The Inverse-conjugate Compositions With Odd Parts

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Abstract

In this paper, we first present a combinatorial proof of the recurrence relation about the number of the inverse-conjugate compositions of $2n+1, n > 1$. And then we get some counting results about the inverse-conjugate compositions for special compositions. In particular, we show that the number of the inverse-conjugate compositions of $4k+1, k > 0$ with odd parts is 2^k, and provide an elegant combinatorial proof. Lastly, we give a relation between the number of the inverse-conjugate *odd compositions* of $4k+1$ and the number of the self-inverse *odd compositions* of $4k+1$.

Key words: compositions; inverse-conjugate compositions; *odd compositions*; *1-2 compositions*; combinatorial proof.

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1 Introduction

A composition of a positive integer n is a representation of n as a sequence of positive integers called parts which sum to n. For example, the compositions of 4 are listed below:

$$(4),\ (3,1),\ (1,3),\ (2,2),\ (2,1,1),\ (1,2,1),\ (1,1,2),\ (1,1,1,1).$$

It is well know that there are 2^{n-1} unrestricted compositions of n. MacMahon's [2] study of compositions was influenced by his pioneering work in partitions. For instance, he devised a graphical representation of a composition, called a *zig-zag graph*, which resembles the partition Ferrers

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graph except that the first dot of each part is aligned with the last part of its predecessor. The *zig-zag graph* of the composition $(6, 3, 1, 2, 2)$ is shown in Figure 1.

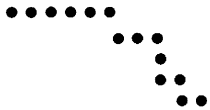

Figure 1

The conjugate of a composition is obtained by reading its graph by columns, from left to right. The Figure 1 gives the conjugate of the composition $(6, 3, 1, 2, 2)$ as $(1, 1, 1, 1, 1, 2, 1, 3, 2, 1)$.

Munagi [3] gave some primary classes of compositions and the relevant theorems. Now we recall some terminologies from [3] herein. Let C denote the composition of n, a k-composition is a composition with k parts, i.e. $C = (c_1, c_2, ..., c_k)$. The conjugate of C is denoted by C', the inverse of C is the reversal composition $\overline{C} = (c_k, c_{k-1}, ..., c_1)$.

C is called *self-inverse* if $C = \overline{C}$.

C is *inverse-conjugate* if it's inverse coincides with its conjugate: $C' = \overline{C}$.

In [3], Munagi defined the following algebraic operations:

Let $A = (a_1, a_2, ..., a_i)$ and $B = (b_1, b_2, ..., b_j)$ be compositions. The concatenation of the parts of A and B is defined as

$A\big|B = (a_1, a_2, ..., a_i, b_1, b_2, ..., b_j)$.

In particular for a nonnegative integer c, $A\big|(c) = (A, c)$ and $(c)\big|A = (c, A)$.

He defined the *join* of A and B as

$A \uplus B = (a_1, a_2, ..., a_i + b_1, b_2, ..., b_j)$.

For the inverse-conjugate of compositions, researchers have obtained some properties [2, 3, 4]. It is well known that if $C = (c_1, c_2, ..., c_k)$ is an inverse-conjugate composition of n, then $k = n - k + 1$ or $n = 2k - 1$. Thus inverse-conjugate compositions are only defined for odd weights. In fact, every odd integer > 1 has a nontrivial inverse-conjugate composition. For example, $(1, 2^{k-1})$ and $(1^{k-1}, k)$ are both inverse-conjugate compositions of $2k - 1$.

Next we shall list some previously known results that will be needed in the next sections.

Lemma 1.1 ([3]) *An inverse-conjugate composition C (or its inverse) has the form:*

$$C = (1^{b_r - 1}, b_1, 1^{b_{r-1} - 2}, b_2, 1^{b_{r-2} - 2}, b_3, ..., b_{r-1}, 1^{b_1 - 2}, b_r), \; b_i > 1. \quad (1)$$

Lemma 1.2 ([3]) *If* $C = (c_1, c_2, ..., c_k)$ *is an inverse-conjugate composition of* $n = 2k - 1 > 1$, *or its inverse, then there is an index* j *such that* $c_1 + c_2 + ... + c_j = k - 1$ *and* $c_{j+1} + ... + c_k = k$ *with* $c_{j+1} > 1$. *Moreover,*

$$\overline{(c_1, c_2, ..., c_j)} = (c_{j+1} - 1, c_{j+2}, ..., c_k)'. \tag{2}$$

Thus C *can be written in the form*

$$C = A\big|(1) \uplus B \qquad suchthat \qquad B' = \overline{A}, \tag{3}$$

where A *and* B *are generally different compositions of* $k - 1$.

Lemma 1.3 ([3]) *There are as many inverse-conjugate compositions of* $2n - 1$ *as there are compositions of* n.

Let $I_C(n)$ be the number of the inverse -conjugate compositions of n. This paper is organized as follows. In Section 2, a combinatorial proof of the recurrence relation of $I_C(n)$ is given. In Section 3, we first study the inverse-conjugate compositions having odd parts, and obtain that the number of the inverse-conjugate compositions of $4k + 1$ with odd parts is 2^k, where $k > 0$. Furthermore, we provide an elegant combinatorial proof. Naturally we get the fact that there are as many inverse-conjugate compositions of $4k+1$ as there are compositions of $k+1$. Lastly, we have a relation between the number of the inverse-conjugate compositions of $4k + 1$ with odd parts and the number of the self-inverse *odd compositions* of $4k + 1$. In addition, we present the counting results about the inverse-conjugate compositions having parts of size 1 or 2.

2 A combinatorial proof

Let $C(n)$ denote the number of compositions of n. By Lemma 1.3 and the recurrence relation of $C(n)$ we easily get the following recurrence relation of $I_C(n)$. In this section, we will present the combinatorial proof.

Theorem 2.1 *We have*

$$I_C(2n + 1) = 2I_C(2n - 1), \quad n > 1, \quad I_C(1) = 1. \tag{4}$$

Proof. Let $C = (c_1, c_2, ..., c_n)$ be an inverse-conjugate composition of $2n - 1 > 1$, By Lemma 1.2, C will belong to one of the following two cases.

Case 1

(1*a*) $c_1 + c_2 + ... + c_j = n - 1$ and $c_{j+1} + c_{j+2} + ... + c_n = n$ with $c_{j+1} > 1$ and

(1*b*) $\overline{(c_1, c_2, ..., c_j, 1)} = (c_{j+1}, c_{j+2}, ..., c_n)'$;

Case 2

(2a) $c_1 + c_2 + ... + c_j = n$ and $c_{j+1} + c_{j+2} + ... + c_n = n - 1$ with $c_j > 1$ and

(2b) $\overline{(c_1, c_2, ..., c_j - 1)} = (c_{j+1}, c_{j+2}, ..., c_n)'$.

We define $T(C) = (1, c_1, c_2, ..., c_n+1)$, then $T(C)$ is an inverse-conjugate composition of $2n+1$ with the first part 1, and it has same structure as C. In fact, if C satisfies (1a) and (1b), then $1+c_1+c_2+...+c_j = n$ and $c_{j+1}+c_{j+2}+...+(c_n+1) = n+1$, and $\overline{(1, c_1, c_2, ..., c_j, 1)} = (c_{j+1}, c_{j+2}, ..., c_n+1)'$. If C satisfies (2a) and (2b), then $1 + c_1 + c_2 + ... + c_j = n + 1$ and $c_{j+1} + c_{j+2} + ... + (c_n + 1) = n$, and $\overline{(1, c_1, c_2, ..., c_j - 1)} = (c_{j+1}, c_{j+2}, ..., c_n + 1)'$.

Next, we again define $S(C) = (c_1 + 1, c_2, ..., c_n, 1)$, then $S(C)$ is an inverse-conjugate composition of $2n + 1$ with the first part > 1, and it has same structure as C using similar discussion to the image T.

From the above discussions, we can see that an inverse-conjugate composition of $2n-1$ can produce two different inverse-conjugate compositions of $2n + 1$.

Conversely, suppose two inverse-conjugate compositions $\alpha = (\alpha_1, \alpha_2, ..., \alpha_{n+1})$ and $\beta = (\beta_1, \beta_2, ..., \beta_{n+1})$ of $2n + 1$ fulfill the following conditions: if $\alpha_1 = 1$, then $\beta_1 > 1$ and $\alpha_1 + \alpha_2 = \beta_1, \alpha_{n+1} = \beta_n + \beta_{n+1}$, and vice versa. We define $T^{-1}(\alpha) = (\alpha_2, \alpha_3, ..., \alpha_{n+1} - 1)$, $S^{-1}(\beta) = (\beta_1 - 1, \beta_2, ..., \beta_n)$. Hence, we get two inverse-conjugate compositions of $2n - 1$ and they are equal.

Therefore, we have the fact that two inverse-conjugate compositions of $2n+1$ correspond to one inverse-conjugate composition of $2n-1$. Thus we have $I_C(2n + 1) = 2I_C(2n - 1)$.

For example, the corresponding relation between the inverse-conjugate composition $(1, 1, 3)$ of 5 and the inverse-conjugate compositions $(1, 1, 1, 4)$, $(2, 1, 3, 1)$ of 7 as follows.

$$(1, 1, 3) \longleftrightarrow (1, 1, 1, 4), \qquad (1, 1, 3) \longleftrightarrow (2, 1, 3, 1).$$

We complete the proof.

3 The inverse-conjugate compositions with odd parts

In this section, we will study the inverse-conjugate compositions with odd parts. We will refer to compositions with odd parts as *odd compositions.* Then we obtain the following counting result. Furthermore, we provide an elegant combinatorial proof.

Theorem 3.1 *Let $I_{(CO)}(n)$ denote the number of inverse-conjugate compositions of n with odd parts. Then*

$$I_{(CO)}(4k+1) = 2^k, \quad k > 0. \tag{5}$$

Proof. By Lemma 1.1, if C is an inverse-conjugate composition of $4k+1$, then C or its inverse has the form: $C = (1^{b_r-1}, b_1, 1^{b_{r-1}-2}, b_2, 1^{b_{r-2}-2}, b_3, \ldots, b_{r-1}, 1^{b_1-2}, b_r), b_i > 1$, and the number of parts is $2k+1$. Since C is always paired with its inverse, we only consider the inverse-conjugate compositions with the first part equal to 1 below.

For each inverse-conjugate composition of $4k+1$ with odd parts we consider the number of 1's: we assume that all the 1's are moved to the left and the parts > 1 are moved to right-end while maintaining their order. Then there are the following cases:

- $(\underbrace{1, 1, \ldots, 1}_{2k}, 2k+1)$;
- $(\underbrace{1, 1, \ldots, 1}_{2k-1}, c_1, c_2)$, and $c_i > 1, \ i = 1, 2$.
- $(\underbrace{1, 1, \ldots, 1}_{2k-2}, c_1, c_2, c_3)$, and $c_i > 1, \ i = 1, 2, 3$.
- ....
- $(\underbrace{1, 1, \ldots, 1}_{k+1}, c_1, c_2, c_3, \ldots c_k)$, and $c_i > 1, \ i = 1, 2, \ldots, k$.

Next we transform the parts > 1 in the above sequences to the following sequences:

- the compositions of $2k+1$ with odd parts having only one part which is $2k+1$;
- the compositions of $2k+2$ with odd parts having 2 parts > 1;
- the compositions of $2k+3$ with odd parts having 3 parts > 1;
- ....
- the compositions of $3k$ with odd parts having k parts > 1 which is $(3, 3, \ldots, 3)$.

In this way, our question becomes to find the corresponding relation between the compositions of k with i parts and the compositions of $2k+i$ with i odd parts > 1, where $i = 1, 2, \ldots, k$. For convenience we stipulate that the first odd is 3, the second odd is 5, and so on.

To establish the desired correspondence we do the following: the composition (k) of k with 1 part corresponds with the composition $(2k+1)$ of $2k+1$ with one odd part > 1, and the part k in the composition of k corresponds to the part in the *odd composition* of $2k+1$ which is the k^{th} odd number, that is, $(k) \longleftrightarrow (2k+1)$. The compositions of k with 2 parts correspond with the compositions of $2k+2$ with 2 odd parts > 1, and the part s in the compositions of k corresponding to the part in the *odd compositions* of $2k+2$ which is the s^{th} odd number. For example, $(1,4) \longleftrightarrow (3,9)$. The compositions of k having 3 parts correspond with the *odd compositions* of $2k+3$ having 3 odd parts > 1, and the part s in the compositions of k corresponding to the part in the *odd compositions* of $2k+3$ which is the s^{th} odd number. For example, $(2,2,1) \longleftrightarrow (5,5,3)$. .... The composition of k having k parts, $(1,1,...,1)$ corresponds with the *odd compositions* of $3k$ with k odd parts, $(3,3,...,3)$, that is, $(1,1,...,1) \longleftrightarrow (3,3,...,3)$.

Obviously the above correspondence is one-to-one.

Hence there are 2^{k-1} inverse-conjugate *odd compositions* of $4k+1$ with the first part equal to 1. So the number of all inverse-conjugate *odd compositions* of $4k+1$ is $2 \times 2^{k-1} = 2^k$.

We complete the proof.

Here we cite an example to illustrate Theorem 3.1.

Example 3.1 *Let $k=3$, then there are 8 inverse-conjugate compositions of 13 with odd parts, and 4 compositions of 3. The corresponding relations in proof of Theorem 3.1 are as follows:*

$$
\begin{aligned}
&(1,1,1,1,1,1,7) \longleftrightarrow (7) \longleftrightarrow (3),\\
&(1,1,5,1,1,1,3) \longleftrightarrow (1,1,1,1,1,8) \longleftrightarrow (8) \longleftrightarrow (3,5) \longleftrightarrow (2,1),\\
&(1,1,1,1,3,1,5) \longleftrightarrow (1,1,1,1,1,8) \longleftrightarrow (8) \longleftrightarrow (5,3) \longleftrightarrow (1,2),\\
&(1,1,3,1,3,1,3) \longleftrightarrow (1,1,1,1,9) \longleftrightarrow (9) \longleftrightarrow (3,3,3) \longleftrightarrow (1,1,1).
\end{aligned}
$$

and

$$
\begin{aligned}
&(1,1,1,1,1,1,7) \longleftrightarrow (3) \longleftrightarrow (7,1,1,1,1,1,1),\\
&(1,1,5,1,1,1,3) \longleftrightarrow (2,1) \longleftrightarrow (3,1,1,1,5,1,1),\\
&(1,1,1,1,3,1,5) \longleftrightarrow (1,2) \longleftrightarrow (5,1,3,1,1,1,1),\\
&(1,1,3,1,3,1,3) \longleftrightarrow (1,1,1) \longleftrightarrow (3,1,3,1,3,1,1).
\end{aligned}
$$

Of course, we easily get the following relation between the number of inverse-conjugate compositions of $4k+1$ with odd parts and the number of the compositions of n.

Corollary 3.1 *Let $I_{(CO)}(n)$ and $C(n)$ denote the number of inverse-conjugate compositions of n with odd parts and the number of the compositions of n, respectively. Then*

$$I_{(CO)}(4k+1) = C(k+1), \quad k \geq 1. \tag{6}$$

In [4], Munagi gave the following relations about the compositions, the inverse-compositions and self-conjugate partitions.

Theorem 3.2 [4] *The following sets of objects are equinumerous:*
(i) Compositions of n.
(ii) Inverse-conjugate compositions of $2n-1$.
(iii) Self-conjugate partitions with largest part equal to n.

By Theorem 3.1 and Corollary 3.1 we easily get the following identities.

Corollary 3.2 *The number of inverse-conjugate compositions of* $4k+1$ *with odd parts equals the number of inverse-conjugate compositions of* $2k+1$.

Corollary 3.3 *The number of inverse-conjugate compositions of* $4k+1$ *with odd parts equals the number of self-conjugate partitions with largest part equal to* $k+1$.

Not unnaturally, using the recurrence relation: $C(n) = 2C(n-1), n > 1$ we obtain the following recurrence relation of the number of inverse-conjugate compositions with odd parts.

Corollary 3.4 *Let* $I_{(CO)}(n)$ *denote the number of inverse-conjugate compositions of* n *with odd parts. Then*

$$I_{(CO)}(4k+1) = 2I_{(CO)}(4k-3), \quad k > 1, \quad I_{(CO)}(1) = 1. \tag{7}$$

Remark. There are no inverse-conjugate compositions of number $4k+3, k > 0$ with odd parts. In fact, if C is an inverse-conjugate composition of $4k+3$ with odd parts, then the number of parts of C is $2k+2$. We know this is impossible.

For the inverse-conjugate compositions of odd integer n, MacMahon demonstrated the following result using **LG** method. And Munagi gave the combinatorial proof in [3].

Theorem 3.3 *(MacMahon) The number of inverse-conjugate compositions of an odd integer* $n > 0$ *equals the number of compositions of* n *which are self-inverse.*

Similar to the method of Munagi, we obtain the following result.

Theorem 3.4 *The number of inverse-conjugate odd compositions of* $4k+1 > 1$ *equals the number of self-inverse odd compositions of* $4k+1 > 1$ *with the middle part is* $4s+1 \geq 1$, *and their conjugate are odd compositions.*

Proof. If C is inverse-conjugate *odd composition* of $4k+1$, then C can be written in the form $C = A\big|(1) \uplus B$ or $C = A \uplus (1)\big|B$ for certain *odd compositions* A, B, of $2k$ satisfying $B' = \overline{A}$.

If $C = A\big|(1) \uplus B$, by (2), we get $T(C) = A\big|[(1) \uplus B]'$, which is a self-inverse odd composition with the middle part is 1.

If $C = A \uplus (1)\big|B$, then there is an odd part $m > 1$ such that $C = X\big|(m)\big|B$, where X is composition of integer $< 2k$. Now split m between the two compositions as follows: $X\big|(m-1)\uplus(1)\big|B = (X, m-1)\uplus(1, B)$, which is in the first case form. Hence $T(C) = (X, m-1)\uplus(1, B)'$, giving a self-inverse *odd composition* with the middle part is d. Where d is $4s+1 > 1$ since $m-1$ is even, and the first part of $(1, B)'$ is m using (2). Because the conjugate of self-inverse composition is self-inverse composition, and C is inverse-conjugate odd composition, so we get the self-inverse *odd compositions* of $4k+1$.

Conversely given a self-inverse *odd composition* $\alpha = (b_1, b_2, ..., b_r) \equiv B\big|(d)\big|\overline{B}$ of $4k+1$ with middle part is $d = 4s+1 > 1$, and the conjugate composition α' also is self-inverse *odd composition*, we first write α as the join of two compositions of $2k$ and $2k+1$ by splitting the middle part. The middle part, by weight, is b_{j+1} such that $s_j = b_1 + ... + b_j \leq 2k$ and $s_j + b_{j+1} \geq 2k+1$. Thus

$$\begin{aligned}\alpha &\mapsto (b_1, b_2, ..., b_j)\big|(2k - s_j) \uplus ((2k+1) - t_j)\big|(b_{j+2}, ..., b_r)\\ &\equiv X\big|(2k - s_j) \uplus ((2k+1) - t_j)\big|\overline{X}.\end{aligned}$$

where $t_j = b_{j+2} + ... + b_r$.

Hence $T^{-1}(\alpha) = (X, (2k - s_j) \uplus (((2k+1) - t_j), \overline{X})'$, which is inverse-conjugate odd composition.

Thus we complete the proof.

We include an example to demonstrate how the bijection actually works.

Example 3.2 *When $k = 4$, the set of inverse-conjugate odd compositions of* 17 *contains the following* 16 *objects.*

$(1,1,1,1,1,1,1,1,9)$, $(9,1,1,1,1,1,1,1,1)$, $(7,1,3,1,1,1,1,1,1)$,
$(3,1,1,1,1,1,7,1,1)$, $(5,1,1,1,5,1,1,1,1)$, $(3,1,3,1,3,1,3,1,1,)$,
$(1,1,1,1,3,1,3,1,5)$, $(1,1,3,1,1,1,5,1,3)(1,1,5,1,3,1,1,1,3)$,
$(1,1,3,1,3,1,3,1,3)$, $(5,1,3,1,3,1,1,1,1)$, $(1,1,1,1,1,1,3,1,7)$
$(3,1,1,1,3,1,5,1,1)$, $(1,1,1,1,5,1,1,1,5)$, $(3,1,5,1,1,1,3,1,1)$,
$(1,1,7,1,1,1,1,1,3)$.

And the second set of compositions contains these 16 *objects:*

$$(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1),\ (7,1,1,1,7),$$
$$(3,1,1,1,1,1,1,1,1,1,1,1,3),\ (5,1,1,1,1,1,1,1,5)$$
$$(3,1,3,1,1,1,3,1,3),\ (1,1,1,1,3,1,1,1,3,1,1,1,1),$$
$$(1,1,3,1,1,1,1,1,1,1,3,1,1),\ (1,1,5,1,1,1,5,1,1,)$$
$$(1,1,3,1,5,1,3,1,1),\ (5,1,5,1,5),\ (1,1,1,1,1,1,5,1,1,1,1,1,1),$$
$$(3,1,1,1,5,1,1,1,3),\ (1,1,1,1,9,1,1,1,1),\ (3,1,9,1,3),$$
$$(1,1,13,1,1),\ (17).$$

In this section, we also study the inverse-conjugate compositions for compositions of n having parts of size 1 or 2. Let *1-2 compositions* be the compositions of n having parts of size 1 or 2. Then we have the following result about the *1-2 compositions.*

Theorem 3.5 *For* $n \geq 1$, *there are exactly two inverse-conjugate of the* 1-2 compositions *of* $2n+1$, *namely* $(1, 2^n)$ *and* $(2^n, 1)$.

Proof. This result follows immediately from Lemma 1.1.
We complete the proof.

Remark. According to Lemma 1.1, there is no inverse-conjugate composition of the compositions of $2n+1$ with parts > 1.

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References

[1] G.E. Andrews and K. Eriksson, Integer Partitions, *Combridge University Press*, 2004.

[2] P.A. MacMahon, Combinatory Analysis, vol.I, *Combridge: at the University press*, 1995.

[3] A.O. Munagi, Primary classes of compositions of numbers, *Annales Mathematicae et Informaticae*, 41(2013), 193-204.

[4] A.O. Munagi, Zig-Zag Graphs and Partitions Identities of A.K.Agarwal, *Annals of Combinatorics*, 19(2015), 557-566.

[5] A.O. Munagi, Euler-type identities for integer compositions via zig-zag graphs, *Integers*, 12(2012), A62, 10pp.

[6] A.O. Munagi, Some inplace identities for integer compositions, *Quaestiones mathematicae*, 38(4)(2015), 535-540.

[7] V.E. Hoggatt and M. Bicknell, Palindromic Composition, *Fibonacci Quarterly,* 13(1975), 350-356.

[8] S. Heubach and T. Mansour, Combinatorics of compositions and words, *Discrete Mathematics and Applications(Boca Raton). CRC Press,Boca Raton, Fl,* 2010.

[9] R.P. Stanley, Enumerative Combinatorics vol.I, *Combridge: at the University press,* 1997.