(u_k) - \{z, t, (u_k, v_3)\}$, we can obtain $\lfloor \frac{2(m-3)}{3} \rfloor$ edge-disjoint S-Steiner paths. These paths together with Q_1, Q_2 are $\lfloor \frac{2m}{3} \rfloor$ edge-disjoint S-Steiner paths, as desired.

Now, we consider the case $k = 4$. Without loss of generality, we may assume that $x, y \in V(H(u_1))$ and $z, t \in V(H(u_4))$, especially, $x = (u_1, v_1), y = (u_1, v_2)$ and $z = (u_4, v_1), t = (u_4, v_2)$. Then we can get 2 edge-disjoint S-Steiner paths, the path Q_1 induced by the edges in $\{x(u_2, v_1), (u_2, v_1)y, y(u_2, v_2), (u_2, v_2)(u_3, v_1), (u_3, v_1)z, z(u_3, v_2), (u_3, v_2)t\}$ and the path Q_2 induced by the edges in $\{y(u_2, v_3), (u_2, v_3)x, x(u_2, v_2), (u_2, v_2)(u_1, v_3), (u_1, v_3)(u_2, v_1), (u_2, v_1)(u_3, v_1), (u_3, v_1)t, t(u_3, v_3), (u_3, v_3)z\}$.

For any three vertices in $H(u_1) - \{x, y, (u_1, v_3)\}$, we say $(u_1, v_{i_1}), (u_1, v_{i_2}), (u_1, v_{i_3})$, where $i_r \in \{4, 5, \ldots, m\}$ and $1 \le r \le 3$, any three vertices in $H(u_2) - \{(u_2, v_1), (u_2, v_2), (u_2, v_3)\}$, we say $(u_2, v_{j_1}), (u_2, v_{j_2}), (u_2, v_{j_3})$, where $j_r \in \{4, 5, \ldots, m\}$ and $1 \le r \le 3$, any three vertices in $H(u_3) - \{(u_3, v_1), (u_3, v_2), (u_3, v_3)\}$, we say $(u_3, v_{k_1}), (u_3, v_{k_2}), (u_3, v_{k_3})$, where $k_r \in \{4, 5, \ldots, m\}$ and $1 \le r \le 3$ and any three vertices in $H(u_4) - \{z, t, (u_4, v_3)\}$, we say $(u_4, v_{s_1}), (u_4, v_{s_2}), (u_4, v_{s_3})$, where $s_r \in \{4, 5, \ldots, m\}$ and $1 \le r \le 3$. Then we can get 2 edge-disjoint S-Steiner paths, the path induced by the edges in $\{x(u_2, v_{j_1}), (u_2, v_{j_1})y, y(u_2, v_{j_2}), (u_2, v_{j_2})(u_3, v_{k_1}), (u_3, v_{k_1})z, z(u_3, v_{k_2}), (u_3, v_{k_2})t\}$ and the path induced by the edges in $\{y(u_2, v_{j_3}), (u_2, v_{j_3}x, x(u_2, v_{j_2}), (u_2, v_{j_2})(u_1, v_{i_3}), (u_1, v_{i_3})(u_2, v_{j_1}), (u_2, v_{j_1})(u_3, v_{k_1}), (u_3, v_{k_1})t, t(u_3, v_{k_3}), (u_3, v_{k_3})z\}$.

Note that the arbitrariness of the three vertices in $H(u_1) - \{x, y, (u_1, v_3)\}$, the three vertices in $H(u_2) - \{(u_2, v_1), (u_2, v_2), (u_2, v_3)\}$, the three vertices in $H(u_3) - \{(u_3, v_1), (u_3, v_2), (u_3, v_3)\}$ and the three vertices in $H(u_4) - \{z, t, (u_4, v_3)\}$, we can obtain $\lfloor\frac{2(m-3)}{3}\rfloor$ edge-disjoint S-Steiner paths. These paths together with Q_1, Q_2 are $\lfloor\frac{2m}{3}\rfloor$ edge-disjoint S-Steiner paths, as desired.

Now, we consider the case $k = 3$. Without loss of generality, we may assume that $x, y \in V(H(u_1))$ and $z, t \in V(H(u_3))$, especially, let $x = (u_1, v_1), y = (u_1, v_2)$ and $z = (u_3, v_1), t = (u_3, v_2)$. Then we can get 2 edge-disjoint S-Steiner paths, the path Q_1 induced by the edges in $\{x(u_2, v_2), (u_2, v_2)y, y(u_2, v_1), (u_2, v_1)z, z(u_2, v_3), (u_2, v_3)t\}$ and the path Q_2 induced by the edges in $\{z(u_2, v_2), (u_2, v_2)t, t(u_2, v_1), (u_2, v_1)x, x(u_2, v_3), (u_2, v_3)y\}$.

For any three vertices in $H(u_1) - \{x, y, (u_1, v_3)\}$, we say $(u_1, v_{i_1}), (u_1, v_{i_2}), (u_1, v_{i_3})$, where $i_r \in \{4, 5, \ldots, m\}$ and $1 \leq r \leq 3$, any three vertices in $H(u_2) - \{(u_2, v_1), (u_2, v_2), (u_2, v_3)\}$, we say $(u_2, v_{j_1}), (u_2, v_{j_2}), (u_2, v_{j_3})$, where $j_r \in \{4, 5, \ldots, m\}$ and $1 \leq r \leq 3$, any three vertices in $H(u_3) - \{u_3, v_1), (u_3, v_2), (u_3, v_3)\}$, we say $(u_3, v_{k_1}), (u_3, v_{k_2}), (u_3, v_{k_3})$, where $k_r \in \{4, 5, \ldots, m\}$ and $1 \leq r \leq 3$. Then we can get 2 edge-disjoint S-Steiner paths, the path induced by the edges in $\{x(u_2, v_{j_2}), (u_2, v_{j_2})y, y(u_2, v_{j_1}), (u_2, v_{j_1})z, z(u_2, v_{j_3}), (u_2, v_{j_3})t\}$ and the path Q_2 induced by the edges in $\{z(u_2, v_{j_2}), (u_2, v_{j_2})t, t(u_2, v_{j_1}), (u_2, v_{j_1})x, x(u_2, v_{j_3}), (u_2, v_{j_3})y\}$.

Note that the arbitrariness of the three vertices in $H(u_1) - \{x, y, (u_1, v_3)\}$, the three vertices in $H(u_2) - \{(u_2, v_1), (u_2, v_2), (u_2, v_3)\}$ and the three vertices in $H(u_3) - \{z, t, (u_3, v_3\}$, we can obtain $\lfloor\frac{2(m-3)}{3}\rfloor$ edge-disjoint S-Steiner paths. These paths together with Q_1, Q_2 are $\lfloor\frac{2m}{3}\rfloor$ edge-disjoint S-Steiner paths, as desired.

Finally, we consider the case $k = 2$. Without loss of generality, we may assume that $x, y \in V(H(u_1))$ and $z, t \in V(H(u_2))$ and $x = (u_1, v_1), y = (u_1, v_2), z = (u_2, v_1), t = (u_2, v_2)$. Then we can get 2 edge-disjoint S-Steiner paths, the path Q_1 induced by the edges in $\{xz, zy, yt\}$ and the path Q_2 induced by the edges in $\{y(u_2, v_3), (u_2, v_3)x, xt, t(u_1, v_3), (u_1, v_3)z\}$.

For any three vertices in $H(u_1) - \{x, y, (u_1, v_3)\}$, we say $(u_1, v_{i_1}), (u_1, v_{i_2}), (u_1, v_{i_3})$, where $i_r \in \{4, 5, \ldots, m\}$ and $1 \leq r \leq 3$ and any three vertices in $H(u_2) - \{z, t, (u_2, v_3\}$, we say $(u_2, v_{j_1}), (u_2, v_{j_2}), (u_2, v_{j_3})$, where $j_r \in \{4, 5, \ldots, m\}$ and $1 \leq r \leq 3$, we can get 2 edge-disjoint S-Steiner paths, the path induced by the edges in $\{x(u_2, v_{j_1}), (u_2, v_{j_1})y, y(u_2, v_{j_2}), (u_2, v_{j_2})(u_1, v_{i_1}), (u_1, v_{i_1})z, z(u_1, v_{i_2}), (u_1, v_{i_2})t\}$ and the path Q_2 induced by the edges in $\{y(u_2, v_{j_3}), (u_2, v_{j_3})x, x(u_2, v_{j_2}), (u_2, v_{j_2})(u_1, v_{i_2}), (u_1, v_{i_2})(u_2, v_{j_1}), (u_2, v_{j_1})(u_1, v_{i_1}), (u_1, v_{i_1})t, t(u_1, v_{i_3}), (u_1, v_{i_3})z\}$.

Note that the arbitrariness of the three vertices in $H(u_1) - \{x, y, (u_1, v_3)\}$ and the three vertices in $H(u_2) - \{z, t, (u_2, v_3)\}$, we can obtain $\lfloor\frac{2(m-3)}{3}\rfloor$ edge-disjoint S-Steiner paths. These paths together with Q_1, Q_2 are $\lfloor \frac{2m}{3} \rfloor$ edge-disjoint S-Steiner paths, as desired.

Lemma 6 *If x, y, z, t are contained in distinct $H(u_i)$s, then there exist $m-1$ edge-disjoint S-Steiner paths.*

Proof. The following cases will be considered.

Case 1. $d_{P_n \circ H}(x, y) = d_{P_n \circ H}(y, z) = d_{P_n \circ H}(z, t) = 1$.

Without loss of generality, we may assume that $x \in V(H(u_1))$, $y \in V(H(u_2))$, $z \in V(H(u_3))$, $t \in V(H(u_4))$ and $x = (u_1, v_1), y = (u_2, v_1), z = (u_3, v_1), t = (u_4, v_1)$. Then the path P_1 induced by the edges in $\{xy, yz, zt\}$, the paths Q_j induced by the edges in $\{x(u_2, v_{2j}), (u_2, v_{2j})(u_1, v_{2j+1}), (u_1, v_{2j+1})y, y(u_3, v_{2j}), (u_3, v_{2j})(u_2, v_{2j+1}), (u_2, v_{2j+1})z, z(u_4, v_{2j}), (u_4, v_{2j})(u_3, v_{2j+1}), (u_3, v_{2j+1})t\}$ $(1 \leq j \leq \lfloor \frac{m-1}{2} \rfloor)$ the paths Q'_j induced by the edges in $\{x(u_2, v_{2j+1}), (u_2, v_{2j+1})(u_1, v_{2j}), (u_1, v_{2j})y, y(u_3, v_{2j+1}), (u_3, v_{2j+1})(u_2, v_{2j}), (u_2, v_{2j})z, z(u_4, v_{2j+1}), (u_4, v_{2j+1})(u_3, v_{2j}), (u_3, v_{2j}t)\}$ $(1 \leq j \leq \lfloor \frac{m-1}{2} \rfloor)$ are $m-1$ or m edge-disjoint S-Steiner paths.

Case 2. $d_{P_n \circ H}(x, y) = d_{P_n \circ H}(y, z) = 1$ and $d_{P_n \circ H}(z, t) \geq 2$.

Without loss of generality, we may assume that $x \in V(H(u_1))$, $y \in (H(u_2))$, $z \in V(H(u_3)), t \in V(H(u_i))$ $(5 \leq i \leq n)$ and $x = (u_1, v_1), y = (u_2, v_1), z = (u_3, v_1)$. Let $P = u_4 u_5 \cdots u_i$. Clearly, $\kappa(P' \circ H) \geq m$. From Lemma 1, there is a t, U-fan in $P' \circ H$, where $U = V(H(u_4)) = \{(u_4, v_r) \mid 1 \leq r \leq m\}$. Thus there exist m pairwise internally disjoint paths $P_1, P_2, \cdots, P_m$ such that each P_r $(1 \leq r \leq m)$ is a path connecting t and (u_4, v_r). Then the path P'_1 induced by the edges in $\{xy, yz, z(u_4, v_1)\} \cup E(P_1)$, the paths Q_j induced by the edges in $\{x(u_2, v_{2j}), (u_2, v_{2j})(u_1, v_{2j+1}), (u_1, v_{2j+1})y, y(u_3, v_{2j}), (u_3, v_{2j})(u_2, v_{2j+1}), (u_2, v_{2j+1})z, z(u_4, v_{2j})\} \cup E(P_{2j})(1 \leq j \leq \lfloor \frac{m-1}{2} \rfloor)$ the paths Q'_j induced by the edges in $\{x(u_2, v_{2j+1}), (u_2, v_{2j+1})(u_1, v_{2j}), (u_1, v_{2j})y, y(u_3, v_{2j+1}), (u_3, v_{2j+1})(u_2, v_{2j}), (u_2, v_{2j})z, z(u_4, v_{2j+1})\} \cup E(P_{2j+1})(1 \leq j \leq \lfloor \frac{m-1}{2} \rfloor)$ are $m-1$ or m edge-disjoint S-Steiner paths.

The other cases $d_{P_n \circ H}(y, z) = d_{P_n \circ H}(z, t) = 1$ and $d_{P_n \circ H}(x, y) \geq 2$ can be proved with similar arguments.

Case 3. $d_{P_n \circ H}(x, y) = 1$, $d_{P_n \circ H}(y, z) \geq 2$ and $d_{P_n \circ H}(z, t) \geq 2$.

Without loss of generality, We may assume that $x \in V(H(u_1))$, $y \in V(H(u_2))$, $z \in V(H(u_i))$ and $t \in V(H(u_j))$, where $3 < i < j$, $|j - i| \geq 2$, $4 \leq i \leq n-2$, $6 \leq j \leq n$. Let $P' = u_3 u_4 \cdots u_i$. Clearly, $\kappa(P' \circ H) \geq m$. From Lemma 1, there is a z, U-fan in $P' \circ H$, where $U = V(H(u_3)) = \{(u_3, v_r) \mid 1 \leq r \leq m\}$. Thus there exist m pairwise internally disjoint paths $P'_1, P'_2, \cdots, P'_m$ such that each P'_r $(1 \leq r \leq m)$ is a path connecting z and (u_3, v_r). Furthermore, let $P'' = u_{i+1} u_{i+2} \cdots u_j$. Then P'' is the path of order at least 2. Since $\kappa(P'' \circ H) \geq m$, it follows from Lemma 2 that, if we add the vertex z to $P'' \circ H$ and join an edge from z to each (u_{i+1}, v_r) $(1 \leq r \leq m)$, then $\kappa((P'' \circ H) \vee \{z, V(H(u_{i+1}))\}) \geq m$. From Menger' s Theorem, there exist m internally disjoint paths connecting z and t in $(P'' \circ H) \vee \{z, V(H(u_{i+1}))\}$, say $P_1'', P_2'', \cdots, P_m''$. We may assume that $x = (u_1, v_1)$ and $y = (u_2, v_1)$. Then the paths Q_1 induced by the edges in $\{xy, y(u_3, v_1)\} \cup E(P_1') \cup E(P_1'')$ and the paths Q_r induced by the edges in $\{x(u_2, v_r), (u_2, v_r)(u_1, v_r), (u_1, v_r)y, y(u_3, v_r)\} \cup E(P_r') \cup E(P_r'')$ $(2 \leq r \leq m)$ are m edge-disjoint S-Steiner paths, as desired.

The other cases $d_{P_n \circ H}(y, z) = 1$, $d_{P_n \circ H}(x, y) \geq 2$ and $d_{P_n \circ H}(z, t) \geq 2$ or $d_{P_n \circ H}(z, t) = 1$, $d_{P_n \circ H}(x, y) \geq 2$ and $d_{P_n \circ H}(y, z) \geq 2$ can be discussed similarly.

Case 4. $d_{P_n \circ H}(x, y) \geq 2$, $d_{P_n \circ H}(y, z) \geq 2$ and $d_{P_n \circ H}(z, t) \geq 2$.

Without loss of generality,we may assume that $x \in V(H(u_1)), y \in V(H(u_i))$, $z \in V(H(u_j))$ and $t \in V(H(u_k))$, where $i < j < k$, $|j - i| \geq 2$, $|k - j| \geq 2$, $3 \leq i \leq n - 4$, $5 \leq j \leq n - 2$ and $7 \leq k \leq n$. Let $P' = u_2 u_3 \cdots u_i$. Clearly, $\kappa(P' \circ H) \geq m$. From Lemma 1, there is a y, U-fan in $P' \circ H$, where $U = V(H(u_2)) = \{(u_2, v_r) \mid 1 \leq r \leq m\}$. Thus there exist m pairwise internally disjoint paths $P_1', P_2', \cdots, P_m'$ such that each P_r' $(1 \leq r \leq m)$ is a path connecting y and (u_2, v_r). Furthermore, let $P'' = u_i, u_{i+1}, \cdots, u_{j-1}$ and $P''' = u_{j+1}, u_{j+2}, \cdots, u_k$. Then P'' and P''' are two paths with order at least 2. Since $\kappa(P'' \circ H) \geq m$, it follows from Lemma 2, if we add the vertex z to $P'' \circ H$ and join an edge from z to each of (u_{j-1}, v_r) $(1 \leq r \leq m)$, then $\kappa((P'' \circ H) \vee \{z, V(H(u_{j-1}))\}) \geq m$. By the same reason, if we add the vertex z to $P''' \circ H$ and join an edge from z to each of (u_{j+1}, v_r) $(1 \leq r \leq m)$, then $\kappa((P''' \circ H) \vee \{z, V(H(u_{j+1}))\}) \geq m$. From Menger' s Theorem, there exist m internally disjoint paths connecting z and y in $(P'' \circ H) \vee \{y, V(H(u_{j-1}))\}$, and we say $P_1'', P_2'', \cdots, P_m''$. And there exist m internally disjoint paths connecting z and t in $(P''' \circ H) \vee \{z, V(H(u_{j+1}))\}$, and we say $P_1''', P_2''', \cdots, P_m'''$. Note that the union of any path in $\{P_r'' \mid 1 \leq r \leq m\}$ with any path in $\{P_r''' \mid 1 \leq r \leq m\}$ is a S-Steiner path. Then the paths Q_r induced by the edges in $\{x(u_2, v_r)\} \cup E(P_r') \cup E(P_r'') \cup E(P_r''')$ $(1 \leq r \leq m)$ are m edge-disjoint S-Steiner paths, as desired. ∎

From Lemmas 3, 4, 5 and 6, we conclude that, for any $S \subseteq V(P_n \circ H)$, there exist $\lfloor \frac{3m}{5} \rfloor$ edge-disjoint S-Steiner paths, and hence $\omega_{P_n \circ H}(S) \geq \lfloor \frac{3m}{5} \rfloor$. From the arbitrariness of S, we have $\omega_4(P_n \circ H) \geq \lfloor \frac{3m}{4} \rfloor$. The proof of Proposition 1 is complete.

2.2 Lexicographic product of two general graphs

After the above preparations, we are ready to prove Theorem 2 in this subsection.

Proof of Theorem 2: Set $\omega_4(G) = \ell$. Recall that $V(G) = \{u_1, u_2, \ldots, u_n\}$,

$V(H) = \{v_1, v_2, \ldots, v_m\}$. From the definition of $\omega_4(G \circ H)$, we need to prove that $\omega_{G\circ H}(S) \geq \ell\lfloor \frac{3m}{5} \rfloor$ for any $S = \{x, y, z, t\} \subseteq V(G \circ H)$. Furthermore, it suffices to show that there exist $\ell\lfloor \frac{3m}{5} \rfloor$ edge-disjoint S-Steiner paths in $G \circ H$. Clearly, $V(G \circ H) = \bigcup_{i=1}^{n} V(H(u_i))$. Without loss of generality, let $x \in V(H(u_i))$, $y \in V(H(u_j))$, $z \in V(H(u_k))$ and $t \in V(H(u_r))$, where $i \leq j \leq k \leq r$.

Suppose that x, y, z, t belong to the same $V(H(u_i))$ $(1 \leq i \leq n)$. Without loss of generality, let $x, y, z, t \in V(H(u_1))$. Since $\lambda(G) \geq \omega_4(G) = \ell$, it follows that the vertex u_1 has ℓ neighbors in G, say $u_2, u_3, \cdots, u_{\ell+1}$. From Proposition 1, there exist $\lfloor \frac{3m}{5} \rfloor$ edge-disjoint S-Steiner paths in $P_i \circ H$ where $P_i = u_1u_i$ $(2 \leq i \leq \ell + 1)$. So there are $\ell\lfloor \frac{3m}{5} \rfloor$ edge-disjoint S-Steiner paths in $G \circ H$, as desired.

Suppose that three vertices of $\{x, y, z, t\}$ belong to some copy $H(u_i)$ $(1 \leq i \leq n)$. Without loss of generality, let $x, y, z \in H(u_1)$ and $t \in H(u_2)$. Note that $\lambda(G) \geq \omega_4(G) = \ell$. Therefore, there exist ℓ edge-disjoint paths connecting u_1 and u_2 in G, say $P_1, P_2, \cdots, P_\ell$. From Proposition 1, there exist $\lfloor \frac{3m}{5} \rfloor$ edge-disjoint S-Steiner paths in $P_j \circ H$ $((1 \leq j \leq \ell)$ by Proposition 1. Observe that $\bigcup_{j=1}^{\ell} P_j$ is a subgraph of G and $(\bigcup_{j=1}^{\ell} P_j) \circ H$ is a subgraph of $G \circ H$. So the total number of the edge-disjoint S-Steiner paths is $\ell\lfloor \frac{3m}{5} \rfloor$, as desired.

Suppose that two vertices of $\{x, y, z, t\}$ belong to some copy $H(u_i)$ $(1 \leq i \leq n)$.

Case 1. $x, y \in V(H(u_i))$, $z \in V(H(u_j))$ and $t \in V(H(u_k))$, where $i < j < k$, $1 \leq i \leq n-2$, $2 \leq j \leq n-1$, $3 \leq k \leq n$.

Without loss of generality, we may assume that $x, y \in V(H(u_1))$,$z \in V(H(u_2))$ and $t \in V(H(u_3))$.

Since $\omega_4(G) = \ell$, it follows that there exist ℓ edge-disjoint Steiner paths connecting $\{u_1, u_2, u_3\}$ in G, say $P_1, P_2, \cdots, P_\ell$. From Proposition 1, there exist $\lfloor \frac{3m}{5} \rfloor$ edge-disjoint S-Steiner paths in $P_k \circ H$ $(1 \leq k \leq \ell)$ by Proposition 1. Observe that $\bigcup_{k=1}^{\ell} P_k$ is a subgraph of G and $(\bigcup_{k=1}^{\ell} P_k) \circ H$ is a subgraph of $G \circ H$. Therefore, the total number of the edge-disjoint S-Steiner paths is $\ell\lfloor \frac{3m}{5} \rfloor$, as desired.

Case 2. $x, y \in V(H(u_i))$, $z, t \in V(H(u_j))$, where $i < j$, $1 \leq i \leq n-1$, $2 \leq j \leq n$.

The case can be discussed similarly.

Suppose that x, y, z, t are contained in distinct $H(u_i)$s. Without loss of generality, let $x \in H(u_1)$, $y \in H(u_2)$, $z \in H(u_3)$ and $t \in H(u_4)$. Since $\omega_4(G) = \ell$, it follows that there exist ℓ edge-disjoint Steiner paths connecting $\{u_1, u_2, u_3, u_4\}$ in G, say $P_1, P_2, \cdots, P_\ell$. From Proposition 1, there exist $\lfloor \frac{3m}{5} \rfloor$ edge-disjoint S-Steiner paths in $P_k \circ H$ $(1 \leq k \leq \ell)$ by Proposition 1. Observe that $\bigcup_{k=1}^{\ell} P_k$ is a subgraph of G and $(\bigcup_{k=1}^{\ell} P_k) \circ H$ is a subgraph of $G \circ H$. Therefore, the total number of the edge-disjoint S-Steiner paths is $\ell\lfloor \frac{3m}{5} \rfloor$, as desired.

From the above argument, we conclude that, for any $S \subseteq V(G \circ H)$, $\omega_{G\circ H}(S) \geq \omega_{(\bigcup_{i=1}^{\ell} P_i)\circ H}(S) \geq \ell\lfloor \frac{3m}{5} \rfloor$, which implies that $\omega_4(G \circ H) \geq \ell\lfloor \frac{3m}{5} \rfloor = \omega_4(G)\lfloor \frac{3|V(H)|}{5} \rfloor$. The proof is now complete. ∎

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