Edge coloring of 1-planar graphs without intersecting triangles and chordal 5-cycles*

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Abstract

A graph G is 1-planar if it can be embedded in the plane R^2 so that each edge of G is crossed by at most one other edge. In this paper, we show that each 1-planar graph of maximum degree Δ at least 7 with neither intersecting triangles nor chordal 5-cycles admits a proper edge coloring with Δ colors.

Key words and phrases: 1-planar graphs, edge coloring, triangles, chordal 5-cycles

MSC(2010): 05C15,05C35

1 Introduction

In this paper, we only consider finite and simple graphs. Let G be a graph. We use $V(G)$, $E(G)$ and $\Delta(G)$ to denote the vertex set, edge set, and the maximum degree of G, respectively. If G is a plane graph, we use $F(G)$ to denote the face set of G.

A proper edge coloring of a graph is a mapping assigning colors to the edges of the graph so that any two adjacent edges receive different colors.

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The edge chromatic number of a graph G, denoted by $\chi'(G)$, is the least number of colors such that G has an edge coloring. The celebrated Vizing Edge-Coloring Theorem [6] states that every graph satisfies $\Delta \leq \chi' \leq \Delta+1$. For planar graphs, Vizing [7] proved that every planar graph with $\Delta \geq 8$ satisfies $\chi'(G) = \Delta$ and asked whether it holds for $6 \leq \Delta \leq 7$. The case $\Delta = 7$ of this problem was settled by Sanders and Zhao [5], and by Zhang [12] independently. For $\Delta = 6$, it remains open, but it has been settled for some special classes of graphs (see [1, 2, 3, 9, 11]). A graph G is said to be of class one if $\chi'(G) = \Delta(G)$, and to be of class two otherwise. A class two graph G of maximum degree Δ is said to be Δ-critical if $\chi'(G-e) = \Delta$ for every $e \in E(G)$.

A graph is 1-planar if it can be drawn on the plane such that each edge is crossed by at most one other edge. Ringel [4] first introduced the notion of 1-planar graphs while trying to simultaneous coloring adjacent/incident vertices and faces of plane graphs. In [14], Zhang and Wu studied the edge coloring of 1-planar graphs and proved that every 1-planar graph with $\Delta(G) \geq 10$ is of class one. There are also some partial results while the maximum degree is at most 9: a 1-planar graph of maximum degree Δ is of class one if (1) it has no chordal 5-cycles and $\Delta \geq 9$ [16], or (2) it has no adjacent triangles and $\Delta \geq 8$ [17], or (3) it has no triangle and $\Delta \geq 7$ [13]. Recently, Zhang [15] constructed class two 1-planar graphs of maximum degree 6 or 7 with adjacent triangles.

In this paper, we study the edge chromatic number of 1-planar graph of maximum degree 7, and prove that every 1-planar graph of maximum degree 7 with neither intersecting triangles nor chordal 5-cycles is of class one.

Theorem 1. *Let G be a 1-planar graph with neither intersecting triangles nor chordal 5-cycles. If $\Delta(G) \geq 7$, then $\chi'(G) = \Delta(G)$.*

The following theorem will be used in our proof.

Theorem 2. (Vizing's Adjacency Lemma) [8] *Let G be a Δ-critical graph and let v and w be adjacent vertices of G with $d_G(v) = k$. Then, w is adjacent to at least $(\Delta - k + 1)$ vertices of maximum degree if $k < \Delta$, and adjacent to at least two vertices of maximum degree otherwise.*

2 Basic definitions and lemmas

In this section, we introduce some notations and lemmas used in our proof. For any 1-planar graph, we always assume that it has been embedded on a plane such that every edge is crossed by at most one other edge and subject to this has the minimum number of crossings. The *associated plane* *graph* $G^\times$ of a 1-plane graph G is the plane graph obtained from G by modifying all crossings of G into new 4-vertices. A *false vertex* of $G^\times$ corresponds to a crossing of G, while a *true vertex* of $G^\times$ belongs to G. A face in $G^\times$ is *false* if it is incident with at least one false vertex, and is *true* otherwise. A vertex of degree k is simply called a k-vertex. A k^+-vertex (resp. k^-vertex) is a vertex of degree at least (resp. at most) k. k-face, k^+-face and k^--face are defined analogously. In [14, 16, 17], Zhang et al. proved some structure properties on a 1-plane graph G and its associated plane graph $G^\times$.

Lemma 3. [14] *Let G be a 1-planar graph. Then the following hold:*

(1) *For any two false vertices u and v in $G^\times$, $uv \notin E(G^\times)$.*

(2) *No false 3-face may be incident with 2-vertex.*

(3) *If $d_G(u) = 3$ and v is a false vertex in $G^\times$, then either $uv \notin E(G^\times)$ or uv is not incident with two 3-faces.*

(4) *If a 3-vertex v in G is incident with two 3-faces and adjacent to two false vertices in $G^\times$, then v must also be incident with a 5^+-face.*

Let G be a 1-planar graph. We say that two cycles of G are *adjacent* (resp. *intersecting*) if they have an edge (resp. a vertex) in common. It is proved in [17] that every 5^+-vertex v of G is incident with at most $\lfloor \frac{4}{5} d_G(v) \rfloor$ 3-faces in $G^\times$ if G has no adjacent triangles. If $d_G(u) \in \{3, 4\}$ and u is incident with $d_G(u)$ 3-faces in $G^\times$, then one can easily find adjacent triangles or multiedges in G. So, we restate the conclusion in the following lemma.

Lemma 4. [17] *Let G be a 1-planar graph without adjacent triangles. Then, every 3^+-vertex v is incident with at most $\lfloor \frac{4}{5} d_G(v) \rfloor$ 3-faces in $G^\times$.*

The following conclusion is implicitly contained in the proof of Lemma 4 in [17]. For completeness, we list it as a lemma and present its proof here.

Lemma 5. [17] *Let G be a 1-planar graph without adjacent triangles. Then every 5^+-vertex is incident with at most four consecutive 3-faces in $G^\times$.*

Proof. Let v be a 5^+-vertex of G. We only prove the case that $d_G(v) \geq 6$ and leave the case $d_G(v) = 5$ to interested readers. Suppose that $d_G(v) \geq 6$, and that v is incident with five consecutive 3-faces, say $vv_iv_{i+1}v$ in $G^\times$, where $1 \leq i \leq 5$. Suppose v_1 is true. If v_2 is true, then v_3 is false, as otherwise two adjacent triangles occur in G. It follows that v_4 is true, and then two adjacent triangles vv_1v_2v and vv_2v_4v exist. Thus v_2 must be false and v_3 is true. By the same argument, v_4 must be false and v_5 is true.

Again two adjacent triangles vv_1v_3v and vv_3v_5v exist in G. So v_1 must be false. Thus v_2 is true. Similarly, we get that v_4 and v_6 are true, and v_3 and v_5 are false. Then vv_2v_4v and vv_4v_6v are adjacent triangles in G. This contradiction completes the proof. □

Let $T = uvwu$ be a triangle with $d_G(v) \geq 4$, and let vx be an edge crossing uw in G. We call the configuration consisting of the triangle and the edge vx as an *umbrella*, and call v the *head* of this umbrella. We show that any three consecutive false 3-faces incident with a same vertex will produce an umbrella.

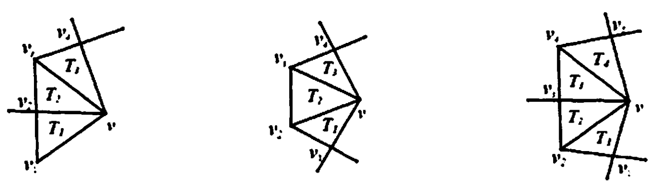
Lemma 6. *Let G be a 1-planar graph without intersecting triangles, and let C be a configuration of $G^\times$ consisting of a vertex v incident with l consecutive 3-faces, say $T_1, T_2 \dots, T_l$.*

(1) *If $l = 3$, then either T_1, T_2 and T_3 are all false and C has an umbrella with v being its head, or T_2 is true.*

(2) *If $l = 4$, then each T_i is false, and T_2 and T_3 form an umbrella with v being its head.*

Proof. Let $vv_iv_{i+1}v$, for $i \in \{1, 2, \dots, l\}$, be the boundary of T_i.

First suppose that $l = 3$ and T_1, T_2 and T_3 are all false. If v_1 is true, then v_2 must be false because of the false 3-face vv_1v_2v. It follows that v_3 is true, otherwise contradicting Lemma 3(1), and so we obtain an umbrella of G, where the triangle is vv_1v_3v and the head is v (see Figure 1(a)). So, suppose that v_1 is false. Then v_2 is trivially true. By the same argument, v_3 must be false and v_4 shall be true. Another umbrella occurs with triangle vv_2v_4v and head v.



(a) (b) (c)

Figure 1: Some consecutive 3-faces incident with a vertex

Now, suppose $l = 3$ and one of T_1, T_2 and T_3 is true. If T_1 is true, then v_1 and v_2 are true. Thus v_3 is false and v_4 is true, two adjacent triangles vv_1v_2v and vv_2v_4v occurs in G, which is a contradiction. Thus T_1 is false and similarly T_3 is also false. Therefore, T_2 must be true (see Figure 1(b)).

Finally suppose that $l = 4$. Since G has no intersecting triangles, T_1 (resp. T_2) cannot be true by considering the three consecutive 3-faces T_1, T_2 and T_3 (resp. T_2, T_3 and T_4) as above. By symmetry, neither T_3 nor T_4 can be true. It follows directly from (1) of the current lemma that $v_2v_4 \in E(G)$ and vv_2v_4v and vv_3 form an umbrella as desired (see Figure 1(c)). □

A 3^+-vertex v of G is said to be *saturated* if v is incident with $\lfloor \frac{4}{5} d_G(v) \rfloor$ 3-faces in $G^\times$, and *unsaturated* otherwise.

Lemma 7. *Let G be a 1-planar graph of maximum degree 7 with neither intersecting triangles nor chordal 5-cycles, and let v be a saturated 7-vertex of $G^\times$. Then, v is incident with a triangle of G which either is a 3-face in $G^\times$ or forms an umbrella, and incident with two 5^+-faces in $G^\times$ of which each is incident with at least one false vertex.*

Proof. By Lemmas 4 and 5, v is incident with five 3-faces in $G^\times$ and the two 4^+-faces incident with v cannot be adjacent. So, the first statement follows immediately from Lemma 6. To prove the latter, we need only considering two possible configurations about the five 3-faces around v: one consists of four consecutive 3-faces and a single 3-face, and the other consists of three consecutive 3-faces and two adjacent 3-faces (see Figure 2).

Let f_1,f_2 be the two 4^+-faces incident with v in $G^\times$. Now we label the neighbors of v in G as v_i for $1 \le i \le 7$ in clockwise order. In the case that the triangle of G incident with v forms an umbrella, we always suppose that v_1v_6 is an edge crossed by vv_7.

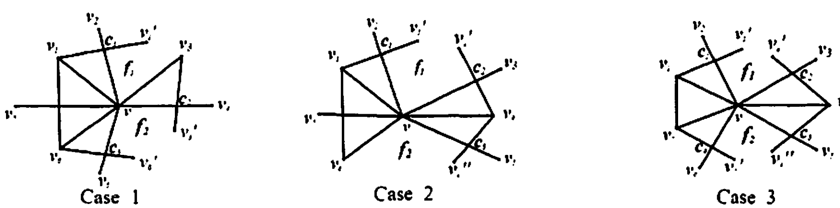


Figure 2: Three cases of saturated 7-vertices in $G^\times$

Case 1: First suppose that v is incident with four consecutive 3-faces. Then, we have a configuration as shown in Case 1 of Figure 2 by Lemma 6(2). We denote by v_1v_1' the edge crossed by edge vv_2, by v_3v_3' the edge crossed by edge vv_4, and by v_6v_6' the edge crossed by edge vv_5. Let the crossings be c_1, c_2, c_3 respectively. If f_1 is a 4-face of $G^\times$, then $v_1'v_3 \in E(G^\times)$, and the 4-cycle $vv_1v_1'v_3v$ and the triangle vv_1v_6v in G form a chordal 5-cycle, a contradiction. If f_2 is a 4-face of $G^\times$, then $v_3' = v_6'$, and the 4-cycle

$vv_3v_3'v_6v$ and the triangle vv_1v_6v in G form a chordal 5-cycle, also a contradiction. Therefore both f_1 and f_2 are 5^+-faces incident with at least one false vertex in $G^\times$.

Case 2: Now we consider the case that v is incident with exact three consecutive false 3-faces. Then, we have a configuration as shown in Case 2 of Figure 2 by Lemma 6(1). We denote by v_1v_1' the edge crossed by edge vv_2, by v_4v_4' the edge crossed by edge vv_3, and by v_4v_4'' the edge crossed by edge vv_5. Let the crossings be c_1, c_2, c_3 respectively. If f_1 is a 4-face of $G^\times$, then $v_1' = v_4'$, the 4-cycle $vv_1v_1'v_4v$ and the triangle vv_1v_6v form a chordal 5-cycle in G. If f_2 is a 4-face of $G^\times$, then $v_4''v_6 \in E(G^\times)$, and the 4-cycle $vv_4v_4'v_6v$ and the triangle vv_1v_6v form a chordal 5-cycle in G. So, both f_1 and f_2 are 5^+-faces incident with at least one false vertex in $G^\times$.

Case 3: Finally, we suppose that v is incident with a true 3-face. Then, we have a configuration as shown in Case 3 of Figure 2. Suppose that vv_1v_7v is the true 3-face incident with v. Let v_1v_1' be the edge crossed by vv_2, v_4v_4' the edge crossed by vv_3, v_4v_4'' the edge crossed by vv_5, and v_7v_7' the edge crossed by vv_6. Let the crossings be c_1, c_2, c_3, c_4 respectively. If f_1 is 4-face of $G^\times$, then $v_1' = v_4'$, the 4-cycle $vv_1v_1'v_4v$ and the triangle vv_1v_7v form a chordal 5-cycle in G. If f_2 is 4-face of $G^\times$, then $v_4'' = v_7'$, the 4-cycle $vv_4v_7'v_7v$ and the triangle vv_1v_7v form a chordal 5-cycle in G. It follows that both f_1 and f_2 are 5^+-faces incident with two false vertices in $G^\times$. □

3 Proof of Theorem 1

Now, we are ready to prove our theorem.

If $\Delta \geq 8$, the conclusion is a consequence of the theorem of [17] asserting that every 1-planar graph H with no adjacent triangles is of class one if $\Delta(H) \geq 8$. So, we need only to prove the case that $\Delta = 7$.

Without loss of generality, we may suppose that G is a Δ-critical 1-planar graph. It follows that $\delta \geq 2$ by Theorem 2.

To derive a contradiction, we use a discharging method. Note that $G^\times$ is a plane graph. The Euler's formula induces that

$$\sum_{v\in V(G)} (d_G(v) - 4) + \sum_{f\in F(G^\times)} (d_{G^\times}(f) - 4)$$
$$= \sum_{v\in V(G^\times)} (d_{G^\times}(v) - 4) + \sum_{f\in F(G^\times)} (d_{G^\times}(f) - 4) = -8$$

We define the initial weight function $\omega(v) = d_G(v) - 4$ for each $v \in V(G)$ and $\omega(f) = d_{G^\times}(f) - 4$ for each $f \in F(G^\times)$. The total sum of weights is equal to -8. We will transfer the weights between elements of $V(G) \cup F(G^\times)$, and then deduce a contradiction by showing that $\omega'(x) \geq 0$ for each $x \in V(G) \cup F(G^\times)$, where $\omega'(x)$ is the resulting weight of x. For convenience, we simply call a k-vertex adjacent to a vertex v as a *k-neighbor* of v. The discharging rules are defined as follows:

R1. A false 3-face receives $\frac{1}{2}$ from each of its incident true vertices. A true 3-face receives $\frac{1}{2}$ from each of its incident 5^+-vertices.

R2. Let f be a 5^+-face of $G^\times$ and let T be the set of true vertices incident with f except unsaturated 7-vertices. Then each element of T receives $\frac{d_{G^\times}(f)-4}{|T|}$ from f.

R3. For $2 \leq k \leq 3$, a k-vertex of G receives $\frac{1}{k-1}$ from each of its neighbors in G.

R4. For $4 \leq k \leq 6$, a k-vertex of G receives $\frac{1}{k-1}$ from each of its 7-neighbors in G.

R5. A 4-vertex of G receives $\frac{1}{4}$ from each of its 6-neighbors in G, and receives $\frac{1}{6}$ from each of its 5-neighbors in G.

It remains to show that $\omega'(x) \geq 0$ for all $x \in V(G) \cup F(G^\times)$.

Let f be a k-face in $F(G^\times)$, where $k \geq 3$.

If $k = 3$, then $\omega(f) = 3 - 4 = -1$. If f is true, then f is incident with at least two 5^+-vertices by Theorem 2. If f is false, then f is incident with two true vertices by Lemma 3(1). So, we have $\omega'(f) \geq -1 + \frac{1}{2} \times 2 = 0$ by R1.

If $k = 4$, then $\omega'(f) = \omega(f) = 0$, since 4-faces don't participate in whole rules.

If $k \geq 5$, then $\omega'(f) = d_{G^\times}(f) - 4 - |T| \times \frac{d_{G^\times}(f)-4}{|T|} = 0$ by R2.

Next, we estimate the weights of vertices. Let v be a d-vertex of G, where $d \geq 2$.

If $d = 2$, then $\omega(v) = 2 - 4 = -2$. By Lemma 3(2), v is incident with no false 3-face in $G^\times$, and hence send out nothing by R1. Therefore, $\omega'(v) = -2 + 1 \times 2 = 0$ by R3.

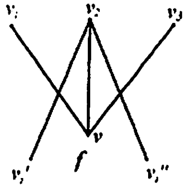


Figure 3: Saturated 3-vertex

Suppose that $d = 3$. Then, $\omega(v) = 3 - 4 = -1$. If v is unsaturated or saturated but incident with a unique false 3-face, then v receives $3 \times \frac{1}{2}$ from its neighbor by R3, and sends out at most $\frac{1}{2}$ by R1, and thus $\omega'(v) \geq -1 - \frac{1}{2} + 3 \times \frac{1}{2} = 0$.

So we suppose that v is saturated and incident with two false 3-faces. Then, v cannot be the head of an umbrella by Lemma 3(3), and is adjacent to two false vertices, and so is incident with a 5^+-face, say f, in $G^\times$ by Lemma 3(4). Let v_i be the neighbor of v in G for $1 \leq i \leq 3$ in a clockwise order, let v_2v_2' be the edge of G crossed by edge vv_1, v_2v_2'' the edge of G crossed by edge vv_3 (see Figure 3). By Theorem 2, $d_G(v_2) \geq 6$, and at least one of v_2' and v_2'' is a 7-vertex. Without loss of generality, we suppose that $d_G(v_2') = 7$. If v_2' is unsaturated, then v receives at least $\frac{1}{2}$ from f by R2. We consider that v_2' is saturated. If f is a 5-face, then $v_2'v_2'' \in E(G)$ and $v_2v_2'v_2''v_2$ is the unique the triangle incident with v_2' that contradicts Lemma 7. So, $d_G(f) \geq 6$ and v also receives at least $\frac{1}{2}$ from f by R2. In both cases, $\omega'(v) \geq -1 - 2 \times \frac{1}{2} + 3 \times \frac{1}{2} + \frac{1}{2} = 0$.

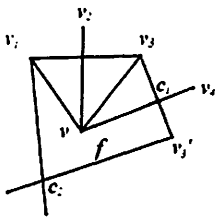


Figure 4: Saturated 4-vertex

Now, let $d = 4$. Then $\omega(v) = 4 - 4 = 0$. Let v_1, v_2, v_3 and $v_4$be the neighbors of v in G in clockwise order.

First we consider the situation that v is saturated and incident with three false 3-faces. By Lemma 6(1), we may suppose that $v_1v_3 \in E(G)$ which together with vv_2 forms an umbrella in G, let v_3v_3' be the edge of G crossed by edge vv_4 with a crossing c_1, and let f be the 4^+-face incident with v in $G^\times$ (see Figure 4). Then, v_3' must be true by Lemma 3(1), and $d_G(f) \geq 5$ as otherwise $v_1v_3' \in E(G)$ forming a triangle adjacent to vv_1v_3v.

If f is a 6^+-face, then v receives at least $\frac{6-4}{6-1} = \frac{2}{5}$ from f by R2. If f is a 5-face, let the boundary of f be $v_1vc_1v_3'c_2$ (see Figure 4), then c_2 must be false (otherwise $vv_3v_3'c_2v_1v$ forms a chordal 5-cycle of G), and so v receives at least $\frac{5-4}{5-2} = \frac{1}{3}$ from f by R2. In either case, v receives at least $\frac{1}{3}$ from f. By R1, v transfers $3 \times \frac{1}{2}$ to these false 3-faces.

By Theorem 2, we may suppose, without loss of generality, that $5 \leq d_G(v_1) \leq d_G(v_2) \leq d_G(v_3) \leq d_G(v_4)$, and $d_G(v_1) = 5$ implying that $d_G(v_2) = d_G(v_3) = d_G(v_4) = 7$, and $d_G(v_1) \geq 6$ implying that $d_G(v_2) \geq 6$ and $d_G(v_3) = d_G(v_4) = 7$. By R4 and R5, $\omega'(v) \geq -\frac{3}{2} + 3 \times \frac{1}{3} + \frac{1}{6} + \frac{1}{3} = 0$ if $d_G(v_1) = 5$, and $\omega'(v) \geq -\frac{3}{2} + 2 \times \frac{1}{3} + 2 \times \frac{1}{4} + \frac{1}{3} = 0$ if $d_G(v_1) \geq 6$.

So, suppose that v is unsaturated or saturated but incident with a true 3-face. Then v transfers at most $2 \times \frac{1}{2} = 1$ to the false 3-faces incident with v by R1. With the same argument as above, either $d_G(v_1) = 5$ implying $\omega'(v) \geq -1 + 3 \times \frac{1}{3} + \frac{1}{6} > 0$, or $d_G(v_1) \geq 6$ implying $\omega'(v) \geq -1 + 2 \times \frac{1}{3} + 2 \times \frac{1}{4} > 0$ (both by R4 and R5).

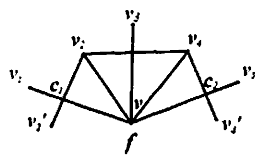


Figure 5: Saturated 5-vertex

Next, suppose that $d = 5$. Then, $\omega(v) = 5 - 4 = 1$. Let v_i be the neighbor of v in G for $1 \leq i \leq 5$ in a clockwise order. If v is saturated, then v is incident with four 3-faces. By Lemma 6(2), each of these 3-faces is false. We denote by v_2v_2' the edge of G crossed by edge vv_1, and denote by v_4v_4' the edge of G crossed by edge vv_5. Let the crossings be c_1 and c_2 respectively. Let f be the 4^+-face incident with v (see Figure 5). If $d_{G^\times}(f) = 4$ or 5, then either $v_2' = v_4'$ producing two adjacent triangles $v_2v_4vv_2$ and $v_2v_4v_2'v_2$, or v_2' is adjacent to v_4' in G producing a chordal 5-cycle on $v_2v_4vv_2$ and $v_2v_4v_4'v_2'v_2$. Therefore, $d_{G^\times}(f) \geq 6$, and furthermore, f is incident with two false vertices c_1 and c_2. Thus, v receives at least $\frac{6-4}{6-2} = \frac{1}{2}$ from f by R2, and transfers $4 \times \frac{1}{2} = 2$ to these false 3-faces by R1. By Theorem 2, v has either a 4-neighbor and four 7-neighbors implying that $\omega'(v) \geq 1 - 2 - \frac{1}{6} + 4 \times \frac{1}{4} + \frac{1}{2} = \frac{1}{3} > 0$ by R4 and R5, or two 5-neighbors and three 7-neighbors implying that $\omega'(v) \geq 1 - 2 + 2 \times \frac{1}{4} + \frac{1}{2} = 0$ by R4 and R5.

If v is unsaturated, then v is incident with at most three false 3-faces, and thus transfers $3 \times \frac{1}{2} = \frac{3}{2}$ to these false 3-faces by R1. With the same argument as above, v has either a 4-neighbor and four 7-neighbors implying that $\omega'(v) \geq 1 - \frac{3}{2} - \frac{1}{6} + 4 \times \frac{1}{4} = \frac{1}{3} > 0$, or two 5-neighbors and three 7-neighbors implying that$\omega'(v) \geq 1 - \frac{3}{2} + 2 \times \frac{1}{4} = 0$.

If $d = 6$, then $\omega(v) = 6 - 4 = 2$. By Lemma 4, v is incident with at most four false 3-faces, and so transfers at most $4 \times \frac{1}{2} = 2$ to these faces by R1. By Theorem 2, v has no neighbor of degree 2. If v has a 3-neighbor then it has five 7-neighbors, and so $\omega'(v) \geq 2 - 2 - \frac{1}{2} + 5 \times \frac{1}{5} = \frac{1}{2} > 0$ by R3 and R4. If v has a 4-neighbor then it has at least four 7-neighbors, and so $\omega'(v) \geq 2 - 2 - 2 \times \frac{1}{4} + 4 \times \frac{1}{5} = \frac{3}{10} > 0$ by R4 and R5. Otherwise, we suppose that v has no neighbors of degree less than 5. Then, v has at least two 7-neighbors by Theorem 2, and so $\omega'(v) \geq 2 - 2 + 2 \times \frac{1}{5} > 0$.

Finally, we deal with the case that $d = 7$. Now, $\omega(v) = 7 - 4 = 3$. If v is saturated then v is incident with two 5^+-faces by Lemma 7 from which v receives totally at least $2 \times \frac{5-4}{5-1} = \frac{1}{2}$ by R2. If v is unsaturated, then v is incident with at most four false 3-faces. In either case, the weight of v is at least 1 after rules R1 and R2. Let v_i be the neighbor of v in G for $1 \leq i \leq 7$, and suppose that $d_G(v_i) \leq d_G(v_j)$ while $i \leq j$. If $d_G(v_1) = 2$, then $d_G(v_i) = 7$ for each $i \geq 2$ by Theorem 2, and $\omega'(v) \geq 1 - 1 = 0$ by R3. If $d_G(v_1) = 3$ then $d_G(v_2) \geq 3$, $d_G(v_i) = 7$ for $i \geq 3$, and so $\omega'(v) \geq 1 - 2 \times \frac{1}{2} = 0$ by R3. If $d_G(v_1) = 4$ then $d_G(v_i) = 7$ for $i \geq 4$, and thus $\omega'(v) \geq 1 - 3 \times \frac{1}{3} = 0$ by R4. If $d_G(v_1) = 5$ then $d_G(v_5) = d_G(v_6) = d_G(v_7) = 7$, and $\omega'(v) \geq 1 - 4 \times \frac{1}{4} = 0$ by R4. If $d_G(v_1) = 6$ then $d_G(v_6) = d_G(v_7) = 7$, and so $\omega'(v) \geq 1 - 5 \times \frac{1}{5} = 0$ by R4. If $d_G(v_1) = 7$, $\omega'(v) \geq 1$.

Now, we have proved that $\omega'(x) \geq 0$ for each $x \in V(G) \cup F(G^\times)$. This completes the proof of Theorem 1.

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