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Note

A Parity Result for Some p-Regular Partitions

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Abstract: Let $p > 2$ be prime and $r \in \{1, 2, \ldots, p-1\}$. Denote by $c_p(n)$ the number of p-regular partitions of n in which parts can occur not more than three times. We prove the following: If $8r+1$ is a quadratic non-residue modulo p, $c_p(pn+r) \equiv 0 \pmod 2$ for all non negative integers n.

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1. Introduction

Let $n \in \mathbb{Z}_{\geq 0}$. A partition of n is a sequence $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ satisfying $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell \geq 1$ and $\sum_{i=1}^{\ell} \lambda_i = n$. The number of partitions of n is usually denoted by $p(n)$ and $p(0)$ is defined to be 1. For instance, there are 7 partitions of 5, namely; $(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1)$ and $(1,1,1,1,1)$. Thus $p(5) = 7$. The function $p(n)$ is called the (unrestricted) partition function.

At times, further restrictions on parts of partitions are imposed. If that happens, the partition enumerating function in question is called restricted partition function. One of such examples is the number of p-regular partitions. A partition is p-regular if none of its parts is divisible by p.

Parity results for the number of p-regular partitions have been recorded. The interested reader is referred to [1–3] for related work. We shall use the following q-series notation

$$(a;q)_\infty = \prod_{n=1}^{\infty}(1 - aq^n).$$

Using the notation above, the generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}. \tag{1}$$

See [4]. Furthermore, we recall the following q-identities:

$$\sum_{j=0}^{\infty}(-1)^j(2j+1)q^{\frac{j(j+1)}{2}} = (q;q)_\infty^3, \tag{2}$$

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{\frac{j(3j-1)}{2}} = (q;q)_\infty. \tag{3}$$

See [4, 5]. For a prime p greater than 3, Sellers [6] gave parity results for p-regular partitions into distinct parts. We relax this condition of parts being distinct, and allow repetitions of parts up to three times. Let $c_p(n)$ denote the number of p-regular partitions of n in which each part cannot appear more than three times. In this note, we prove the following result.

Theorem 1. *Let $p > 2$ be prime and $r \in \{1, 2, \ldots, p-1\}$ such that $8r + 1$ is a quadratic non-residue modulo p. Then $c_p(pn + r) \equiv 0 \pmod 2$ for all nonnegative integers n.*

2. Proof of Theorem 1

The generating function of $c_p(n)$ is

$$\sum_{n=0}^{\infty} c_p(n)q^n = \prod_{n=1}^{\infty} \frac{1 + q^n + q^{2n} + q^{3n}}{1 + q^{pn} + q^{2pn} + q^{3pn}}.$$

Thus, by (1), (2) and (3),

$$\begin{aligned}
\sum_{n=0}^{\infty} c_p(n)q^n &= \frac{(q^4; q^4)_\infty}{(q; q)_\infty} \frac{(q^p; q^p)_\infty}{(q^{4p}; q^{4p})_\infty} \\
&\equiv \frac{(q; q)_\infty^4}{(q; q)_\infty} \frac{(q^p; q^p)_\infty}{(q^{4p}; q^{4p})_\infty} \pmod 2 \\
&= \frac{(q; q)_\infty^3 (q^p; q^p)_\infty}{(q^{4p}; q^{4p})_\infty} \\
&\equiv \sum_{j=0}^{\infty} q^{\frac{j(j+1)}{2}} \sum_{k=-\infty}^{\infty} q^{\frac{pk(3k-1)}{2}} \sum_{l=0}^{\infty} p(l) q^{4pl} \pmod 2.
\end{aligned}$$

Suppose $pn + r = \frac{j(j+1)}{2} + \frac{pk(3k-1)}{2} + 4pl$ for some integers $j, l \geq 0$ and k. Then

$$r \equiv \frac{j(j+1)}{2} \pmod p$$

which implies

$$8r + 1 \equiv (2j + 1)^2 \pmod p.$$

Hence, if $8r + 1$ is a quadratic non-residue modulo p, $pn + r$ cannot be represented as a sum $\frac{j(j+1)}{2} + \frac{pk(3k-1)}{2} + 4pl$ for some integers $j, l \geq 0$ and k. Therefore, we must have

$$c_p(pn + r) \equiv 0 \pmod 2.$$

Conflict of Interest

The authors declare no conflict of interest

References

1. Alladi, K., 1997. Partition identities involving gaps and weights. *Transactions of the American Mathematical Society, 349*(12), pp.5001-5019.
2. Gordon, B. and Ono, K., 1997. Divisibility of certain partition functions by powers of primes. *The Ramanujan Journal, 1*(1), pp.25-34.
3. MacMahon, P.A., 1921. Note on the parity of the number which enumerates the partitions of a number. In *Proc. Cambridge Philos. Soc* (Vol. 20, pp. 281-283).

4. Andrews, G.E., 1998. *The Theory of Partitions*. Cambridge University Press.
5. Andrews, G.E. and Eriksson, K., 2004. *Integer Partitions*. Cambridge University Press.
6. Sellers, J.A., 2003. Parity results for p-regular partitions with distinct parts. *Ars Combinatoria, 69,* pp.143-146.