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Article

Decomposition of Complete Tripartite Graphs into Short Cycles

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Abstract: For a graph G and for non-negative integers p, q and r, the triplet (p, q, r) is said to be an admissible triplet, if $3p + 4q + 6r = |E(G)|$. If G admits a decomposition into p cycles of length 3, q cycles of length 4 and r cycles of length 6 for every admissible triplets (p, q, r), then we say that G has a $\{C_3^p, C_4^q, C_6^r\}$-decomposition. In this paper, the necessary conditions for the existence of $\{C_3^p, C_4^q, C_6^r\}$-decomposition of $K_{\ell,m,n}(\ell \leq m \leq n)$ are proved to be sufficient. This affirmatively answers the problem raised in [Decomposing complete tripartite graphs into cycles of lengths 3 and 4, Discrete Math. 197/198 (1999), 123-135]. As a corollary, we deduce the main results of [Decomposing complete tripartite graphs into cycles of lengths 3 and 4, Discrete Math., 197/198, 123-135 (1999)] and [Decompositions of complete tripartite graphs into cycles of lengths 3 and 6, Austral. J. Combin., 73(1), 220-241 (2019)].

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1. Introduction

All graphs considered here are simple, finite and undirected. Let K_m and C_m denote the complete graph and a cycle on m vertices. Let P_{m+1} denotes a path on m edges. If $H_1, H_2, ..., H_n$ are edge disjoint subgraphs of G such that $E(G) = E(H_1) \cup E(H_2) \cup ... \cup E(H_n)$, where $\cup$ denotes the disjoint union of graphs, then we say that $H_1, H_2, ..., H_n$ decomposes G. If each $H_i \simeq H$, then we say that H decomposes G and it is denoted by $H|G$. If each H is a cycle C_m, then we say that G admits a C_m-decomposition or m-cycle decomposition and is denoted by $C_m|G$. For non-negative integers p, q and r, the triplet (p, q, r) is said to be an admissible triplet for the graph G, if $3p + 4q + 6r = |E(G)|$. Similarly, the triplet (p', q', r') is said to be an admissible triplet for the sub-graph H, if $3p' + 4q' + 6r' = |E(H)|$. If G admits a decomposition into p cycles of length 3, q cycles of length 4 and r cycles of length 6 for every admissible triplets (p, q, r), then we say that G has a $\{C_3^p, C_4^q, C_6^r\}$-decomposition. For terms not defined here one can refer to [1, 2].

A *latin square* of order n is a $n \times n$ array, each cell of which contains exactly one of the symbols in $\{1, 2, ..., n\}$, such that each row and each column of the array contains each of the symbols in $\{1, 2, ..., n\}$ exactly once. A latin square is said to be *idempotent* if the cell (i, i) contains the symbol i, $1 \leq i \leq n$. A latin square of order n is said to be *cyclic* if it's first row entries are $a_1, a_2, \cdots, a_n$, then the p^{th} row entries are $a_p, a_{p+1}, a_{p+2}, \cdots, a_{p-1}$ in order, where the subscripts are taken modulo n with residues $1, 2, ..., n$, see [3]. A latin square is said to be a latin rectangle, if there exists a rectangular $\ell \times m$ array with entries from the set $N = \{1, 2, ..., n\}$ such that each entry appears at most once in each row and column based on n elements [4].

It is worth mentioning that cycle decomposition problems are NP - complete in general, see [5]. Recently, Paulraja and Srimathi [6, 7] proved the necessary and sufficient conditions for the existence of $\{C_3^p, C_6^r\}$-decomposition of some product of complete graphs. Ganesamurthy and Paulraja [8] gave the necessary and sufficient conditions for some classes of dense graph to admit a $\{C_4^p, C_8^q\}$-decomposition. Very recently, Ezhilarasi and Muthusamy [9], proved the necessary and sufficient conditions for the existence of $\{P_{2p+1}, C_{2p}\}$-decomposition of even regular complete equipartite graphs for all prime p.

The problem of decomposing complete tripartite graphs into cycles have been studied by different authors [4, 10–16]. The necessary and sufficient conditions for the existence of $\{C_3^p, C_4^q\}$-decomposition of complete tripartite graph were given by Billington [17] in 1999. Recently, Ganesamurthy and Paulraja [3] proved the necessary and sufficient conditions for the existence of $\{C_3^p, C_6^r\}$-decomposition of complete tripartite graphs. Billington suggested finding the necessary and sufficient conditions for the existence of $\{C_3^p, C_4^q, C_6^r\}$-decomposition of $K_{\ell,m,n}(\ell \le m \le n)$. The main theorem of this paper answer this question in the affirmative.

Theorem 1. *The complete tripartite graph $K_{\ell,m,n}(\ell \le m \le n)$ admits a $\{C_3^p, C_4^q, C_6^r\}$-decomposition if and only if the partite sets are of same parity and $3p + 4q + 6r = \ell m + mn + \ell n$.*

The main results of [17] can be deduced as a corollary by substituting $r = 0$ in Theorem 1.

Corollary 1. [17] *The complete tripartite graph $K_{\ell,m,n}(\ell \le m \le n)$ has an edge disjoint decomposition into p cycles of length 3 and q cycles of length 4 if and only if,*

(i) ℓ, m, n are all even or odd.

(ii) If ℓ is even or if ℓ is odd and $m - \ell \equiv 0 (mod\ 4)$, then $p \le \ell m$.

(iii) If ℓ is odd and $m - \ell \equiv 2 (mod\ 4)$, then $p \le \ell m - 2$.

(iv) The value of p decreases from its maximum value in steps of size 4, down to 0 if ℓ is even and to 1, if ℓ is odd.

If we put $q = 0$ in Theorem 1, we have the following

Corollary 2. *Let $K_{\ell,m,n}(\ell \le m \le n)$ be the complete tripartite. Then this complete tripartite graph admits a $\{C_3^p, C_6^r\}$-decomposition whenever the partite sets are of same parity and $3p + 6r = \ell m + mn + \ell n$.*

The corollary 2 subsumes the main result of [3].

Corollary 3. [3] *Let $K_{\ell,m,n}(\ell \le m \le n)$ be the complete tripartite graph and let $K_{\ell,m,n} \ne K_{1,1,n}$ when $n \equiv 1 (mod\ 6)$ and $n > 1$. If $\ell \equiv m \equiv n (mod\ 6)$, then $K_{\ell,m,n}$ admits a $\{C_3^p, C_6^r\}$-decomposition for any $p \equiv \ell (mod\ 2)$, with $0 \le p \le \ell m$.*

In order to prove our result, we make use of the following

Theorem 2. [18] *Let m and n be positive integers. Then the complete bipartite graph $K_{2m,2n}$ and $K_{2n+1,2n+1} - F$ admits a $\{C_4^p, C_6^q, C_8^r\}$ - decomposition whenever $4p + 6q + 8r = |E(K_{2m,2n})|$ or $4p + 6q + 8r = |E(K_{2n+1,2n+1} - F)|$, where F is a 1-factor of $K_{2n+1,2n+1}$.*

Lemma 1. [4] *Let ℓ, m and n be integers such that $\ell \le m \le n$. A latin rectangle of order $\ell \times m$ based on n elements is equivalent to the existence of ℓm edge-disjoint triangles sitting inside the complete tripartite graph $K_{\ell,m,n}$.*

Remark 1. *Since a cycle of length 3 in a $\{C_3^p, C_4^q, C_6^r\}$-decomposition of $K_{\ell,m,n}(\ell \leq m \leq n)$ needs to visit all three partite sets, in any $\{C_3^p, C_4^q, C_6^r\}$-decomposition of $K_{\ell,m,n}$, maximum number of 3-cycles is ℓm.*

Throughout this paper, we denote $V(K_{\ell,m,n}) = A \cup B \cup C$ where $A = \{a_1, a_2, ..., a_\ell\}$, $B = \{b_1, b_2, ..., b_m\}$ and $C = \{c_1, c_2, ..., c_n\}$.

2. When Partite Sets are of Same Size

In this section, we prove the necessary conditions for the existence of $\{C_3^p, C_4^q, C_6^r\}$-decomposition of the complete tripartite graphs $K_{\ell,m,n}$ are sufficient whenever $\ell = m = n$.

Remark 2. [17] *A C_3-decomposition of the complete tripartite graph $K_{m,m,m}$ can be achieved using a latin square as follows: an entry k in the cell (i, j) corresponds to a C_3, given by (a_i, b_j, c_k) .*

Lemma 2. *The graph $K_{2,2,2}$ admits a $\{C_3^p, C_4^q, C_6^r\}$-decomposition.*

Proof. In this case, all the possible triplets are: $(p, q, r) \in \{(4, 0, 0), (0, 3, 0), (0, 0, 2), (2, 0, 1)\}$. The decomposition is given below.

(4, 0, 0): (a_1, b_1, c_2), (a_1, b_2, c_1), (a_2, b_1, c_1) and (a_2, b_2, c_2).

(0, 3, 0): (a_1, b_2, a_2, b_1), (b_1, c_2, b_2, c_1) and (a_1, c_2, a_2, c_1).

(0, 0, 2): $(a_1, b_1, c_1, b_2, a_2, c_2)$ and $(a_1, b_2, c_2, b_1, a_2, c_1)$.

(2, 0, 1):(a_1, b_1, c_1), (a_2, b_2, c_2) and $(a_1, b_2, c_1, a_2, b_1, c_2)$.

Thus, the graph $K_{2,2,2}$ admits a $\{C_3^p, C_4^q, C_6^r\}$-decomposition. □

Lemma 3. *The graph $K_{3,3,3}$ admits a $\{C_3^p, C_4^q, C_6^r\}$-decomposition.*

Proof. Consider a cyclic idempotent latin square of order 3. By Remark 2, every entry k in the latin square corresponds to a C_3 in $K_{3,3,3}$. For a $\{C_3^p, C_4^q, C_6^r\}$-decomposition of $K_{3,3,3}$, it is obvious that $p \neq 0$, since the total number of edges is odd. We fix a C_3 namely (a_1, b_1, c_1), in all possible decompositions given below:

Now, $(p, q, r) \in \{(7, 0, 1), (5, 0, 2), (5, 3, 0), (3, 3, 1), (3, 0, 3), (1, 3, 2), (1, 6, 0), (1, 0, 4)\}$ are the set of admissible triplets in the required decomposition.

(7, 0, 1): $(a_1, b_1, c_1), (a_2, b_2, c_3), (a_1, b_3, c_2), (a_2, b_3, c_1), (a_3, b_1, c_2), (a_3, b_2, c_1)$, (a_3, b_3, c_3) and $(a_1, b_2, c_2, a_2, b_1, c_3)$.

(5, 0, 2): (a_1, b_1, c_1), (a_1, b_2, c_2), (a_2, b_2, c_3), (a_2, b_1, c_2), (a_3, b_3, c_2), $(a_1, b_3, a_2, c_1, a_3, c_3)$ and $(a_3, b_1, c_3, b_3, c_1, b_2)$.

(5, 3, 0): (a_1, b_1, c_1), (a_2, b_3, c_1), (a_3, b_1, c_2), (a_3, b_2, c_1), (a_3, b_3, c_3), (a_1, b_2, c_2, b_3), (a_1, c_2, a_2, c_3) and (a_2, b_1, c_3, b_2).

(3, 3, 1): (a_1, b_1, c_1), (a_2, b_1, c_3), (a_3, b_1, c_2), (a_1, b_2, c_1, b_3), (a_2, b_2, c_2, b_3), (a_3, b_3, c_3, b_2) and $(a_1, c_2, a_2, c_1, a_3, c_3)$.

(3, 0, 3): (a_1, b_1, c_1), (a_1, b_2, c_3), (a_1, b_3, c_2), $(a_2, b_1, c_3, a_3, b_2, c_1)$, $(a_2, b_3, c_1, a_3, b_1, c_2)$ and $(a_2, b_2, c_2, a_3, b_3, c_3)$.

(1, 3, 2): (a_1, b_1, c_1), (a_1, b_2, a_2, b_3), (a_1, c_2, b_2, c_3), (a_2, c_1, a_3, c_3), $(a_2, b_1, c_3, b_3, a_3, c_2)$ and $(a_3, b_1, c_2, b_3, c_1, b_2)$.

(1, 6, 0): (a_1, b_1, c_1), (a_1, b_2, a_2, b_3), (a_1, c_2, b_2, c_3), (a_2, c_1, a_3, c_3), (a_3, b_2, c_1, b_3), (a_2, b_1, a_3, c_2) and (b_1, c_2, b_3, c_3).

(1, 0, 4): (a_1, b_1, c_1), $(a_1, b_2, a_2, c_1, a_3, c_3)$, $(a_1, b_3, a_2, c_3, b_2, c_2)$, $(a_2, b_1, c_3, b_3, a_3, c_2)$ and $(a_3, b_1, c_2, b_3, c_1, b_2)$.

The above cases guarantees the existence of $\{C_3^p, C_4^q, C_6^r\}$-decomposition of $K_{3,3,3}$ for all admissible triplets. □

Theorem 3. *The graph $K_{\ell,\ell,\ell}$, admits a $\{C_3^p, C_4^q, C_6^r\}$-decomposition.*

Proof. Let the partite sets of $K_{\ell,\ell,\ell}$ be $A \cup B \cup C$ where, $A = \{a_1, a_2, ..., a_\ell\}$, $B = \{b_1, b_2, ..., b_\ell\}$ and $\{c_1, c_2, ..., c_\ell\}$. We consider the following two cases.

Case 1. ℓ is even.

Consider a cyclic latin square of order ℓ. This latin square is partitioned into 2×2 partial latin squares (with rows i, $i+1$ and columns j, $j+1$) of the form, The partial latin square of the above

	j	$j+1$
i	k	$k+1$
$i+1$	$k+1$	$k+2$

form corresponds to 12 edges and can be decomposed into 3-cycles, 4-cycles and 6-cycles for the following admissible triplets $(p, q, r) \in \{(4, 0, 0), (2, 0, 1), (0, 3, 0), (0, 0, 2)\}$.

(4, 0, 0): The four 3-cycles can be obtained directly by using Remark 2.

(2, 0, 1): The two 3-cycles are (a_i, b_j, c_{k+1}) and $(a_{i+1}, b_{j+1}, c_{k+2})$. The required 6-cycle is $(a_i, c_k, b_j, a_{i+1}, c_{k+1}, b_{j+1})$.

(0, 3, 0): The required 4-cycles are given by (a_i, c_k, b_j, c_{k+1}), $(a_i, b_j, a_{i+1}, b_{j+1})$ and $(a_{i+1}, c_{k+1}, b_{j+1}, c_{k+2})$.

(0, 0, 2): $(a_i, c_k, b_j, c_{k+1}, a_{i+1}, b_{j+1})$ and $(a_i, c_{k+1}, b_{j+1}, c_{k+2}, a_{i+1}, b_j)$ are the required 6-cycles.

Thus each of these 2×2 partial latin squares can be decomposed into 3, 4 and 6 cycles for all admissible triplets.

Hence $K_{\ell,\ell,\ell}$, where ℓ is even, admits a $\{C_3^p, C_4^q, C_6^r\}$-decomposition.

Case 2. ℓ is odd.

Consider a cyclic latin square of order ℓ. As ℓ is odd, $p \neq 0$. Hence, we fix a 3-cycle, (a_1, b_1, c_1) that will be present in all possible decompositions. For $1 \leq i \leq \frac{\ell-1}{2}$, with the first row and first column entries of this latin square, we first partitioned the 2×2 partial latin square entries of the form, The edges corresponding to partial latin square of the above form can be de-

	1	$2i$	$2i+1$
1		$2i$	$2i+1$
$2i$	$2i$	$4i-1$	$4i$
$2i+1$	$2i+1$	$4i$	$4i+1$

composed into 3-cycles, 4-cycles and 6-cycles for all admissible triplets (p, q, r), where $(p, q, r) \in \{(8, 0, 0), (6, 0, 1), (4, 3, 0), (4, 0, 2), (2, 3, 1), (2, 0, 3), (0, 6, 0), (0, 0, 4), (0, 3, 2)\}$.

(8, 0, 0): This can be achieved directly from Remark 2.

(6, 0, 1): (a_1, b_{2i}, c_{2i}), $(a_1, b_{2i+1}, c_{2i+1})$, (a_{2i}, b_1, c_{2i}), $(a_{2i+1}, b_1, c_{2i+1})$, (a_{2i}, b_{2i}, c_{4i}), $(a_{2i+1}, b_{2i+1}, c_{4i+1})$ and $(a_{2i}, c_{4i-1}, b_{2i}, a_{2i+1}, c_{4i}, b_{2i+1})$.

(4, 3, 0): (a_1, b_{2i}, c_{2i}), $(a_1, b_{2i+1}, c_{2i+1})$, (a_{2i}, b_1, c_{2i}), $(a_{2i+1}, b_1, c_{2i+1})$, $(a_{2i}, c_{4i-1}, b_{2i}, c_{4i})$, $(a_{2i}, b_{2i}, a_{2i+1}, b_{2i+1})$ and $(a_{2i+1}, c_{4i}, b_{2i+1}, c_{4i+1})$.

(4, 0, 2): (a_1, b_{2i}, c_{2i}), $(a_1, b_{2i+1}, c_{2i+1})$, (a_{2i}, b_1, c_{2i}), $(a_{2i+1}, b_1, c_{2i+1})$, $(a_{2i}, c_{4i-1}, b_{2i}, c_{4i}, a_{2i+1}, b_{2i+1})$ and $(a_{2i}, c_{4i}, b_{2i+1}, c_{4i+1}, a_{2i+1}, b_{2i})$.

(2, 3, 1): (a_{2i}, b_{2i}, c_{2i}), $(a_{2i+1}, b_{2i+1}, c_{2i})$, $(a_1, b_{2i}, c_{4i}, b_{2i+1})$, $(a_1, c_{2i}, b_1, c_{2i+1})$, $(a_{2i}, b_1, a_{2i+1}, c_{4i})$ and $(a_{2i}, b_{2i+1}, c_{4i+1}, a_{2i+1}, b_{2i}, c_{4i-1})$.

(2, 0, 3): (a_{2i}, b_{2i}, c_{4i}), $(a_{2i+1}, b_{2i+1}, c_{4i+1})$, $(a_{2i}, c_{4i-1}, b_{2i}, a_{2i+1}, c_{4i}, b_{2i+1})$, $(a_1, b_{2i}, c_{2i}, a_{2i}, b_1, c_{2i+1})$ and $(a_1, b_{2i+1}, c_{2i+1}, a_{2i+1}, b_1, c_{2i})$.

(0, 6, 0): $(a_1, b_{2i}, a_{2i}, b_{2i+1})$, $(a_{2i}, b_1, a_{2i+1}, c_{4i})$, $(a_{2i}, c_{2i}, b_{2i}, c_{2i+1})$, $(a_{2i+1}, b_{2i}, c_{4i}, b_{2i+1})$, $(a_1, c_{2i}, b_1, c_{2i+1})$ and $(a_{2i+1}, c_{4i+1}, b_{2i+1}, c_{2i+1})$.

(0, 0, 4): $(a_1, b_{2i+1}, c_{2i+1}, a_{2i+1}, b_{2i}, c_{2i})$, $(a_1, b_{2i}, a_{2i}, c_{2i}, b_1, c_{2i+1})$, $(a_{2i}, b_1, a_{2i+1}, c_{4i+1}, b_{2i+1}, c_{4i})$ and $(a_{2i}, b_{2i+1}, a_{2i+1}, c_{4i}, b_{2i}, c_{4i-1})$.

(0, 3, 2): $(a_{2i}, b_{2i}, a_{2i+1}, b_{2i+1})$, $(a_{2i}, c_{4i-1}, b_{2i}, c_{4i})$, $(a_{2i+1}, c_{4i}, b_{2i+1}, c_{4i+1})$, $(a_1, b_{2i}, c_{2i}, a_{2i}, b_1, c_{2i+1})$ and $(a_1, b_{2i+1}, c_{2i+1}, a_{2i+1}, b_1, c_{2i})$.

The remaining entries of the latin square can be partitioned into 2×2 partial latin squares where the edges corresponding to each of the 2×2 partial latin square can be decomposed into all possible (C_3, C_4, C_6) as in Case 1.

Hence for all admissible triplets (p, q, r), the graph $K_{\ell,\ell,\ell}$ admits a $\{C_3^p, C_4^q, C_6^r\}$-decomposition. □

3. When Partite Sets are of Different Size

In this section, we have proved the necessary conditions for the existence of $\{C_3^p, C_4^q, C_6^r\}$-decomposition of the complete tripartite graphs $K_{\ell,m,n}(\ell \le m \le n)$ are sufficient.

Lemma 4. *The graph $K_{1,3,3}$ admits a $\{C_3^p, C_4^q, C_6^r\}$-decomposition.*

Proof. The graph $K_{1,3,3}$ has 15 edges. The maximum possible 3-cycles in the required decomposition will be three. Hence, the following are the admissible triplets $(p, q, r) \in \{(3, 0, 1), (1, 3, 0), (1, 0, 2)\}$.

(3, 0, 1): (a_1, b_1, c_1), (a_1, b_2, c_2), (a_1, b_3, c_3) and $(b_1, c_2, b_3, c_1, b_2, c_3)$.

(1, 3, 0): (a_1, b_1, c_1), (a_1, b_2, c_1, b_3), (a_1, c_2, b_2, c_3) and (b_1, c_2, b_3, c_3).

(1, 0, 2): $(a_1, b_2, c_3, b_1, c_2, b_3)$, $(a_1, c_2, b_2, c_1, b_3, c_3)$ and (a_1, b_1, c_1).

Thus, the graph $K_{1,3,3}$ admits a $\{C_3^p, C_4^q, C_6^r\}$-decomposition. □

Lemma 5. *The graph $K_{1,5,5}$ admits a $\{C_3^p, C_4^q, C_6^r\}$-decomposition.*

Proof. The graph $K_{1,5,5}$ has 35 edges for which the set of admissible triplets are given by $(p, q, r) \in \{(5, 5, 0), (5, 2, 2), (3, 5, 1), (3, 2, 3), (1, 8, 0), (1, 5, 2), (1, 2, 4)\}$.

(5, 5, 0): (a_1, b_1, c_1), (a_1, b_2, c_2), (a_1, b_3, c_3), (a_1, b_4, c_4), (a_1, b_5, c_5), (b_1, c_2, b_3, c_4), (b_1, c_3, b_4, c_5), (b_2, c_1, b_3, c_5), (b_2, c_3, b_5, c_4) and (b_4, c_1, b_5, c_2).

(5, 2, 2): (a_1, b_1, c_1), (a_1, b_2, c_2), (a_1, b_3, c_3), (a_1, b_4, c_4), (a_1, b_5, c_5), (b_2, c_3, b_5, c_4), (b_4, c_1, b_5, c_2), $(b_1, c_2, b_3, c_1, b_2, c_5)$ and $(b_1, c_3, b_4, c_5, b_3, c_4)$.

(3, 5, 1): (a_1, b_1, c_1), (a_1, b_4, c_4), (a_1, b_5, c_5), (b_2, c_3, b_5, c_4), (b_4, c_1, b_5, c_2), (a_1, c_2, b_3, c_3), (b_1, c_3, b_4, c_5), (a_1, b_2, c_5, b_3) and $(b_1, c_2, b_2, c_1, b_3, c_4)$.

(3, 2, 3): (a_1, b_1, c_1), (a_1, b_4, c_4), (a_1, b_5, c_5), (b_2, c_3, b_5, c_4), (b_4, c_1, b_5, c_2), $(a_1, b_2, c_5, b_4, c_3, b_3)$, $(b_1, c_2, b_2, c_1, b_3, c_4)$ and $(a_1, c_2, b_3, c_5, b_1, c_3)$.

(1, 8, 0): (a_1, b_1, c_1), (a_1, b_2, c_1, b_3), (a_1, b_4, c_5, b_5), (b_1, c_2, b_3, c_4), (b_2, c_3, b_3, c_5), (b_4, c_4, b_5, c_1), (a_1, c_3, b_1, c_5), (a_1, c_2, b_2, c_4) and (b_4, c_2, b_5, c_3).

(1, 5, 2): (a_1, b_1, c_1), (a_1, b_2, c_1, b_3), (a_1, b_4, c_5, b_5), (b_1, c_2, b_3, c_4), (b_2, c_3, b_3, c_5), (b_4, c_4, b_5, c_1), $(b_1, c_3, b_4, c_2, a_1, c_5)$ and $(b_2, c_2, b_5, c_3, a_1, c_4)$.

(1, 2, 4): (a_1, b_1, c_1), (b_1, c_2, b_3, c_4), (b_4, c_4, b_5, c_1), $(b_1, c_3, b_4, c_2, a_1, c_5)$, $(b_2, c_2, b_5, c_3, a_1, c_4)$, $(a_1, b_2, c_1, b_3, c_5, b_4)$ and $(a_1, b_3, c_3, b_2, c_5, b_5)$.

Thus there exists a $\{C_3^p, C_4^q, C_6^r\}$-decomposition of the graph $K_{1,5,5}$ for all admissible triplets (p, q, r). □

Lemma 6. *There exists a $\{C_3^p, C_4^q, C_6^r\}$-decomposition of $K_{1,7,7}$.*

Proof. In order to prove the existence of $\{C_3^p, C_4^q, C_6^r\}$-decomposition of $K_{1,7,7}$ we consider the following admissible triplets:

(7, 0, 7): Seven 3-cycles are as follows: by (a_1, b_1, c_1), (a_1, b_2, c_2), (a_1, b_3, c_3), (a_1, b_4, c_4), (a_1, b_5, c_5), (a_1, b_6, c_6) and (a_1, b_7, c_7). Seven 6-cycles are $(b_1, c_2, b_7, c_6, b_5, c_3)$, $(b_1, c_4, b_5, c_7, b_2, c_5)$, $(b_1, c_7, b_6, c_1, b_2, c_6)$, $(b_3, c_2, b_6, c_5, b_4, c_7)$, $(b_2, c_3, b_7, c_5, b_3, c_4)$, $(b_3, c_1, b_5, c_2, b_4, c_6)$ and $(b_4, c_1, b_7, c_4, b_6, c_3)$.

(7, 3, 5): Seven 3-cycles are same as above. Required 4-cycles are (b_3, c_1, b_5, c_2), (b_3, c_6, b_4, c_7) and (b_4, c_2, b_6, c_5). Five edge disjoint 6-cycles are given by, $(b_1, c_2, b_7, c_6, b_5, c_3)$, $(b_1, c_4, b_5, c_7, b_2, c_5)$, $(b_1, c_7, b_6, c_1, b_2, c_6)$, $(b_2, c_3, b_7, c_5, b_3, c_4)$ and $(b_4, c_1, b_7, c_4, b_6, c_3)$.

(7, 6, 3): The seven 3-cycles are as follows: (a_1, b_1, c_1), (a_1, b_2, c_2), (a_1, b_3, c_3), (a_1, b_4, c_4), (a_1, b_5, c_5), (a_1, b_6, c_6) and (a_1, b_7, c_7). Six 4-cycles are (b_3, c_1, b_5, c_2), (b_3, c_6, b_4, c_7), (b_4, c_2, b_6, c_5), (b_1, c_4, b_5, c_7), (b_1, c_6, b_2, c_5) and (b_2, c_1, b_6, c_7). 6-cycles in the required decomposition are given by, $(b_1, c_2, b_7, c_6, b_5, c_3)$, $(b_2, c_3, b_7, c_5, b_3, c_4)$ and $(b_4, c_1, b_7, c_4, b_6, c_3)$.

(7, 9, 1): (b_3, c_1, b_5, c_2), (b_3, c_6, b_4, c_7), (b_4, c_2, b_6, c_5), (b_1, c_4, b_5, c_7), (b_1, c_6, b_2, c_5), (b_2, c_1, b_6, c_7), (b_3, c_4, b_7, c_5), (b_4, c_1, b_7, c_3) and (b_2, c_3, b_6, c_4) are the nine 4-cycles and the required 6-cycle is given by $(b_1, c_2, b_7, c_6, b_5, c_3)$. Required 3-cycles are same as above.

(5, 0, 8): (a_1, b_1, c_1), (a_1, b_4, c_4), (a_1, b_5, c_5), (a_1, b_6, c_6) and (a_1, b_7, c_7) are the five copies of C_3. Required 6-cycles are given by, $(a_1, c_2, b_7, c_6, b_5, c_3)$, $(a_1, b_2, c_2, b_1, c_3, b_3)$, $(b_1, c_4, b_5, c_7, b_2, c_5)$, $(b_1, c_7, b_6, c_1, b_2, c_6)$, $(b_3, c_2, b_6, c_5, b_4, c_7)$, $(b_2, c_3, b_7, c_5, b_3, c_4)$, $(b_3, c_1, b_5, c_2, b_4, c_6)$ and $(b_4, c_1, b_7, c_4, b_6, c_3)$.

(5, 3, 6): Three copies of 4-cycles are (b_3, c_1, b_5, c_6), (b_4, c_2, b_7, c_6) and (a_1, c_2, b_5, c_3). The six copies of C_6 are $(a_1, b_2, c_2, b_1, c_3, b_3)$, $(b_1, c_4, b_5, c_7, b_2, c_5)$, $(b_1, c_7, b_6, c_1, b_2, c_6)$, $(b_3, c_2, b_6, c_5, b_4, c_7)$, $(b_2, c_3, b_7, c_5, b_3, c_4)$ and $(b_4, c_1, b_7, c_4, b_6, c_3)$. Five copies of 3-cycles are same as above.

(5, 6, 4): Five copies of 3-cycles are (a_1, b_1, c_1), (a_1, b_4, c_4), (a_1, b_5, c_5), (a_1, b_6, c_6) and (a_1, b_7, c_7). Six copies of C_4 are given by, (b_3, c_1, b_5, c_6), (b_4, c_2, b_7, c_6), (a_1, c_2, b_5, c_3), (a_1, b_2, c_4, b_3), (b_1, c_2, b_2, c_3) and (b_3, c_3, b_7, c_5). Four edge disjoint copies of 6-cycles are $(b_1, c_4, b_5, c_7, b_2, c_5)$, $(b_1, c_7, b_6, c_1, b_2, c_6)$, $(b_3, c_2, b_6, c_5, b_4, c_7)$ and $(b_4, c_1, b_7, c_4, b_6, c_3)$.

(5, 9, 2): Five copies of 3-cycles are same as above. Nine copies of 4-cycles are (b_3, c_1, b_5, c_6), (b_4, c_2, b_7, c_6), (a_1, c_2, b_5, c_3), (a_1, b_2, c_4, b_3), (b_1, c_2, b_2, c_3), (b_3, c_3, b_7, c_5), (b_1, c_4, b_5, c_7), (b_1, c_6, b_2, c_5) and (b_2, c_1, b_6, c_7). Two copies of 6-cycles are $(b_3, c_2, b_6, c_5, b_4, c_7)$ and $(b_4, c_1, b_7, c_4, b_6, c_3)$.

(5, 12, 0): (b_3, c_1, b_5, c_6), (b_4, c_2, b_7, c_6), (a_1, c_2, b_5, c_3), (a_1, b_2, c_4, b_3), (b_1, c_2, b_2, c_3), (b_3, c_3, b_7, c_5), (b_1, c_4, b_5, c_7), (b_1, c_5, b_2, c_6), (b_2, c_1, b_4, c_7), (b_3, c_2, b_6, c_7), (b_6, c_1, b_7, c_4) and (b_4, c_3, b_6, c_5) are the required 4-cycles. Five copies of 3-cycles are (a_1, b_1, c_1), (a_1, b_4, c_4), (a_1, b_5, c_5), (a_1, b_6, c_6) and (a_1, b_7, c_7).

(3, 0, 9): Three copies of 3-cycles are (a_1, b_1, c_1), (a_1, b_6, c_6) and (a_1, b_7, c_7). Nine edge disjoint copies of 6-cycles are given by, $(a_1, c_2, b_3, c_1, b_5, c_3)$, $(b_3, c_6, b_5, c_2, b_6, c_7)$, $(b_1, c_5, b_3, c_3, b_7, c_6)$, $(b_2, c_5, b_7, c_2, b_4, c_6)$, $(a_1, b_2, c_2, b_1, c_4, b_3)$, $(b_1, c_3, b_2, c_4, b_5, c_7)$, $(b_2, c_1, b_7, c_4, b_4, c_7)$, $(a_1, b_4, c_1, b_6, c_5, b_5)$ and $(a_1, c_4, b_6, c_3, b_4, c_5)$.

(3, 3, 7): Required copies of 3-cycles are same as above. Three copies of 4-cycles are (b_3, c_1, b_5, c_6), (b_3, c_2, b_6, c_7) and (a_1, c_2, b_5, c_3). 6-cycles in the required decomposition are $(b_1, c_5, b_3, c_3, b_7, c_6)$, $(b_2, c_5, b_7, c_2, b_4, c_6)$, $(a_1, b_2, c_2, b_1, c_4, b_3)$, $(b_1, c_3, b_2, c_4, b_5, c_7)$, $(b_2, c_1, b_7, c_4, b_4, c_7)$, $(a_1, b_4, c_1, b_6, c_5, b_5)$ and $(a_1, c_4, b_6, c_3, b_4, c_5)$.

(3, 6, 5): (b_3, c_1, b_5, c_6), (b_3, c_2, b_6, c_7), (a_1, c_2, b_5, c_3), (b_4, c_2, b_7, c_6), (b_1, c_5, b_2, c_6) and (b_3, c_3, b_7, c_5) are the required copies of 4-cycles. Five copies of 6-cycles are given by, $(a_1, b_2, c_2, b_1, c_4, b_3)$, $(b_1, c_3, b_2, c_4, b_5, c_7)$, $(b_2, c_1, b_7, c_4, b_4, c_7)$, $(a_1, b_4, c_1, b_6, c_5, b_5)$ and $(a_1, c_4, b_6, c_3, b_4, c_5)$. Two copies of 3-cycles in the required decomposition are (a_1, b_1, c_1), (a_1, b_6, c_6) and (a_1, b_7, c_7).

(3, 9, 3): Three copies of 3-cycles are same as above. Nine edge disjoint copies of 4-cycles are given by (b_3, c_1, b_5, c_6), (b_3, c_2, b_6, c_7), (a_1, c_2, b_5, c_3), (b_4, c_2, b_7, c_6), (b_1, c_5, b_2, c_6), (b_3, c_3, b_7, c_5), (a_1, b_2, c_4, b_3), (b_1, c_2, b_2, c_3) and (b_1, c_4, b_5, c_7). Required copies of 6-cycles are $(b_2, c_1, b_7, c_4, b_4, c_7)$, $(a_1, b_4, c_1, b_6, c_5, b_5)$ and $(a_1, c_4, b_6, c_3, b_4, c_5)$.

(3, 12, 1): Twelve edge disjoint copies of 4-cycles are (b_3, c_1, b_5, c_6), (b_3, c_2, b_6, c_7), (a_1, c_2, b_5, c_3), (b_4, c_2, b_7, c_6), (b_1, c_5, b_2, c_6), (b_3, c_3, b_7, c_5), (a_1, b_2, c_4, b_3), (b_1, c_2, b_2, c_3), (b_1, c_4, b_5, c_7), (b_2, c_1, b_4, c_7), (a_1, c_4, b_4, c_5) and (b_6, c_1, b_7, c_4). Required C_6 is $(a_1, b_4, c_3, b_6, c_5, b_5)$. Three copies of 3-cycles are (a_1, b_1, c_1), (a_1, b_6, c_6) and (a_1, b_7, c_7).

(1, 0, 10): (a_1, b_1, c_1) is the required C_3. Ten edge disjoint copies of 6-cycles are $(a_1, c_2, b_3, c_1, b_5, c_3)$, $(b_3, c_6, b_5, c_2, b_6, c_7)$, $(b_1, c_5, b_3, c_3, b_7, c_6)$, $(b_2, c_5, b_7, c_2, b_4, c_6)$, $(a_1, b_2, c_2, b_1, c_4, b_3)$, $(b_1, c_3, b_2, c_4, b_5, c_7)$, $(a_1, c_4, b_4, c_3, b_6, c_6)$, $(a_1, b_4, c_7, b_2, c_1, b_6)$, $(a_1, c_5, b_6, c_4, b_7, c_7)$ and $(a_1, b_5, c_5, b_4, c_1, b_7)$.

(1, 3, 8): (b_3, c_1, b_5, c_6), (b_3, c_2, b_6, c_7) and (a_1, c_2, b_5, c_3) are the 3 edge disjoint copies of 4-cycles. Required 6-cycles are $(b_1, c_5, b_3, c_3, b_7, c_6)$, $(b_2, c_5, b_7, c_2, b_4, c_6)$, $(a_1, b_2, c_2, b_1, c_4, b_3)$, $(b_1, c_3, b_2, c_4, b_5, c_7)$, $(a_1, c_4, b_4, c_3, b_6, c_6)$, $(a_1, b_4, c_7, b_2, c_1, b_6)$, $(a_1, c_5, b_6, c_4, b_7, c_7)$ and $(a_1, b_5, c_5, b_4, c_1, b_7)$. The required C_3 is (a_1, b_1, c_1).

(1, 6, 6): One copy of C_3 is given by, (a_1, b_1, c_1). Required 4-cycles are as follows: (b_3, c_1, b_5, c_6), (b_3, c_2, b_6, c_7), (a_1, c_2, b_5, c_3), (b_4, c_2, b_7, c_6), (b_1, c_5, b_2, c_6) and (b_3, c_3, b_7, c_5). 6-cycles in the required decomposition are given by, $(a_1, b_2, c_2, b_1, c_4, b_3)$, $(b_1, c_3, b_2, c_4, b_5, c_7)$, $(a_1, c_4, b_4, c_3, b_6, c_6)$, $(a_1, b_4, c_7, b_2, c_1, b_6)$, $(a_1, c_5, b_6, c_4, b_7, c_7)$ and $(a_1, b_5, c_5, b_4, c_1, b_7)$.

(1, 9, 4): (a_1, b_1, c_1) is the required C_3. Nine copies of 4-cycles are as follows: (b_3, c_1, b_5, c_6), (b_3, c_2, b_6, c_7), (a_1, c_2, b_5, c_3), (b_4, c_2, b_7, c_6), (b_1, c_5, b_2, c_6), (b_3, c_3, b_7, c_5), (a_1, b_2, c_4, b_3), (b_1, c_2, b_2, c_3) and (b_1, c_4, b_5, c_7). Required 6-cycles are as follows: $(a_1, c_4, b_4, c_3, b_6, c_6)$, $(a_1, b_4, c_7, b_2, c_1, b_6)$, $(a_1, c_5, b_6, c_4, b_7, c_7)$ and $(a_1, b_5, c_5, b_4, c_1, b_7)$.

(1, 12, 2): (a_1, b_1, c_1) is the required C_3. Twelve copies of 4-cycles are as follows: (b_3, c_1, b_5, c_6), (b_3, c_2, b_6, c_7), (a_1, c_2, b_5, c_3), (b_4, c_2, b_7, c_6), (b_1, c_5, b_2, c_6), (b_3, c_3, b_7, c_5), (a_1, b_2, c_4, b_3), (b_1, c_2, b_2, c_3), (b_1, c_4, b_5, c_7), (a_1, c_4, b_7, c_7), (a_1, b_4, c_3, b_6) and (a_1, c_5, b_6, c_6). 6-cycles in the required decomposition is given by, $(b_2, c_1, b_6, c_4, b_4, c_7)$ and $(a_1, b_5, c_5, b_4, c_1, b_7)$.

(1, 15, 0): Required 4-cycles are given by: (b_3, c_1, b_5, c_6), (b_3, c_2, b_6, c_7), (a_1, c_2, b_5, c_3), (b_4, c_2, b_7, c_6), (b_1, c_5, b_2, c_6), (b_3, c_3, b_7, c_5), (a_1, b_2, c_4, b_3), (b_1, c_2, b_2, c_3), (b_1, c_4, b_5, c_7), (a_1, c_5, b_6, c_6), (b_4, c_3, b_6, c_4), (a_1, c_4, b_7, c_7), (a_1, b_4, c_5, b_5), (a_1, b_6, c_1, b_7) and (b_2, c_1, b_4, c_7). The C_3 in the required decomposition is (a_1, b_1, c_1).

Thus the graph $K_{1,7,7}$ admits a $\{C_3^p, C_4^q, C_6^r\}$-decomposition for all admissible triplets (p, q, r). □

Theorem 4. *The graph $K_{1,m,m}$ where m is odd, admits a $\{C_3^p, C_4^q, C_6^r\}$-decomposition where $1 \le p \le m$ and $3p + 4q + 6r = m^2 + 2m$.*

Proof. The graph $K_{1,m,m}$ has $m^2 + 2m$ edges. Since m is odd, here $p \neq 0$. Consider the case $m \equiv 1 (\text{mod } 4)$. Let $m = 4n + 1$. Here,

$$K_{1,m,m} = (a_1, b_1, c_1) \oplus \underbrace{(K_{1,5,5} - C_3) \oplus (K_{1,5,5} - C_3) \oplus ... \oplus (K_{1,5,5} - C_3)}_{\text{n copies}} \oplus \underbrace{(K_{4,4}) \oplus (K_{4,4}) \oplus ... \oplus (K_{4,4})}_{\text{n(n - 1) copies}}.$$

By Lemma 5, the graph $K_{1,5,5} - C_3$ admits a (C_3, C_4, C_6) decomposition for all admissible triplets. Theorem 2 guarantees the existence of (C_4, C_6)- cycle decomposition of $K_{4,4}$ for all admissible pairs (q', r'). Now consider the case $m \equiv 3(\text{mod } 4)$. Let $m = 4n + 3$. In this case,

$$K_{1,m,m} = (a_1, b_1, c_1) \oplus (K_{1,7,7} - C_3) \oplus \underbrace{(K_{1,5,5} - C_3) \oplus (K_{1,5,5} - C_3) \oplus ... \oplus (K_{1,5,5} - C_3)}_{\text{(n - 1) copies}}$$
$$\oplus \underbrace{(K_{4,6}) \oplus (K_{4,6}) \oplus ... \oplus (K_{4,6})}_{\text{2(n - 1) copies}}.$$

By Lemmas 5 and 6, the graph $K_{1,5,5} - C_3$ and $K_{1,7,7} - C_3$ can be decomposed into copies of 3-cycles, 4-cycles and 6-cycles for all admissible triplets. Theorem 2 guarantees the existence of 4-cycles and 6-cycles for all possible pairs (q', r'). Thus, the graph $K_{1,m,m}$ can be decomposed into $\{C_3^p, C_4^q, C_6^r\}$ for all admissible triplets (p, q, r). □

Lemma 7. *There exists a $\{C_3^p, C_4^q, C_6^r\}$-decomposition of the graph $K_{\ell,\ell,m}$.*

Proof. The graph $K_{\ell,\ell,m} = K_{\ell,\ell,\ell} \oplus K_{2\ell,m-\ell}$. By Theorem 3, the graph $K_{\ell,\ell,\ell}$ admits a 3-cycle, 4-cycle and 6-cycle decomposition for all possible values of p, q and r. Theorem 2 guarantees the existence of 4-cycles and 6-cycles in $K_{2\ell,m-\ell}$ for all possible pair (q', r').

It is easy to verify that whenever $m - \ell = 2$ and $p = \ell^2$ then $r = 0$. When $p < \ell^2$, then there exists 4-cycles and 6-cycles for all possible triplets (p, q, r).

Thus the graph $K_{\ell,\ell,m}$ can be decomposed into p copies of C_3, q copies of C_4 and r copies of C_6 for all admissible triplets (p, q, r). □

Lemma 8. *The graph $K_{\ell,m,m}$ with $m - \ell \equiv 0(mod\ 4)$ has a $\{C_3^p, C_4^q, C_6^r\}$-decomposition.*

Proof. Let $\{a_1, a_2, ..., a_l\}, \{b_1, b_2, ..., b_m\}$ and $\{c_1, c_2, ..., c_m\}$ be the partite sets of $K_{\ell,m,m}$. In order to prove this lemma, consider a cyclic latin square of order m.

By Lemma 1, the edges corresponding to the entries in the first ℓ rows of the latin square corresponds to the maximum possible cycles of length 3. Thus $p = \ell m$ is achieved. Further, the entries in the first ℓ rows of the latin square can be then partitioned into 2×2 partial latin squares and the corresponding edges can be decomposed into copies of 3-cycles, 4-cycles and 6-cycles depending upon the values of (p', q', r') similar to Case 1 or Case 2 of Theorem 3, according as ℓ even or odd.

Next, we consider the remaining $m - \ell$ rows of the latin square, where the entries will be of the form,

1	2	3	4	$\cdots$	$m-1$	m
$\ell+1$	$\boxed{\ell+2}$	$\underline{\ell+3}$	$\ell+4$	$\cdots$	$\ell-1$	ℓ
$\ell+2$	$\boxed{\ell+3}$	$\underline{\ell+4}$	$\ell+5$	$\cdots$	ℓ	$\ell+1$
$\underline{\ell+3}$	$\ell+4$	$\ell+5$	$\ell+6$	$\cdots$	$\ell+1$	$\boxed{\ell+2}$
$\underline{\ell+4}$	$\ell+5$	$\ell+6$	$\ell+7$	$\cdots$	$\ell+2$	$\boxed{\ell+3}$

Note that each entry in the remaining $m - \ell$ rows represent an edge between the second and third partite sets. We first decompose the edges corresponding to the entries in these $m - \ell$ rows of the latin square into C_4. Consider a block of first four rows, say rows $\ell + 1$, $\ell + 2$, $\ell + 3$, $\ell + 4$. The entries in the rows correspond to $4m$ edges and are decomposed into copies of C_4 as follows: For example, we consider the bold entries as shown above, which corresponds to a 4-cycle $(b_1, c_{\ell+1}, b_{m-1}, c_{\ell+2})$. Similarly, the underlined entries and the entries in the rectangular box corresponds to the 4-cycles $(b_1, c_{\ell+3}, b_3, c_{\ell+4})$ and $(b_2, c_{\ell+2}, b_m, c_{\ell+3})$, respectively. These three cycles of length four are taken together to have two copies of C_6 and are given by $(b_1, c_{\ell+1}, b_{m-1}, c_{\ell+2}, b_m, c_{\ell+3})$ and $(b_1, c_{\ell+2}, b_2, c_{\ell+3}, b_3, c_{\ell+4})$. Similarly, the remaining entries in this block can be decomposed into 4-cycles and 6-cycles accordingly. This can be repeated for all the block of four consecutive rows. After converting a group of 4-cycles into required number of 6-cycles, if there are unused 4-cycles in a block of four rows and if there are three such blocks, then it is straight forward to see that they can be transformed into 6-cycles using edge trading.

This proves the existence of $\{C_3^p, C_4^q, C_6^r\}$-decomposition of the graph $K_{\ell,m,m}$ with $m-\ell \equiv 0(\text{mod } 4)$. □

Lemma 9. *For $p = \ell(\ell + 2)$ and $4q + 6r = 2(\ell + 2)$, the graph $K_{\ell,\ell+2,\ell+2}$ admits a $\{C_3^p, C_4^q, C_6^r\}$-decomposition.*

Proof. Consider the bipartite graph $K_{\ell+2,\ell+2}$, a proper subgraph of $K_{\ell,\ell+2,\ell+2}$. The degree of each vertex in $K_{\ell+2,\ell+2}$ is $\ell + 2$. From this complete bipartite graph, we first construct a 2-factor $\mathcal{F}$ consisting q copies of C_4 and r copies of C_6. For this, we consider base cycles $C = b_1c_1b_2c_2$ and

$C' = b_{2q+1}c_{2q+1}b_{2q+2}c_{2q+2}b_{2q+3}c_{2q+3}$. Then the 2-factor $\mathcal{F}$ is given by

$$\{\rho^0(C), \rho^2(C), ..., \rho^{2q-2}(C)\} \bigcup \{\rho^0(C'), \rho^3(C'), ..., \rho^{\ell-2q-1}(C')\}.$$

Now, if we decompose the graph $(K_{\ell,\ell+2,\ell+2} - \mathcal{F})$ into $\ell(\ell+2)$ copies of 3-cycles, then we are done. This can be achieved as follows: after the removal of $\mathcal{F}$ and $\ell(\ell+2)$ copies of 3-cycles from $K_{\ell,\ell+2,\ell+2}$, the edges in between second and third partite sets can be decomposed into 1-factors $F_1, F_2, ..., F_\ell$. Now, for $1 \leq i \leq \ell$, the edges incident with a vertex a_i together with a 1-factor F_i would yield a C_3-factor, which completes the proof of this lemma. □

In order to prove the existence of $\{C_3^p, C_4^q, C_6^r\}$-decomposition of $K_{\ell,m,m}$ with $m - \ell \equiv 2(mod\,4)$, we use a latin square which is constructed from an idempotent latin square. Since there is no idempotent latin square of order 2×2, we now prove the existence of $\{C_3^p, C_4^q, C_6^r\}$-decomposition of the graph $K_{3,5,5}$.

Lemma 10. *The graph $K_{3,5,5}$ admits a $\{C_3^p, C_4^q, C_6^r\}$-decomposition.*

Proof. In order to prove the existence of $\{C_3^p, C_4^q, C_6^r\}$-decomposition of $K_{3,5,5}$ for all possible values of p, q and r, the following cases are considered.

(15, 1, 1): The maximum number of possible 3-cycles in the required decomposition of $K_{3,5,5}$ is 15 which are as follows: (a_1, b_1, c_3), (a_1, b_2, c_4), (a_1, b_3, c_1), (a_1, b_4, c_5), (a_1, b_5, c_2), (a_2, b_1, c_5), (a_2, b_2, c_3), (a_2, b_3, c_4), (a_2, b_4, c_2), (a_2, b_5, c_1), (a_3, b_1, c_4), (a_3, b_2, c_5), (a_3, b_3, c_2), (a_3, b_4, c_1) and (a_3, b_5, c_3). The remaining 10 edges from second and third partite which can be decomposed into a C_4 and C_6 given by, (b_1, c_2, b_2, c_1) and $(b_3, c_3, b_4, c_4, b_5, c_5)$.

(13, 4, 0): Required edge disjoint copies of 3-cycles are (a_1, b_2, c_4), (a_1, b_3, c_1), (a_1, b_5, c_2), (a_2, b_1, c_5), (a_2, b_2, c_3), (a_2, b_3, c_4), (a_2, b_4, c_2), (a_2, b_5, c_1), (a_3, b_1, c_4), (a_3, b_2, c_5), (a_3, b_3, c_2), (a_3, b_4, c_1) and (a_3, b_5, c_3). Four copies of 4-cycles are (b_1, c_1, b_2, c_2), (b_4, c_4, b_5, c_5), (a_1, b_1, c_3, b_4) and (a_1, c_3, b_3, c_5).

(13, 1, 2): Edge disjoint copies of 3-cycles are same as above. Required C_4 is given by (b_1, c_1, b_2, c_2). Two copies of 6-cycles are $(a_1, b_1, c_3, b_3, c_5, b_4)$ and $(a_1, c_3, b_4, c_4, b_5, c_5)$.

(11, 4, 1): Required copies of 3-cycles are given by, (a_1, b_2, c_4), (a_1, b_3, c_1), (a_1, b_5, c_2), (a_2, b_2, c_3), (a_2, b_3, c_4), (a_2, b_5, c_1), (a_3, b_1, c_4), (a_3, b_2, c_5), (a_3, b_3, c_2), (a_3, b_4, c_1) and (a_3, b_5, c_3). Four copies of 4-cycles are (a_2, c_2, b_1, c_5), (b_4, c_4, b_5, c_5), (a_2, b_1, c_3, b_4) and (a_1, c_3, b_3, c_5). Required C_6 is $(a_1, b_1, c_1, b_2, c_2, b_4)$.

(11, 1, 3): Required copies of 3-cycles will be the same as given above. Three copies of 6-cycles are $(b_3, c_3, b_4, c_4, b_5, c_5)$, $(a_2, c_2, b_4, a_1, b_1, c_5)$ and $(a_1, c_3, b_1, a_2, b_4, c_5)$. Required C_4 is (b_1, c_1, b_2, c_2).

(9, 7, 0): Seven copies of 4-cycles are as follows, (b_1, c_1, b_2, c_2), (a_1, c_3, b_4, c_5), (a_1, b_4, c_2, b_5), (a_1, b_1, a_2, c_2), (b_1, c_3, b_3, c_5), (a_2, b_3, c_4, b_4) and (a_2, c_4, b_5, c_5). Required copies of 3-cycles are given by (a_1, b_2, c_4), (a_1, b_3, c_1), (a_2, b_2, c_3), (a_2, b_5, c_1), (a_3, b_1, c_4), (a_3, b_2, c_5), (a_3, b_3, c_2), (a_3, b_4, c_1) and (a_3, b_5, c_3).

(9, 4, 2): Nine copies of 3-cycles are given by (a_1, b_2, c_4), (a_1, b_3, c_1), (a_2, b_2, c_3), (a_2, b_5, c_1), (a_3, b_1, c_4), (a_3, b_2, c_5), (a_3, b_3, c_2), (a_3, b_4, c_1) and (a_3, b_5, c_3). Four edge disjoint copies of 4-cycles are (b_1, c_1, b_2, c_2), (a_1, c_3, b_4, c_5), (a_2, b_3, c_4, b_4) and (b_1, c_3, b_3, c_5). Required 6-cycles are $(a_1, b_1, a_2, c_4, b_5, c_2)$ and $(a_1, b_4, c_2, a_2, c_5, b_5)$.

(9, 1, 4): Required copies of 3-cycles are same as given above. (b_1, c_1, b_2, c_2) is the required C_4. Edge disjoint copies of 6-cycles are as follows: $(a_1, b_1, a_2, c_4, b_5, c_2)$, $(a_1, b_4, c_2, a_2, c_5, b_5)$, $(a_1, c_3, b_3, a_2, b_4, c_5)$ and $(b_1, c_3, b_4, c_4, b_3, c_5)$.

(7, 7, 1): (a_1, b_2, c_4), (a_1, b_3, c_1), (a_2, b_2, c_3), (a_2, b_5, c_1), (a_3, b_1, c_4), (a_3, b_2, c_5) and (a_3, b_4, c_1) are the seven edge disjoint copies of 3-cycles and the required C_6 is $(a_2, b_3, c_5, b_1, c_3, b_4)$. Seven copies of 4-cycles are as follows: (b_1, c_1, b_2, c_2), (a_1, b_1, a_2, c_5), (a_1, b_4, c_5, b_5), (a_2, c_2, b_4, c_4), (a_1, c_2, b_3, c_3), (a_3, c_2, b_5, c_3) and (a_3, b_3, c_4, b_5).

(7, 4, 3): Four copies of 4-cycles are (b_1, c_1, b_2, c_2), (a_1, b_1, a_2, c_5), (a_1, b_4, c_5, b_5) and (a_2, c_2, b_4, c_4). Required 6-cycles are $(a_3, c_2, b_5, c_4, b_3, c_3)$, $(a_1, c_2, b_3, a_3, b_5, c_3)$ and $(a_2, b_3, c_5, b_1, c_3, b_4)$. Seven copies of 3-cycles are same as given above.

(7, 1, 5): $(a_3, c_2, b_5, c_4, b_3, c_3)$, $(a_1, c_2, b_3, a_3, b_5, c_3)$, $(a_2, b_3, c_5, b_1, c_3, b_4)$, $(a_1, b_4, c_2, a_2, c_5, b_5)$ and $(a_1, b_1, a_2, c_4, b_4, c_5)$ are the five edge disjoint copies of 6-cycles required and one copy of C_4 is (b_1, c_1, b_2, c_2). Required seven copies of 3-cycles are (a_1, b_2, c_4), (a_1, b_3, c_1), (a_2, b_2, c_3), (a_2, b_5, c_1), (a_3, b_1, c_4), (a_3, b_2, c_5) and (a_3, b_4, c_1).

(5, 10, 0): Five copies of 3-cycles are (a_1, b_2, c_4), (a_2, b_2, c_3), (a_3, b_1, c_4), (a_3, b_2, c_5) and (a_3, b_4, c_1). Ten edge disjoint copies of 4-cycles are (b_1, c_3, b_3, c_5), (b_3, c_1, b_5, c_4), (a_1, c_2, a_3, c_3), (a_2, b_4, c_3, b_5), (a_1, c_1, a_2, b_3), (a_3, b_3, c_2, b_5), (a_1, b_4, c_5, b_5), (a_1, b_1, a_2, c_5), (a_2, c_2, b_4, c_4) and (b_1, c_1, b_2, c_2).

(5, 7, 2): Five copies of 3-cycles are same as given above. Required 4-cycle are as follows: (a_1, b_4, c_5, b_5), (a_1, c_2, a_3, c_3), (a_2, b_4, c_3, b_5), (b_3, c_1, b_5, c_4), (a_2, c_2, b_4, c_4), (a_3, b_3, c_2, b_5) and (b_1, c_1, b_2, c_2). 6-cycles in the required decomposition are given by $(a_1, b_1, c_3, b_3, a_2, c_5)$ and $(a_1, b_3, c_5, b_1, a_2, c_1)$.

(5, 4, 4): (a_1, b_2, c_4), (a_2, b_2, c_3), (a_3, b_1, c_4), (a_3, b_2, c_5) and (a_3, b_4, c_1) are the five copies of 3-cycles. 4-cycles in the required decomposition are (a_1, b_4, c_5, b_5), (b_3, c_1, b_5, c_4), (a_3, b_3, c_2, b_5) and (b_1, c_1, b_2, c_2). Four copies of 6-cycles are given by, $(a_1, b_1, c_3, b_3, a_2, c_5)$, $(a_1, b_3, c_5, b_1, a_2, c_1)$, $(a_1, c_2, b_4, a_2, b_5, c_3)$ and $(a_2, c_2, a_3, c_3, b_4, c_4)$.

(5, 1, 6): Five copies of 3-cycles are same as given above. Six copies of 6-cycles are $(a_1, b_1, c_3, b_3, a_2, c_5)$, $(a_1, b_3, c_5, b_1, a_2, c_1)$, $(a_1, c_2, b_4, a_2, b_5, c_3)$, $(a_2, c_2, a_3, c_3, b_4, c_4)$, $(a_3, b_3, c_1, b_1, c_2, b_5)$ and $(b_2, c_1, b_5, c_4, b_3, c_2)$ and one copy of C_4 in the required decomposition is (a_1, b_4, c_5, b_5).

(3, 1, 7): Edge disjoint copies of 3-cycles are (a_3, b_1, c_4), (a_3, b_2, c_5) and (a_2, b_2, c_4). Required copies of 6-cycles are as follows: $(a_1, b_1, c_3, b_3, a_2, c_5)$, $(a_1, b_3, c_5, b_1, a_2, c_1)$, $(a_1, c_2, b_4, a_2, b_5, c_3)$, $(b_2, c_1, b_5, c_4, b_3, c_2)$, $(a_2, c_2, a_3, c_1, b_4, c_3)$, $(a_3, b_3, c_1, b_1, c_2, b_5)$ and $(a_1, b_2, c_3, a_3, b_4, c_4)$. Required 4-cycle is given by (a_1, b_4, c_5, b_5).

(3, 4, 5): Edge disjoint copies of 3-cycles is same as given above. Required 4-cycles are (a_1, b_1, c_3, b_3), (a_1, c_1, a_2, c_5), (a_2, b_1, c_5, b_3) and (a_1, b_4, c_5, b_5). Five copies of 6-cycles are given by $(a_1, c_2, b_4, a_2, b_5, c_3)$, $(b_2, c_1, b_5, c_4, b_3, c_2)$, $(a_2, c_2, a_3, c_1, b_4, c_3)$, $(a_3, b_3, c_1, b_1, c_2, b_5)$ and $(a_1, b_2, c_3, a_3, b_4, c_4)$.

(3, 7, 3): Seven copies of 4-cycles are (a_1, b_1, c_3, b_3), (a_1, c_1, a_2, c_5), (a_2, b_1, c_5, b_3), (a_1, b_4, c_5, b_5), (a_1, c_2, a_2, c_3), (a_2, b_4, c_3, b_5) and (a_3, c_1, b_4, c_2). Required 6-cycles are given by, $(b_2, c_1, b_5, c_4, b_3, c_2)$, $(a_3, b_3, c_1, b_1, c_2, b_5)$ and $(a_1, b_2, c_3, a_3, b_4, c_4)$. Edge disjoint copies of 3-cycles are (a_3, b_1, c_4), (a_3, b_2, c_5) and (a_2, b_2, c_4).

(3, 10, 1): (a_1, b_1, c_3, b_3), (a_1, c_1, a_2, c_5), (a_2, b_1, c_5, b_3), (a_1, b_4, c_5, b_5), (a_1, c_2, a_2, c_3), (a_2, b_4, c_3, b_5), (a_3, c_1, b_4, c_2), (b_1, c_1, b_2, c_2), (b_3, c_1, b_5, c_4) and (a_3, b_3, c_2, b_5) are the ten edge disjoint copies of 4-cycles. Required 6-cycle is given by $(a_1, b_2, c_3, a_3, b_4, c_4)$. Edge disjoint copies of 3-cycles are same as given above.

(1, 10, 2): Ten copies of 4-cycles are (a_1, b_1, c_3, b_3), (a_1, c_1, a_2, c_5), (a_2, b_1, c_5, b_3), (a_1, b_4, c_5, b_5), (a_3, b_1, c_4, b_4), (a_1, b_2, a_2, c_4), (a_3, c_4, b_2, c_3), (b_1, c_1, b_2, c_2), (b_3, c_1, b_5, c_4) and (a_3, b_3, c_2, b_5). Two copies of 6-cycles are $(a_1, c_2, b_4, a_2, b_5, c_3)$ and $(a_2, c_2, a_3, c_1, b_4, c_3)$. Required C_3 is (a_3, b_2, c_5).

(1, 7, 4): Required copies of 4-cycles are as follows: (a_1, b_1, c_3, b_3), (a_1, c_1, a_2, c_5), (a_2, b_1, c_5, b_3), (a_1, b_4, c_5, b_5), (a_3, b_1, c_4, b_4), (a_1, b_2, a_2, c_4) and (a_3, c_4, b_2, c_3). Four copies of 6-cycles are $(a_1, c_2, b_4, a_2, b_5, c_3)$, $(a_2, c_2, a_3, c_1, b_4, c_3)$, $(a_3, b_3, c_1, b_1, c_2, b_5)$ and $(b_2, c_1, b_5, c_4, b_3, c_2)$. Required C_3 is (a_3, b_2, c_5).

(1, 4, 6): (a_1, b_4, c_5, b_5), (a_3, b_1, c_4, b_4), (a_1, b_2, a_2, c_4) and (a_3, c_4, b_2, c_3) gives the required 4-cycles. Edge disjoint copies of 6-cycles are given by $(a_1, c_2, b_4, a_2, b_5, c_3)$, $(a_2, c_2, a_3, c_1, b_4, c_3)$, $(a_3, b_3, c_1, b_1, c_2, b_5)$, $(b_2, c_1, b_5, c_4, b_3, c_2)$, $(a_1, b_1, c_3, b_3, a_2, c_5)$ and $(a_1, c_1, a_2, b_1, c_5, b_3)$. Required 3-cycle is (a_3, b_2, c_5).

(1, 1, 8): $(a_1,c_2,b_4,a_2,b_5,c_3)$, $(a_2,c_2,a_3,c_1,b_4,c_3)$, $(a_3,b_3,c_1,b_1,c_2,b_5)$, $(b_2,c_1,b_5,c_4,b_3,c_2)$, $(a_1,b_1,c_3,b_3,a_2,c_5)$, $(a_1,c_1,a_2,b_1,c_5,b_3)$, $(a_1,b_2,a_2,c_4,a_3,b_4)$ and $(a_1,c_4,b_1,a_3,c_5,b_5)$ are the 8 edge disjoint copies of 6-cycles. Required 4-cycle is given by (b_2,c_4,b_4,c_5). One copy of C_3 is given by (a_3,b_2,c_5).

Thus the graph $K_{3,5,5}$ admits a $\{C_3^p, C_4^q, C_6^r\}$-decomposition for all possible triplets. □

Definition 1. [17] *In a $n \times n$ latin square, if each of the 2×2 subsquare has entries of the form,*

x	$x+1$
$x+1$	x

is called a subsquare of the form (x).

Next to prove the existence of $\{C_3^p, C_4^q, C_6^r\}$-decomposition of $K_{\ell,m,m}$ with $m-\ell \equiv 2 (\mathrm{mod}\ 4)$, we use the following construction given by Elizabeth J. Billington [17].

Recall that if the cell (i,i) of a latin square of order n contains an entry i then the latin square is called idempotent latin square. When n is odd, an idempotent latin square can be constructed easily by using the entries in a cyclic order. But when n is even, an idempotent latin square can be constructed by using the stripping the transversal technique which is explained in [19].

Lemma 11. [17] *For any $P > 2$, there exists a latin square of order $2p+1$ possessing $p(p-1)$ 2×2 cell disjoint subsquares of the form (x).*

In the following example, using an idempotent latin square of order 5, we construct an idempotent latin square of order 11 by using Lemma 11 which consists of 20 cell disjoint 2×2 subsquares of the form (x).

Example 1. *Consider the latin square L_5.*

$L_5 =$

1	4	2	5	3
4	2	5	3	1
2	5	3	1	4
5	3	1	4	2
3	1	4	2	5

We can obtain the required latin square, L_{11} using Lemma 11 as given below.

$L_{11} =$

0	2	1	4	3	6	5	8	7	10	9
2	1	0	7	8	3	4	9	10	5	6
1	0	2	8	7	4	3	10	9	6	5
4	7	8	3	0	9	10	5	6	1	2
3	8	7	0	4	10	9	6	5	2	1
6	3	4	9	10	5	0	1	2	7	8
5	4	3	10	9	0	6	2	1	8	7
8	9	10	5	6	1	2	7	0	3	4
7	10	9	6	5	2	1	0	8	4	3
10	5	6	1	2	7	8	3	4	9	0
9	6	5	2	1	8	7	4	3	0	10

Lemma 12. *For $m-\ell \equiv 2(mod\ 4)$, there exists a $\{C_3^p, C_4^q, C_6^r\}$-decomposition of $K_{\ell,m,m}$.*

Proof. The proof is splitted into 2 cases.

Case 1. ℓ is odd.

The graph $K_{\ell,m,m}$ with $m - \ell \equiv 2 (\text{mod } 4)$ has $m^2 + 2\ell m$ edges. Let $m = 2M + 1$ and $\ell = 2L + 1$. Here, the number of edges is odd and hence $p \neq 0$. Let us fix one C_3 as (a_0, b_0, c_0) in all possible decomposition. In order to prove this result, we use the latin square as described in Lemma 11, say L_m. This latin square is of order m, which will be of the form,

0	2	1	4	3	6	5	$\ldots$	$2L$	$2L-1$	...	$2M$	$2M-1$
2	1	0										
1	0	2										
4			3	0								
3			0	4								
6					5	0						
5					0	6						
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	
$2L$								$2L-1$	0			
$2L-1$								0	$2L$			
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	
$2M$											$2M-1$	0
$2M-1$											0	$2M$

Clearly, $p \leq \ell m$ and equality can be achieved by considering the entries in the first ℓ rows of L_m. These $3\ell m$ edges can be decomposed into all possible 3, 4 and 6 cycles as follows:

It may be noted that the edges corresponding to the entry k in the cell (i, j) of the first ℓ rows correspond to a 3-cycle, (a_i, b_j, c_k). Similarly, an entry c in the cell (a, b) after first ℓ rows correspond to a single edge from partite set 2 to partite set 3. Now, the entries in the first ℓ rows of the latin square L_m other than row 0 and column 0 can be partitioned into $L(M - 1)$ 2×2 subsquares of the form (x) as given in Definition 1 together with L 2×2 partial latin square of the form: Observe

	$2i-1$	$2i$
$2i-1$	$2i-1$	0
$2i$	0	$2i$

that the edges corresponding to each of the 2×2 subsquares is isomorphic to $K_{2,2,2}$ which admits a (C_3, C_4, C_6)-decomposition by Lemma 2. Now consider each of the L partial latin squares together

	$2i-1$	$2i$
$2i-1$	$2i-1$	0
$2i$	0	$2i$

with the corresponding entries of row 0 and column 0, that is:

	0	$2i-1$	$2i$
0		$2i$	$2i-1$
$2i-1$	$2i$	$2i-1$	0
$2i$	$2i-1$	0	$2i$

The corresponding edges induce a graph isomorphic to $K_{3,3,3} - C_3$. By Lemma 3, the graph $K_{3,3,3} - C_3$ admits a (C_3, C_4, C_6)-decomposition for all admissible triplets. Observe that the edges corresponding to the entries in the following cells are not used so far in the decomposition

$$\bigcup_{i=\ell+1}^{m}\{(0,i)\}\bigcup_{i=\ell+1}^{m}\{(i,0)\}\bigcup_{i=\ell+1}^{m}\{(i,1),(i,2),...,(i,m)\}.$$

Now consider the edges corresponding to the entries of the cells

$$\bigcup_{i=\ell+1}^{m}\{(0,i)\}\bigcup_{i=\ell+1}^{m}\{(i,0)\}\bigcup_{i=\ell+1}^{m}\{(i,i)\}\bigcup_{i=\ell+1}^{m}\{(i,i+1),(i+1,i)\}.$$

That is, for some k with $\ell+1 \leq k \leq m$, the entries will be of the form:

	0	$2k-1$	$2k$
0		**2k**	**2k − 1**
$2k-1$	$2k$	$2k-1$	0
$2k$	$2k-1$	0	$2k$

Table 1. Partial Latin Square along with Row 0 and Column 0 Entries

The edges corresponding to the entries given in Table 1 can be either decomposed into three 4-cycles $(a_0, b_{2k-1}, c_0, b_{2k})$, $(a_0, c_{2k-1}, b_{2k-1}, c_{2k})$ and $(b_0, c_{2k-1}, b_{2k}, c_{2k})$ or into two 6-cycles $(a_0, c_{2k-1}, b_{2k-1}, c_0, b_{2k}, c_{2k})$ and $(a_0, b_{2k-1}, c_{2k}, b_0, c_{2k-1}, b_{2k})$.

The remaining edges, corresponding to the last $(m-\ell)$ rows are decomposed into required (C_4, C_6) by grouping three 2×2 subsquares(note that each 2×2 subsquare corresponds to a 4-cycle) such that these subsquares are from 4 columns of L_m and contains four symbols. For example, see Tables 2 and 3.

1	2	3	4
$M-2$	$M-1$	$2M-1$	$2M$
$M-1$	$M-2$	$2M$	$2M-1$
$2M-1$	$2M$		
$2M$	$2M-1$		

Table 2. Partial Latin Square 1

1	2	3	4
		$2M-5$	$2M-4$
		$2M-4$	$2M-5$
$2M-5$	$2M-4$	$M-4$	$M-3$
$2M-4$	$2M-5$	$M-3$	$M-4$

Table 3. Partial Latin Square 2

The edges corresponding to the entries as shown in Table 2 can be decomposed into two 6-cycles $(b_1, c_{M-2}, b_2, c_{2M-1}, b_3, c_{2M})$ and $(b_1, c_{M-1}, b_2, c_{2M}, b_4, c_{2M-1})$. Similarly, the edges corresponding to the entries in Table 3 can be decomposed into two 6-cycles $(b_1, c_{2M-4}, b_3, c_{M-3}, b_4, c_{2M-5})$ and $(b_2, c_{2M-4}, b_4, c_{M-4}, b_3, c_{2M-5})$.

Similarly, the edges corresponding to other groups with the above mentioned condition(4 column and 4 symbols) admits a (C_4, C_6)-decomposition for all admissible pairs.

Now it remains to show that when $m-\ell \equiv 2(mod\, 4)$, the last $m-\ell$ rows of L_m are partitioned into any of the form of Table 2 or 3. First, we consider M is odd. The case when $m-\ell = 2$ has been dealt in Lemma 9. Consider $m-\ell = 6$, by the construction of the latin square L_m, there are $3(M-1)$ of

2×2 subsquares each of which corresponds to a 4-cycle. The entries in the last 6 rows of the latin square is grouped as shown in Figure 1(Note that a box in Figure 1 correspond to a subsquare in the latin square). Hence, we are done with $m - \ell = 6$. Next, the case $m - \ell = 10$ is considered. By the

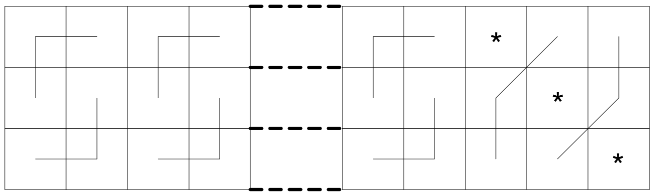

Figure 1. Partition of the Latin Square

construction of the latin square, there are $5(M-1)$ subsquares and 5 partial latin square in the last 10 rows of the latin square. Note that, each subsquare corresponds to a 4-cycle. Thus, there are $5(M-1)$ 4-cycles available corresponding to the entries in the last 10 rows. In order to construct 6-cycles, we may trade certain set of three 4-cycles for two 6-cycles. Here, depending upon m, the following 3 cases arise. when $m \equiv 1 \,(mod\, 6)$, then $q \geq 1$. Similarly, when $m \equiv 3 \,(mod\, 6)$, then $q \geq 0$ and when $m \equiv 5 \,(mod\, 6)$, then $q \geq 1$. For instance, consider the case $m - \ell \equiv 3(mod\, 6)$. The entries in these 10 rows are grouped as shown in Figure 2. It is easy to verify that each of the partial latin square shown in Figure 2 either corresponds to three 4-cycles or two 6-cycles. Thus the edges corresponding to the entries in the last 10 rows of the latin square can be decomposed into (C_4, C_6) for all admissible pairs.

A similar approach can be used to partition the last 10 rows of L_m in the case $m \equiv 1(mod\, 6)$ and $m \equiv 5(mod\, 6)$. Thus, the case $m - \ell = 10$ is done.

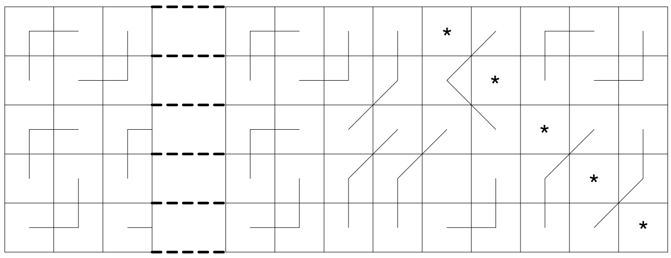

Figure 2. Partition of the Last 10 Rows of the Latin Square

Next, we consider the case $m - \ell = 14$. These 14 rows are made up of $7(M-1)$ subsquares where each subsquare corresponds to a 4-cycle and 7 partial latin square. Depending upon the value of m, the following 3 cases arise. when $m \equiv 1 \,(mod\, 6)$, then $q \geq 2$. Similarly, when $m \equiv 3 \,(mod\, 6)$, then $q \geq 0$ and when $m \equiv 5 \,(mod\, 6)$, then $q \geq 2$. For instance, consider the case $m - \ell \equiv 3(mod\, 6)$. The entries in these 14 rows can be partitioned into partial latin squares as shown in Figure 3. Observe

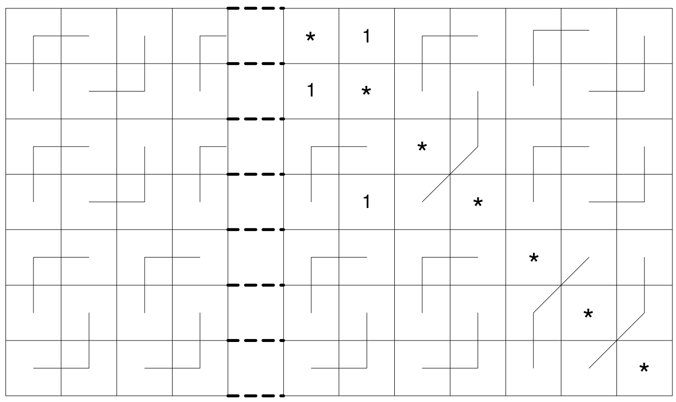

Figure 3. $m - \ell = 14$

that each of the partial latin square considered in Figure 3 corresponds to either three 4-cycles or two

6-cycles. Thus the edges corresponding to the last 14 rows of the latin square can be decomposed into copies of (C_4, C_6) for all admissible pairs.

The same approach can be used to partition the last 14 rows of the latin square in cases when $m \equiv 1(mod\,6)$ and $m \equiv 5(mod\,6)$.

In the case when $p = \ell m$, the edges corresponding to the last $m - \ell$ rows can be decomposed into (C_4, C_6) using edge trading as follows. The edges corresponding to the entries in the subsquare corresponds to 4-cycles and by grouping three 4-cycles with the above mentioned condition(4 columns and 4 entries) can be decomposed into two 6-cycles. The partial latin square together with corresponding column 0 entry corresponds to a 6-cycle. For some k, this partial latin square will be of the form:

	0	$2k-1$	$2k$
$2k-1$	$2k$	$2k-1$	0
$2k$	$2k-1$	0	$2k$

Two such 6-cycles can be decomposed into three 4-cycles as follows. For instance, consider $m-\ell = 6$, then the last 6 rows of the latin square will be of the form; See Table 4.

0	1	2	3	4	$\cdots$	$2M-5$	$2M-4$	$2M-3$	$2M-2$	$2M-1$	$2M$
$2M-4$	$M-2$	$M-1$	$2M-1$	$2M$	$\cdots$	$2M-5$	0	$M-4$	$M-3$	$2M-3$	$2M-2$
$2M-5$	$M-1$	$M-2$	$2M$	$2M-1$	$\cdots$	0	$2M-4$	$M-3$	$M-4$	$2M-2$	$2M-3$
$2M-2$	$2M-1$	$2M$	$2M-5$	$2M-4$	$\cdots$	$M-4$	$M-3$	$2M-3$	0	$M-2$	$M-1$
$2M-3$	$2M$	$2M-1$	$2M-4$	$2M-5$	$\cdots$	$M-3$	$M-4$	0	$2M-2$	$M-1$	$M-2$
$2M$	$2M-5$	$2M-4$	$M-4$	$M-3$	$\cdots$	$2M-3$	$2M-2$	$M-2$	$M-1$	$2M-1$	0
$2M-1$	$2M-4$	$2M-5$	$M-3$	$M-4$	$\cdots$	$2M-2$	$2M-3$	$M-1$	$M-2$	0	$2M$

Table 4. Last 6 Rows of $m - \ell$ Rows

Consider the highlighted entries in the above Table 4 which correspond to three 4-cycles and two 6-cycles. There are 24 edges corresponding to the considered entries and can be decomposed into 6 copies of C_4, given by, $(b_0, c_{2M-5}, b_{2M-5}, c_{2M-3})$, $(b_{2M-2}, c_0, b_{2M-5}, c_{2M-2})$, $(b_0, c_{2M-4}, b_{2M-4}, c_{2M-2})$, $(b_{2M-4}, c_{M-4}, b_{2M-5}, c_{M-3})$, $(b_{2M-3}, c_0, b_{2M-4}, c_{2M-3})$ and $(b_{2M-2}, c_{M-4}, b_{2M-3}, c_{M-3})$.

It is straightforward to check that similar edge trading is possible to have all possible (C_4, C_6) corresponding to the edges of the entries in these $(m - \ell)$ rows.

When $m - \ell > 14$, $m - \ell = 6x + 10y + 14z$ where $x, y, z \geq 0$ and the entries in the last $m - \ell$ rows of the latin square can be partitioned as above and the corresponding edges can be decomposed into (C_4, C_6). Thus, there exists a $\{C_3^p, C_4^q, C_6^r\}$-decomposition of $K_{\ell,m,m}$ with $p \leq \ell m$ and $m - \ell \equiv 2(mod\,4)$ when M is odd.

Similarly, when M is even, the entries in the last $m - \ell$ rows of the latin square can be grouped using the above mentioned conditions(4 columns and 4 entries).

Thus, there exists a $\{C_3^p, C_4^q, C_6^r\}$-decomposition of $K_{\ell,m,m}$ with $p \leq \ell m$ and $m - \ell \equiv 2(mod\,4)$.

Case 2. ℓ is even.

In order to prove this case, we consider a latin square of order m,

The first ℓ rows of the above latin square can be partitioned into 2×2 subsquares each of which correspond to $K_{2,2,2}$. Lemma 2 guarantees the existence of $3, 4$ and 6 cycle decomposition of $K_{2,2,2}$ for all admissible triplets. By the structure of the latin square, the edges corresponding to each 2×2 subsquare in the remaining $(m - \ell)$ rows give rise to C_4. As in previous case three 4 cycles can be used to construct two 6-cycles. Hence the proof of this lemma. □

Theorem 5. *The graph $K_{\ell,m,n}(\ell \leq m \leq n)$, admits a $\{C_3^p, C_4^q, C_6^r\}$-decomposition.*

Proof. The graph $K_{\ell,m,n} = K_{\ell,m,m} \bigoplus K_{\ell+m,n-m}$. By Lemmas 8, 9, 10, 12, there exists a $3, 4$ and 6 cycle decomposition of $K_{\ell,m,m}$ for all admissible triplets. Theorem 2 assures the existence of 4 and 6 cycle

1	2	3	4	$\cdots$	$m-1$	m
2	1	4	3	$\cdots$	m	$m-1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$\ell-1$	ℓ	$\ell+1$	$\ell+2$	$\cdots$	$\ell-2$	$\ell-3$
ℓ	$\ell-1$	$\ell+2$	$\ell+1$	$\cdots$	$\ell-3$	$\ell-2$
$\ell+1$	$\ell+2$	$\ell+3$	$\ell+4$	$\cdots$	$\ell-1$	ℓ
$\ell+2$	$\ell+1$	$\ell+4$	$\ell+3$	$\cdots$	ℓ	$\ell-1$
$\ell+3$	$\ell+4$	$\ell+5$	$\ell+6$	$\cdots$	$\ell+1$	$\ell+2$
$\ell+4$	$\ell+3$	$\ell+6$	$\ell+5$	$\cdots$	$\ell+2$	$\ell+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$m-1$	m	1	2	$\cdots$	$m-3$	$m-2$
m	$m-1$	2	1	$\cdots$	$m-2$	$m-3$

decomposition of $K_{\ell+m,n-m}$ for all m and n, where $n-m>2$. Hence we consider the case $n-m=2$ to complete the proof of this theorem.

Case 1. $m-\ell \equiv 0(mod\,4)$.

Consider the graph $K_{\ell,m,n}$ with $m-\ell \equiv 0(mod\,4)$. In order to prove this result, it is enough to consider the graph $K_{\ell,\ell+4,\ell+6}$. The graph $K_{\ell,\ell+4,\ell+6}$ can be represented using a partial latin square of order $\ell+6$, as shown in Figure 4. The first $\ell \times (\ell+4)$ entries form a latin rectangle. Entries outside

	1	2	3	4	$\cdots$	$\ell+3$	$\ell+4$	$\ell+5$	$\ell+6$
1	1	2	3	4	$\cdots$	$\ell+3$	$\ell+4$	$\ell+5$	$\ell+6$
2	2	3	4	5	$\cdots$	$\ell+4$	$\ell+5$	$\ell+6$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
ℓ	ℓ	$\ell+1$	$\ell+2$	$\ell+3$	$\cdots$	$\ell-4$	$\ell-3$	$\ell-2$	$\ell-1$
$\ell+1$	$\boldsymbol{\ell+1}$	$\boldsymbol{\ell+2}$	$\boldsymbol{\ell+3}$	$\ell+4$	$\cdots$	$\ell-3$	$\ell-2$		
$\ell+2$	$\boldsymbol{\ell+2}$	$\boldsymbol{\ell+3}$	$\boldsymbol{\ell+4}$	$\ell+5$	$\cdots$	$\ell-2$	$\ell-1$		
$\ell+3$	$\boldsymbol{\ell+3}$	$\ell+4$	$\ell+5$	$\ell+6$	$\cdots$	$\ell-1$	ℓ		
$\ell+4$	$\boldsymbol{\ell+4}$	$\ell+5$	$\ell+6$	1	$\cdots$	ℓ	$\ell+1$		
$\ell+5$	$\ell+5$	$\ell+6$	1	2	$\cdots$	$\boldsymbol{\ell+1}$	$\boldsymbol{\ell+2}$		
$\ell+6$	$\ell+6$	1	2	3	$\cdots$	$\boldsymbol{\ell+2}$	$\boldsymbol{\ell+3}$		

Figure 4. Partial Latin Square Corresponding to $K_{\ell,\ell+4,\ell+6}$

the latin rectangle are separated by double line. Each entry of column $\ell+5$ and $\ell+6$ denote an edge from partite set 1 to 3. Similarly, each entry of rows $\ell+1$ to $\ell+6$ denote an edge from partite set 2 to 3. That is, if the cell $(\ell, \ell+5)$ contains the entry $\ell+5$, then the corresponding edge is $a_\ell c_{\ell+5}$.

The edges corresponding to the latin rectangle can be decomposed into cycles of length 3, 4 and 6 for all admissible triplets depending upon p, q and r similar to Case 1 or Case 2 of Theorem 3.

Now, we consider the edges corresponding to the entries outside the latin rectangle (the remaining edges from partite set 1 to 3 and the edges from partite set 2 and 3). We decompose these edges into C_4 using two different construction which are as follows:

Construction 1. In this type of construction, we use the edges between partite set 1 to 3 and partite set 2 to 3 to construct a C_4. For example, consider the four underlined entries as shown in table below. These entries correspond to a C_4 namely $(a_1, c_{\ell+5}, b_1, c_{\ell+6})$ in $K_{\ell,\ell+4,\ell+6}$.

Construction 2. In this type of construction, we consider only the edges between the partite set 2 to

	1	$\ell+5$	$\ell+6$
1		$\underline{\underline{\ell+5}}$	$\underline{\underline{\ell+6}}$
$\ell+5$	$\underline{\underline{\ell+5}}$		
$\ell+6$	$\underline{\underline{\ell+6}}$		

3 to construct a C_4. For example, consider the four bold entries as shown in the table below. These entries also correspond to a C_4 namely, $(b_1, c_{\ell+3}, b_3, c_{\ell+4})$.

	1	3
$\ell+1$		$\ell+3$
$\ell+2$		$\ell+4$
$\ell+3$	$\ell+3$	
$\ell+4$	$\ell+4$	

Thus by using these two types of construction, all the remaining edges can be decomposed into 4-cycles. Thus, we have a C_4-decomposition of the remaining edges.

In order to obtain all possible 4 and 6-cycles, we use two different types of edge trading, say, Type 1 and Type 2.

Type 1. This edge trading is similar to Construction 1, where we use edges between partite set 1 to 3 and partite set 2 to 3. For instance, consider the entries in rectangular box shown in Table 4. These entries correspond to three 4-cycles $(a_1, c_{\ell+5}, b_1, c_{\ell+6})$, $(a_2, c_{\ell+6}, b_2, c_1)$ and $(b_2, c_{\ell+4}, b_4, c_{\ell+5})$ which can be decomposed into two copies of C_6 $(a_1, c_{\ell+5}, b_4, c_{\ell+4}, b_2, c_{\ell+6})$ and $(a_2, c_1, b_2, c_{\ell+5}, b_1, c_{\ell+6})$.

Type 2. This edge trading is similar to Construction 2, where we use only the edges between partite set 2 to 3. For instance, consider the bold entries in Table 4. These entries correspond to three 4-cycles $(b_1, c_{\ell+1}, b_{\ell+3}, c_{\ell+2})$, $(b_1, c_{\ell+3}, b_3, c_{\ell+4})$ and $(b_2, c_{\ell+2}, b_{\ell+4}, c_{\ell+3})$ which can then be decomposed into 2 copies of C_6 given by $(b_1, c_{\ell+1}, b_{\ell+3}, c_{\ell+2}, b_{\ell+4}, c_{\ell+3})$ and $(b_1, c_{\ell+2}, b_2, c_{\ell+3}, b_3, c_{\ell+4})$.

By using Type 1 and Type 2 edge trading, all the remaining edges can be decomposed into copies of (C_4, C_6).

Thus, all the remaining edges corresponding to the entries outside the latin rectangle can be decomposed into copies of 4 and 6 cycles.

Thus the graph $K_{\ell,m,n}$ with $m - \ell \equiv 0(mod\,4)$ admits a $\{C_3^p, C_4^q, C_6^r\}$-decomposition.

Case 2. $m - \ell = 2(mod\,4)$.

In this case, let $K_{\ell,m,m+2} = K_{\ell,m,m} \bigoplus K_{\ell+m,2}$. By Theorem 2, all the edges corresponding to $K_{\ell+m,2}$ can be decomposed into edge disjoint copies of C_4. In order to obtain cycles of length 6, we use edge trading. Let ℓ be even. In order to prove this result, it is enough to consider the graph $K_{\ell,\ell+2,\ell+4}$. Then the graph $K_{\ell,\ell+2,\ell+2}$ along with the entries corresponding to the bipartite graph $K_{2\ell+2,2}$ can be represented using the partial latin square of order $\ell + 4$. See Figure 5.

Similar to Case 1, the first $\ell\times(\ell+2)$ entries form a latin rectangle. Entries outside the latin rectangle are separated by double line. Each entry outside the latin rectangle represent a single edge. The edges corresponding to the entries in the latin rectangle can be decomposed into copies of 3-cycles, 4-cycles and 6-cycles similar to Case 1 of Theorem 3.

By the structure of the latin square, the edges corresponding to the entries in rows $\ell + 1$ and $\ell + 2$ can be decomposed into 4-cycles. Now in order to obtain all possible 4 and 6-cycles, we use the following edge trading.

Here, we take $\frac{\ell+2}{2}$ C_4 from $K_{\ell,\ell+2,\ell+2}$(the edges corresponding to the entries in the last 2 rows of the latin square $K_{\ell,\ell+2,\ell+2}$) together with the edges of $K_{2\ell+2,2}$ which can be then decomposed into 6-cycles. For instance, consider the highlighted entries in Table 5. The edges corresponding to these entries gives rise to a C_6 given by $(b_1, c_{\ell+1}, b_2, c_{\ell+3}, a_1, c_{\ell+4})$. Similarly, the entries in the rectangular box correspond to a C_6 given by $(b_1, c_{\ell+2}, b_2, c_{\ell+4}, a_2, c_{\ell+3})$. By proceeding this way, the remaining

	1	2	3	4	$\cdots$	$\ell+1$	$\ell+2$	$\ell+3$	$\ell+4$
1	1	2	3	4	$\cdots$			$\boldsymbol{\ell+3}$	$\boldsymbol{\ell+4}$
2	2	1	4	3	$\cdots$			$\ell+3$	$\ell+4$
3	3	4	5	6	$\cdots$			$\ell+3$	$\ell+4$
4	4	3	6	5	$\cdots$			$\ell+3$	$\ell+4$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\ell-1$	$\ell-1$	ℓ	$\ell+1$	$\ell+2$		$\ell-3$	$\ell-2$	$\ell+3$	$\ell+4$
ℓ	ℓ	$\ell-1$	$\ell+2$	$\ell+1$	$\cdots$	$\ell-2$	$\ell-3$	$\ell+3$	$\ell+4$
$\ell+1$	$\boldsymbol{\ell+1}$	$\ell+2$	$\ell+3$	$\ell+4$	$\cdots$	$\ell-1$	ℓ		
$\ell+2$	$\ell+2$	$\boldsymbol{\ell+1}$	$\ell+4$	$\ell+3$	$\cdots$	ℓ	$\ell-1$		
$\ell+3$	$\ell+3$	$\boldsymbol{\ell+3}$	$\ell+3$	$\ell+3$	$\cdots$	$\ell+3$	$\ell+3$		
$\ell+4$	$\boldsymbol{\ell+4}$	$\ell+4$	$\ell+4$	$\ell+4$	$\cdots$	$\ell+4$	$\ell+4$		

Figure 5. The Latin Square Corresponding to $K_{\ell,m,m} \bigoplus K_{\ell+m,n-m}$

edges can be decomposed into copies of C_6.

When ℓ is odd, the complete tripartite graph $K_{\ell,m,n}$ can be represented using a partial latin square similar to the even case where the edges corresponding to the entries in the latin rectangle can be decomposed into 3, 4 and 6 cycles similar to Case 2 of Theorem 3. The remaining edges corresponding to the entries outside the latin rectangle can be decomposed into 4 and 6 cycles using the above edge trading technique.

Thus the graph $K_{\ell,m,n}(\ell \leq m \leq n)$ can be decomposed into p copies of C_3, q copies of C_4 and r copies of C_6 for all admissible triplets (p, q, r). □

Theorem 1. The complete tripartite graph $K_{\ell,m,n}(\ell \leq m \leq n)$ admits a $\{C_3^p, C_4^q, C_6^r\}$-decomposition if and only if the partite sets are of same parity and $3p + 4q + 6r = \ell m + mn + \ell n$.

Proof. The proof follows from Lemma 7, Theorem 3, Theorem 4 and Theorem 5. □

4. Conclusion

In this paper, the necessary condition for the existence of $\{C_3^p, C_4^q, C_6^r\}$-decomposition of complete tripartite graph $K_{\ell,m,n}(\ell \leq m \leq n)$ has been proved to be sufficient. This answers the problem posted by Billington in the affirmative. The problem of $\{C_3^p, C_4^q, C_6^r\}$-decomposition of $K_m \circ \bar{K}_n$ is still open for $m > 3$.

Declaration of Competing Interest

There is no conflict of interest related to this work.

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