The Combinatorial Representation of Jacobsthal and Jacobsthal Lucas Matrix Sequences

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Abstract

In this study, by using *Jacobsthal and Jacobsthal Lucas matrix sequences we define k-Jacobsthal, k*-Jacobsthal Lucas *matrix* sequences depending on one parameter k. After that by using two parameters (s,t), we define (s,t) *Jacobsthal and* (s,t)*-Jacobsthal Lucas matrix sequences.* And then, we establish combinatoric representations of all of these matrices.

Keywords: Jacobsthal numbers, Jacobsthal Lucas numbers, matrix sequences, generalized sequences.

AMS Classifications: 11B39, 11B83, 15A24, 15B36

1 Introduction and Preliminaries

There are many articles in the literature that study on the different number sequences. There are a lot of identities of number sequences described in all our references. From these sequences, Jacobsthal and Jacobsthal Lucas numbers are given by the recurrence relations $j_n = j_{n-1} + 2j_{n-2}$, $j_0 = 0$, $j_1 = 1$ and $c_n = c_{n-1} + 2c_{n-2}$, $c_0 = 2$, $c_1 = 1$ for $n \geq 2$, respectively. We can generalize the sequences depending on one parameter. For any positive real numbers k; the k-Jacobsthal $\{\hat{\jmath}_{k,n}\}_{n\in\mathbb{N}}$ and the k-Jacobsthal Lucas $\{\hat{c}_{k,n}\}_{n\in\mathbb{N}}$ number sequences are defined in [7] recurrently by

$$\begin{aligned} \hat{\jmath}_{k,n} &= k\hat{\jmath}_{k,n-1} + 2\hat{\jmath}_{k,n-2}, \ \ \hat{\jmath}_{k,0} = 0, \ \hat{\jmath}_{k,1} = 1, \quad n \geq 2, \\ \hat{c}_{k,n} &= k\hat{c}_{k,n-1} + 2\hat{c}_{k,n-2}, \quad \hat{c}_{k,0} = 2, \ \hat{c}_{k,1} = k, \quad n \geq 2. \end{aligned} \tag{1}$$

If we generalize the sequences depending on two parameters we obtain *the* (s,t)*-Jacobsthal and* (s,t)*-Jacobsthal Lucas sequences* are defined recurrently by

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$$\begin{aligned} \hat{j}_n(s,t) &= s\hat{j}_{n-1}(s,t) + 2t\hat{j}_{n-2}(s,t), \quad \hat{j}_0(s,t) = 0, \hat{j}_1(s,t) = 1 \qquad (2) \\ \hat{c}_n(s,t) &= s\hat{c}_{n-1}(s,t) + 2t\hat{c}_{n-2}(s,t), \quad \hat{c}_0(s,t) = 2,\ \hat{c}_1(s,t) = s \end{aligned}$$

where $s > 0,\ t \neq 0\ ,\ s^2 + 8t > 0,\ n \geq 1$ any integer [6].

1.1 The Jacobsthal and Jacobsthal Lucas Matrix Sequences

Jacobsthal $\{J_n\}_{n\in\mathbb{N}}$ and Jacobsthal Lucas $\{C_n\}_{n\in\mathbb{N}}$ matrix sequences are defined as given by the recurrence relations

$$J_{n+1} = J_n + 2J_{n-1}, \quad J_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J_1 = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \tag{3}$$

and

$$C_{n+1} = C_n + 2C_{n-1}, \quad C_0 = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}, \ C_1 = \begin{pmatrix} 5 & 2 \\ 1 & 4 \end{pmatrix}, \tag{4}$$

respectively in [3].

The relation between Jacobsthal and Jacobsthal Lucas number and matrix sequences is given as in [4]

$$J_n = \begin{pmatrix} \hat{j}_{n+1} & 2\hat{j}_n \\ \hat{j}_n & 2\hat{j}_{n-1} \end{pmatrix}, \quad C_n = \begin{pmatrix} \hat{c}_{n+1} & 2\hat{c}_n \\ \hat{c}_n & 2\hat{c}_{n-1} \end{pmatrix}.$$

$k-$Jacobsthal $\{J_{k,n}\}_{n\in\mathbb{N}}$ and $k-$Jacobsthal Lucas $\{C_{k,n}\}_{n\in\mathbb{N}}$ matrix sequences are defined as given by the recurrence relations

$$J_{k,n+1} = kJ_{k,n} + 2J_{k,n-1}, \qquad J_{k,0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J_{k,1} = \begin{pmatrix} k & 2 \\ 1 & 0 \end{pmatrix}, \tag{5}$$

$$C_{k,n+1} = kC_{k,n} + 2C_{k,n-1}, C_{k,0} = \begin{pmatrix} k & 4 \\ 2 & -k \end{pmatrix}, C_{k,1} = \begin{pmatrix} k^2+4 & 2k \\ k & 4 \end{pmatrix}, \tag{6}$$

respectively in [7].

The relation between $k-$Jacobsthal and $k-$Jacobsthal Lucas number and matrix sequences is given as in [7]

$$J_{k,n} = \begin{pmatrix} \hat{j}_{k,n+1} & 2\hat{j}_{k,n} \\ \hat{j}_{k,n} & 2\hat{j}_{k,n-1} \end{pmatrix}, C_{k,n} = \begin{pmatrix} \hat{c}_{k,n+1} & 2\hat{c}_{k,n} \\ \hat{c}_{k,n} & 2\hat{c}_{k,n-1} \end{pmatrix}.$$

(s,t) Jacobsthal and (s,t) Jacobsthal Lucas matrix sequences are defined as given by the recurrence relations

$$\begin{aligned} J_{n+1}(s,t) &= sJ_n(s,t) + 2tJ_{n-1}(s,t), \qquad (7) \\ J_0(s,t) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J_1(s,t) = \begin{pmatrix} s & 2 \\ t & 0 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} C_{n+1}(s,t) &= sC_n(s,t) + 2tC_{n-1}(s,t), \qquad (8) \\ C_0(s,t) &= \begin{pmatrix} s & 4 \\ 2t & -s \end{pmatrix}, \ C_1(s,t) = \begin{pmatrix} s^2+4t & 2s \\ st & 4t \end{pmatrix}, \end{aligned}$$

respectively in[5].

The relation between (s,t) Jacobsthal and (s,t) Jacobsthal Lucas number and matrix sequences is given as in [5]

$$J_n(s,t)=\begin{pmatrix} \hat{j}_{n+1}(s,t) & 2\hat{j}_n(s,t) \\ t\hat{j}_n(s,t) & 2t\hat{j}_{n-1}(s,t) \end{pmatrix},\ C_n(s,t)=\begin{pmatrix} \hat{c}_{n+1}(s,t) & 2\hat{c}_n(s,t) \\ t\hat{c}_n(s,t) & 2t\hat{c}_{n-1}(s,t) \end{pmatrix}$$

1.2 Combinatorial Represantations of Jacobsthal, Jacobsthal Lucas and Their Generalized Matrix Sequences

Lemma 1 *For $n \in N$ the sequence of $\{y_n\}_{n\geq 0}$ is defined as follows provides the recurrence relation* $y_{n+1} = y_n + 2y_{n-1}$,

$$y_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} 2^i. \tag{9}$$

Proof. For $n \in N$, it is obtained that

$$\begin{aligned} y_k + 2y_{k-1} &= \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-i}{i} 2^i + \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k-1-i}{i} 2^{i+1} \\ &= \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-i}{i} 2^i + \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k-i}{i-1} 2^i. \end{aligned}$$

If k is an even integer, then $\lfloor k/2 \rfloor = \lfloor (k+1)/2 \rfloor$ and

$$\begin{aligned} y_k + 2y_{k-1} &= \binom{k}{0} + \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \left[\binom{k-i}{i} + \binom{k-i}{i-1} \right] 2^i \\ &= 1 + \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k+1-i}{i} 2^i \\ &= \sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k+1-i}{i} 2^i = y_{k+1}. \end{aligned}$$

If k is an odd integer, then $\lfloor k/2 \rfloor = \lfloor (k-1)/2 \rfloor$

$$\begin{aligned} y_k + 2y_{k-1} &= \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-i}{i} 2^i + \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k-1-i}{i} 2^{i+1} \\ &= \binom{k}{0} + \sum_{i=1}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k-i}{i} 2^i + \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k-i}{i-1} 2^i \end{aligned}$$

$$
\begin{aligned}
&= 1+\sum_{i=1}^{\lfloor\frac{k-1}{2}\rfloor}\binom{k+1-i}{i}2^i+\binom{k-\lfloor\frac{k+1}{2}\rfloor}{\lfloor\frac{k+1}{2}\rfloor-1}2^{\lfloor\frac{k+1}{2}\rfloor}\\
&= \sum_{i=0}^{\lfloor\frac{k-1}{2}\rfloor}\binom{k+1-i}{i}2^i+\binom{k+1-\lfloor\frac{k+1}{2}\rfloor}{\lfloor\frac{k+1}{2}\rfloor}2^{\lfloor\frac{k+1}{2}\rfloor}\\
&= \sum_{i=0}^{\lfloor\frac{k+1}{2}\rfloor}\binom{k+1-i}{i}2^i=y_{k+1}.
\end{aligned}
$$

■

In the following theorem we give a combinatoric presentation of Jacobsthal matrix and the relation between $\{y_n\}_{n\geq 0}$.

Theorem 2 *Let $n\geq 1$, and be integer, then it is obtained that*

$$J_n=y_nI_2+y_{n-1}\begin{bmatrix}0 & 2\\ 1 & -1\end{bmatrix}. \tag{10}$$

Proof. We use induction method for the proof. Because of $y_0=1,\ y_1=1$ the assertion is true for $n=1$,

$$J_1=y_1I_2+y_0\begin{bmatrix}0 & 2\\ 1 & -1\end{bmatrix}.$$

We assume that the assertion is true for $n\leq k$. For $n=k+1$, we have

$$
\begin{aligned}
J_{k+1} &= J_k+2J_{k-1}=\left(y_kI_2+y_{k-1}\begin{bmatrix}0 & 2\\ 1 & -1\end{bmatrix}\right)\\
&\qquad +2\left(y_{k-1}I_2+y_{k-2}\begin{bmatrix}0 & 2\\ 1 & -1\end{bmatrix}\right)\\
&= (y_k+2y_{k-1})I_2+(y_{k-1}+2y_{k-2})\begin{bmatrix}0 & 2\\ 1 & -1\end{bmatrix}\\
&= y_{k+1}I_2+y_k\begin{bmatrix}0 & 2\\ 1 & -1\end{bmatrix}.
\end{aligned}
$$

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Corollary 3 *For Jacobsthal sequences we obtain*

$$j_{n+1}=\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-i}{i}2^i=y_n,\qquad n\geq 1.$$

Proof. By the equality of the matrices in the Theorem 2, it is easily seen. ■

Corollary 4 *Let $n\geq 1$, and integer*

$$C_{n+1}=y_n\begin{bmatrix}5 & 2\\ 1 & 4\end{bmatrix}+y_{n-1}\begin{bmatrix}2 & 8\\ 4 & -2\end{bmatrix} \tag{11}$$

Proof. By the product of matrices, it is clearly seen

$$\begin{aligned} C_{n+1} &= C_1 J_n = \begin{bmatrix} 5 & 2 \\ 1 & 4 \end{bmatrix} \left(y_n I_2 + y_{n-1} \cdot \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 5y_n + 2y_{n-1} & 10y_{n-1} + 4y_{n-2} \\ y_n + 4y_{n-1}1 & 2y_{n-1} + 8y_{n-2} \end{bmatrix} \\ &= \begin{bmatrix} 5y_n + 2y_{n-1} & 2y_n + 8y_{n-1} \\ y_n + 4y_{n-1}1 & 4y_n - 2y_{n-1} \end{bmatrix}. \end{aligned}$$

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Corollary 5 *For Jacobsthal Lucas sequences we obtain*

$$c_{n+1} = 1 + \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \left[\binom{n+1-i}{i} + \binom{n-i}{i-1} \right] 2^i$$

Proof. By the definition of $\{y_n\}$, it is seen that

$$\begin{aligned} c_{n+1} &= j_{n+1} + 4j_n = y_n + 4y_{n-1} = y_{n+1} + 2y_{n-1} \\ &= \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-i}{i} 2^i + \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i} 2^{i+1} \\ &= 1 + \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-i}{i} 2^i + \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-i}{i-1} 2^i \\ &= 1 + \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \left[\binom{n+1-i}{i} + \binom{n-i}{i-1} \right] 2^i. \end{aligned}$$

■

Now we want to use these results for $k-$Jacobsthal and $k-$ Jacobsthal Lucas sequences by using the same procedure.

Lemma 6 *The sequence of* $\{ y_{k,m} \}_{m \geq 0}$ *is defined as follows provides the recurrence relation* $y_{k,m+1} = k y_{k,m} + 2 y_{k,m-1}$,

$$y_{k,m} = \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-i}{i} k^{m-2i} 2^i.$$

Proof. For $m \in N$, by using the definition of $\{ y_{k,m} \}_{m \geq 0}$, we have

$$\begin{aligned} \Omega &= k y_{k,m} + 2 y_{k,m-1} = \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-i}{i} k^{m+1-2i} 2^i \\ &\quad + \sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-1-i}{i} k^{m-2i} 2^{i+1} \\ &= \left(\sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-i}{i} + \sum_{i=1}^{\lfloor \frac{m+1}{2} \rfloor} \binom{m-i}{i-1} \right) k^{m+1-2i} 2^i. \end{aligned}$$

For m is an even integer, then $\lfloor m/2 \rfloor = \lfloor (m+1)/2 \rfloor$

$$\begin{aligned}
\Omega &= k^{m+1} + \sum_{i=1}^{\lfloor \frac{m+1}{2} \rfloor} \left[\binom{m-i}{i} + \binom{m-i}{i-1} \right] k^{m+1-2i} 2^i \\
&= k^{m+1} + \sum_{i=1}^{\lfloor \frac{m+1}{2} \rfloor} \binom{m+1-i}{i} k^{m+1-2i} 2^i \\
&= \sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{m+1-i}{i} k^{m+1-2i} 2^i = y_{k,m+1}.
\end{aligned}$$

For k is an odd integer, then $\lfloor k/2 \rfloor = \lfloor (k-1)/2 \rfloor$

$$\begin{aligned}
\Omega &= \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-i}{i} k^{m+1-2i} 2^i \\
&\quad + \sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-1-i}{i} k^{m-1-2i} 2^{i+1} \\
&= k^{m+1} + \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-i}{i} k^{m+1-2i} 2^i \\
&\quad + \sum_{i=1}^{\lfloor \frac{m+1}{2} \rfloor} \binom{m-i}{i-1} k^{m+1-2i} 2^i \\
&= k^{m+1} + \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m+1-i}{i} k^{m+1-2i} 2^i \\
&\quad + \binom{m - \lfloor \frac{m+1}{2} \rfloor}{\lfloor \frac{m+1}{2} \rfloor - 1} 2^{\lfloor \frac{m+1}{2} \rfloor} \\
&= \sum_{i=0}^{\lfloor \frac{m+1}{2} \rfloor} \binom{m+1-i}{i} k^{m+1-2i} 2^i = y_{k,m+1}.
\end{aligned}$$

■

Theorem 7 *For $n \geq 1$, $n \in N$, it's obtained that*

$$J_{k,n} = y_{k,n} I_2 + y_{k,n-1} \begin{bmatrix} 0 & 2 \\ 1 & -k \end{bmatrix}.$$

Proof. The assumption is true for $n = 1$ because of $y_{k,0} = 1,\ \ y_{k,1} = k$,

$$J_{k,1} = y_{k,1} I_2 + y_{k,0} \begin{bmatrix} 0 & 2 \\ 1 & -k \end{bmatrix}. \tag{12}$$

Let the statement is true for $n \leq m$. For $n = m + 1$, we have

$$\begin{aligned}
J_{k,m+1} &= kJ_{k,m} + 2J_{k,m-1} = k\left(y_{k,m}I_2 + y_{k,m-1}\begin{bmatrix} 0 & 2 \\ 1 & -k \end{bmatrix}\right) \\
&\quad +2\left(y_{k,m-1}I_2 + y_{k,m-2}\begin{bmatrix} 0 & 2 \\ 1 & -k \end{bmatrix}\right) \\
&= \left(ky_{k,m} + 2y_{k,m-1}\right)I_2 + \left(ky_{k,m-1} + 2y_{k,m-2}\right)\begin{bmatrix} 0 & 2 \\ 1 & -k \end{bmatrix} \\
&= y_{k,m+1}I_2 + y_{k,m}\begin{bmatrix} 0 & 2 \\ 1 & -k \end{bmatrix}.
\end{aligned}$$

■

Corollary 8 *For $k-$Jacobsthal sequences, we obtain*

$$\hat{\jmath}_{k.n+1} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} k^{n-2i}2^i.$$

Corollary 9 *For $k-$Jacobsthal Lucas matrix sequences we obtain*

$$C_{k,n+1} = y_{k,n}\begin{bmatrix} k^2+4 & 2k \\ k & 4 \end{bmatrix} + y_{k,n-1}\begin{bmatrix} 2k & 8 \\ 4 & -2k \end{bmatrix}. \tag{13}$$

Proof. By using the product matrices it is seen that

$$C_{k,n+1} = C_{k,1}J_{k,n} = \begin{pmatrix} k^2+4 & 2k \\ k & 4 \end{pmatrix}\left(y_nI_2 + y_{n-1}\cdot\begin{bmatrix} 0 & 2 \\ t & -s \end{bmatrix}\right).$$

■

Corollary 10 *For $k-$Jacobsthal Lucas sequences we have*

$$\hat{c}_{k,n+1} = k^{n+1} + \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor}\left[\binom{n+1-i}{i} + \binom{n-i}{i-1}\right]k^{n+1-2i}2^i.$$

Proof. By using the relation between $k-$Jacobsthal Lucas sequences and $\{y_{k,n}\}$, we have

$$\begin{aligned}
\hat{c}_{k,n+1} &= s\hat{\jmath}_{k,n+1} + 4t\hat{\jmath}_{k,n} = sy_{k,n} + 4ty_{k,n-1} = y_{k,n+1} + 2ty_{k,n-1} \\
&= \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-i}{i} k^{n+1-2i}2^i \\
&\quad + \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i} k^{n-1-2i}2^{i+1} \\
&= k^{n+1} + \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-i}{i} k^{n+1-2i}2^i \\
&\quad + \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-i}{i-1} k^{n+1-2i}2^i.
\end{aligned}$$

Therefore we complete the proof. ■

Now we want to use these results for (s,t) Jacobsthal and (s,t) Jacobsthal Lucas sequences by using the same procedure.

Lemma 11 *For $n \in N$, the sequence of $\{\hat{y}_n\}_{n\geq 0}$ is defined as follows provides the recurrence relation $\hat{y}_{n+1} = \hat{y}_n + 2\hat{y}_{n-1}$,*

$$\hat{y}_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} s^{n-2i}(2t)^i.$$

Proof. For $s,t \in C$ and $n \in N$

$$\begin{aligned} s\hat{y}_k + 2t\hat{y}_{k-1} &= \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-i}{i} s^{k+1-2i}(2t)^i \\ &\quad + \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k-1-i}{i} s^{k+1-2i}(2t)^{i+1} \\ &= \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-i}{i} s^{k+1-2i}(2t)^i \\ &\quad + \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k-i}{i-1} s^{k+1-2i}(2t)^i \end{aligned}$$

For k is an even integer, it is true that $\lfloor k/2 \rfloor = \lfloor (k+1)/2 \rfloor$ and we have

$$\begin{aligned} s\hat{y}_k + 2t\hat{y}_{k-1} &= s^{k+1} + \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \left[\binom{k-i}{i} + \binom{k-i}{i-1} \right] s^{k+1-2i}(2t)^i \\ &= s^{k+1} + \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k+1-i}{i} s^{k+1-2i}(2t)^i \\ &= \sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k+1-i}{i} s^{k+1-2i}(2t)^i = \hat{y}_{k+1}. \end{aligned}$$

For k is an odd integer, it is true that $\lfloor k/2 \rfloor = \lfloor (k-1)/2 \rfloor$ and we have

$$\begin{aligned} s\hat{y}_k + 2t\hat{y}_{k-1} &= \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-i}{i} s^{k+1-2i}(2t)^i \\ &+ \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k-1-i}{i} s^{k-1-2i}(2t)^{i+1} \\ &= s^{k+1} + \sum_{i=1}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k-i}{i} s^{k+1-2i}(2t)^i \\ &+ \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k-i}{i-1} s^{k+1-2i}(2t)^i \\ &+ \binom{k-\lfloor \frac{k+1}{2} \rfloor}{\lfloor \frac{k+1}{2} \rfloor - 1} (2t)^{\lfloor \frac{k+1}{2} \rfloor} \\ &= \sum_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k-i}{i-1} s^{k+1-2i}(2t)^i = \hat{y}_{k+1}. \end{aligned}$$

■

Theorem 12 *For $n \geq 1, n \in N$, it is obtained that*

$$J_n(s,t) = \hat{y}_n I_2 + \hat{y}_{n-1} \begin{bmatrix} 0 & 2 \\ t & -s \end{bmatrix}. \tag{14}$$

Proof. The assumption is true for $n = 1$ because of $\hat{y}_0 = 1$, $\hat{y}_1 = s$ then we have

$$J_1(s,t) = \hat{y}_1 I_2 + \hat{y}_0 \begin{bmatrix} 0 & 2 \\ t & -s \end{bmatrix}.$$

Assume that the statement is true for $n < k$. For $n = k+1$, we have

$$\begin{aligned} J_{k+1}(s,t) &= sJ_k(s,t) + 2tJ_{k-1}(s,t) \\ &= s\left(\hat{y}_k I_2 + \hat{y}_{k-1} \begin{bmatrix} 0 & 2 \\ t & -s \end{bmatrix}\right) + 2t\left(\hat{y}_{k-1} I_2 + \hat{y}_{k-2} \begin{bmatrix} 0 & 2 \\ t & -s \end{bmatrix}\right) \\ &= (s\hat{y}_k + 2t\hat{y}_{k-1}) I_2 + (s\hat{y}_{k-1} + 2t\hat{y}_{k-2}) \begin{bmatrix} 0 & 2 \\ t & -s \end{bmatrix} \\ &= \hat{y}_{k+1} I_2 + \hat{y}_k \begin{bmatrix} 0 & 2 \\ t & -s \end{bmatrix}. \end{aligned}$$

■

Corollary 13 *For (s,t) Jacobsthal sequences we have*

$$\jmath_{n+1}(s,t) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} s^{n-2i}(2t)^i.$$

Corollary 14 *For (s,t) Jacobsthal Lucas matrix sequences we have*

$$\begin{aligned} C_{n+1}(s,t) &= C_1(s,t)J_n(s,t) \\ &= \hat{y}_n \begin{bmatrix} s^2+4t & 2s \\ st & 4t \end{bmatrix} + \hat{y}_{n-1} \cdot \begin{bmatrix} 2st & 8t \\ 4t^2 & -2st \end{bmatrix}. \end{aligned} \tag{15}$$

Corollary 15 *For $(s,t)-$ Jacobsthal Lucas sequences we have*

$$\hat{c}_{n+1}(s,t) = s^{n+1} + \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \left[\binom{n+1-i}{i} + \binom{n-i}{i-1} \right] s^{n+1-2i}(2t)^i.$$

Proof. By using the relation between (s,t) $-$Jacobsthal Lucas sequences and $\{\hat{y}_n\}$, we have

$$\begin{aligned} \hat{c}_{n+1}(s,t) &= s\hat{j}_{n+1}(s,t) + 4t\hat{j}_n(s,t) = s\hat{y}_n + 4t\hat{y}_{n-1} = \hat{y}_{n+1} + 2t\hat{y}_{n-1} \\ &= \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-i}{i} s^{n+1-2i}(2t)^i \\ &\quad + \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i} s^{n-1-2i}(2t)^{i+1} \\ &= s^{n+1} + \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-i}{i} s^{n+1-2i}(2t)^i \\ &\quad + \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-i}{i-1} s^{n+1-2i}(2t)^i. \end{aligned}$$

The proof is completed. ∎

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