Ars Combinatoria, 160: 73–83
DOI:10.61091/ars-160-08
http://www.combinatorialpress.com/ars
Received 15 March 2020, Accepted 17 September 2020, Published 30 September 2024

Article

Decomposition of the λ-Fold Complete Equipartite Graph into Unicyclic Graphs of Order Five

T. Sivakaran[1,*]

[1] Department of Mathematics, Sri Sai Ram Engineering College, Sai Leo Nagar, West Tambaram 600 044, India

* **Correspondence:** shivaganesh1431991@gmail.com

Abstract: For a graph G and a subgraph H of a graph G, an H-decomposition of the graph G is a partition of the edge set of G into subsets E_i, $1 \leq i \leq k$, such that each E_i induces a graph isomorphic to H. In this paper, it is proved that every simple connected unicyclic graph of order five decomposes the λ-fold complete equipartite graph whenever the necessary conditions are satisfied. This generalizes a result of Huang, Utilitas Math. 97 (2015), 109–117.

Keywords: Decomposition, λ-fold equipartite graph
2020 Mathematics Subject Classification: 05C51, 05C12

---

1. Introduction

For a graph G and a positive integer λ, $G(\lambda)$ is the graph obtained from G by replacing each of its edges by λ parallel edges. Let C_k denote the cycle of length k. The complete graph on m vertices is denoted by K_m and its complement is denoted by $\overline{K}_m$. If $H_1, H_2, \ldots, H_k$ are edge-disjoint subgraphs of a graph G such that $E(G) = \bigcup_{i=1}^{k} E(H_i)$, then $H_1, H_2, \ldots, H_k$ *decompose* G; we write it as $G = H_1 \oplus H_2 \oplus \ldots \oplus H_k$. If each $H_i \cong H$, then G has an *H-decomposition* and we denote it by $H \,|\, G$. A graph G has a *C_k-decomposition* or a *k-cycle decomposition* whenever $C_k \,|\, G$.

For two graphs G and H their *wreath product*, denoted by $G \circ H$, has vertex set $V(G) \times V(H)$ in which two vertices (g_1, h_1) and (g_2, h_2) are adjacent whenever $g_1g_2 \in E(G)$ or, $g_1 = g_2$ and $h_1h_2 \in E(H)$; see Figure 1. Clearly, if $G = H_1 \oplus H_2 \oplus \ldots \oplus H_k$, then $G \circ \overline{K}_n = H_1 \circ \overline{K}_n \oplus H_2 \circ \overline{K}_n \oplus \ldots \oplus H_k \circ \overline{K}_n$. It can be observed that $K_m \circ \overline{K}_n$ is isomorphic to the complete m-partite graph in which each partite set has n vertices. For graphs G and H, and $x \in V(G)$, $x \times V(H) = \{(x, v) \,|\, v \in V(H)\}$ is called the *layer* of vertices of $G \circ H$ corresponding to x.

A *latin square* L of order n is an $n \times n$ array, each cell of which contains exactly one of the symbols in $\{0, 1, 2, \ldots, n-1\}$, such that each row and each column of the array contains each of the symbols in $\{0, 1, 2, \ldots, n-1\}$ exactly once, see [1]. A *quasigroup* of order n is a pair $(Q, *)$, where Q is a set of size n and $*$ is a binary operation on Q such that for every pair of elements $a, b \in Q$, the equations $a * x = b$ and $y * a = b$ have *unique* solutions. We consider a quasigroup is just a latin square with a headline and a sideline, see [1].

Let G be a bipartite graph with bipartition (X, Y), where $X = \{x_0, x_1, x_2, \ldots, x_{n-1}\}$, $Y = \{y_0, y_1, y_2, \ldots, y_{n-1}\}$; if G contains the set of edges $F_i(X, Y) = \{x_j y_{j+i} \,|\, 0 \leq j \leq n-1$, where addition in the subscript is taken modulo $n\}$, $0 \leq i \leq n-1$, then G has the 1-*factor*

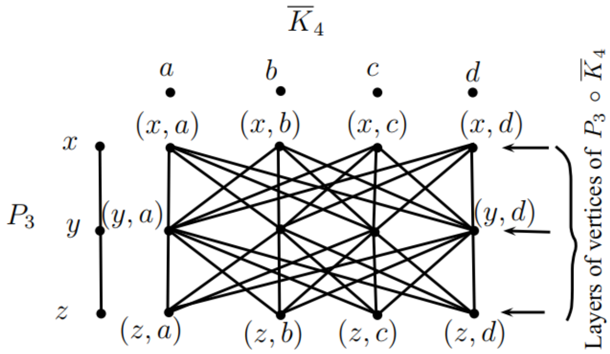

Figure 1. The Graph $P_3 \circ \overline{K}_4$.

of distance i from X to Y. It is important to note that for $0 \leq i \leq n-1$, $F_i(X, Y) = F_{n-i}(Y, X)$. An edge $e \in F_i(X, Y)$ is an edge of *distance i from X to Y* or it is an *edge of distance* $n - i$ *from Y to X*. Clearly, if $G = K_{n,n}$, then $E(G) = \bigcup_{i=0}^{n-1} F_i(X, Y)$.

We denote the graphs of Figure 2 by H_i, $1 \leq i \leq 4$ and C_5. For all i such that $1 \leq i \leq 4$, H_i has the vertex set $\{a, b, c, d, e\}$. The graph H_1 with the edge set $\{ab, bc, ca, bd, ce\}$ is denoted by $((a, b, c); bd, ce)$ or $(C; bd, ce)$, where C denotes the cycle (a, b, c); the graph H_2 with the edge set $\{ab, bc, ca, cd, ce\}$ is denoted by $((a, b, c); cd, ce)$ or $(C; cd, ce)$, where C denotes the cycle (a, b, c); the graph H_3 with the edge set $\{ab, bc, ca, cd, de\}$ is denoted by $((a, b, c); cd, de)$ or $(C; cd, de)$, where C denotes the cycle (a, b, c); the graph H_4 with the edge set $\{ab, bc, cd, da, de\}$ is denoted by $((a, b, c, d); de)$ or $(C; de)$, where C denotes the cycle (a, b, c, d) and the cycle C_5 with the edge set $\{ab, bc, cd, de, ea\}$ is denoted by (a, b, c, d, e).

In the future, for $1 \leq i \leq 4$, H_i, stands for the graphs in Figure 2.

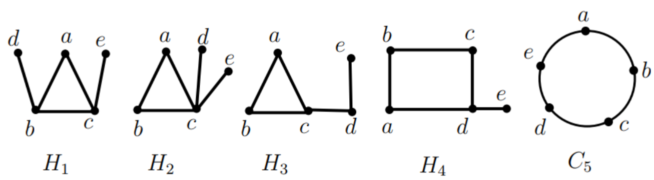

Figure 2

Decomposition of a graph into a specified subgraph is an interesting area of research in graph theory. In particular K_k-decomposition of K_n (*BIBD*) has received much attention, see [2]. The K_3-design of order n is known as the Steiner triple system. Decompositions of K_n into complete subgraphs, complete bipartite graphs, complete equipartite graphs, linear forests have been studied, see [3–6]. Decomposition of $(K_m \circ \overline{K}_n)(\lambda)$ (*GDD*) into K_3 (resp. K_4) is studied in [7, 8]. Cycle decompositions of the graphs $K_n(\lambda)$, $K_n - F$, where F is a perfect matching of K_n, $K_{n,m}(\lambda)$ and $(K_m \circ \overline{K}_n)(\lambda)$ are considered in [9–13].

Bermond et al. [14] studied the decompositions of complete graphs into isomorphic subgraphs with five vertices. Further, Bermond and Schönheim [15] obtained G-decompositions of K_n, where G has four vertices or less. Moreover, in [16], Huang obtained decompositions of the complete equipartite graphs into connected unicyclic graphs of size five. Here we obtain decompositions of the λ-fold complete equipartite graphs into connected unicyclic graphs of size five, whenever the necessary conditions are satisfied. This generalizes a result of Huang [16].

The main result of this paper is the following:

Theorem 1. *If m and n are at least* 3, *then for* $1 \leq i \leq 4$, $H_i \mid (K_m \circ \overline{K}_n)(\lambda)$ *if and only if* $5 \mid \lambda nm(m-1)$.

2. Decompositions of λ-Fold Complete Equipartite Graph Into Unicylic Graphs

In this section, we prove that every simple connected unicyclic graph on five vertices decomposes the graph $(K_m \circ \overline{K}_n)(\lambda)$, whenever the necessary conditions are satisfied.

Lemma 1. *If $n \geq 3$ and $H_i \,|\, G$, $1 \leq i \leq 3$, then H_i decomposes the graph $G \circ \overline{K}_n$.*

Proof. Consider the graph $G \circ \overline{K}_n = (H_i \oplus H_i \oplus \cdots \oplus H_i) \circ \overline{K}_n = H_i \circ \overline{K}_n \oplus H_i \circ \overline{K}_n \oplus \cdots \oplus H_i \circ \overline{K}_n$. We need to prove that for all i such that $1 \leq i \leq 3$, $H_i \,|\, H_i \circ \overline{K}_n$. Let $(L, *)$ be a quasigroup of order n, where $L = \{0, 1, 2, \ldots, n-1\}$. Let the vertices of H_i be as shown in Figure 2 and let the vertex set of $\overline{K}_n$ be $\{0, 1, 2, \ldots, n-1\}$. Let $\{(a, j); 0 \leq j \leq n-1\}$ be the layer of $H_i \circ \overline{K}_n$ corresponding to the vertex a in $V(H_i)$. Then the graphs in $\{(((a,\ell),(b,k),(c,\ell * k)); (b,k)(d,\ell),(c,\ell * k)(e,\ell)) \,|\, \forall\, \ell, k \in L\}$, each one of them is isomorphic to H_1, decompose the graph $H_1 \circ \overline{K}_n$, the graphs in $\{(((a,\ell),(b,k),(c,\ell * k)); (c,\ell * k)(d,\ell),(c,\ell * k)(e,\ell)) \,|\, \forall\, \ell, k \in L\}$, each one of them is isomorphic to H_2, decompose the graph $H_2 \circ \overline{K}_n$ and the graphs in $\{(((a,\ell),(b,k),(c,\ell * k)); (c,\ell * k)(d,\ell),(d,\ell)(e,k)) \,|\, \forall\, \ell, k \in L\}$, each one of them is isomorphic to H_3, decompose the graph $H_3 \circ \overline{K}_n$. □

Lemma 2. *If $n \geq 2$ and $H_4 \,|\, G$, then H_4 decomposes the graph $G \circ \overline{K}_n$.*

Proof. Consider the graph $G \circ \overline{K}_n = (H_4 \oplus H_4 \oplus \cdots \oplus H_4) \circ \overline{K}_n = H_4 \circ \overline{K}_n \oplus H_4 \circ \overline{K}_n \oplus \ldots \oplus H_4 \circ \overline{K}_n$. We need to prove that $H_4 \,|\, H_4 \circ \overline{K}_n$. Let $(L, *)$ be a quasigroup of order n, where $L = \{0, 1, 2, \ldots, n-1\}$. Let the vertices of H_4 be as shown in Figure 2 and let the vertex set of $\overline{K}_n$ be $\{0, 1, 2, \ldots, n-1\}$. Let $\{(a, j); 0 \leq j \leq n-1\}$ be the layer of $H_4 \circ \overline{K}_n$ corresponding to the vertex a in $V(H_4)$. Then the graphs in $\{(((a,\ell),(b,k),(c,\ell),(d,k)); (d,k)(e,\ell * k)) \,|\, \forall\, \ell, k \in L\}$, each one of them is isomorphic to H_4, decompose the graph $H_4 \circ \overline{K}_n$. □

Lemma 3. $K_4 \setminus \{e\} \,|\, K_4(5)$*, where e is an edge of K_4.*

Proof. Let $V(K_4) = \{a, b, c, d\}$. A $K_4 \setminus \{e\}$ decomposition of $K_4(5)$ is given by the edge induced subgraphs $\langle bc, cd, da, ac, bd \rangle$, $\langle ab, cd, da, ac, bd \rangle$, $\langle ab, bc, da, ac, bd \rangle$, $\langle ab, bc, cd, ac, bd \rangle$, $\langle ab, bc, cd, da, bd \rangle$, $\langle ab, bc, cd, da, ac \rangle$. □

Lemma 4. *For $i \in \{1, 3, 4\}$, H_i decomposes the graph $(K_4 \circ \overline{K}_n)(5)$.*

Proof. Let $V(K_4) = \{a, b, c, d\}$ and let $V(\overline{K}_n) = \{0, 1, 2, \ldots, n-1\}$. By Lemma 3, $K_4 \setminus \{e\} \,|\, K_4(5)$; hence it is enough to prove that for $i \in \{1, 3, 4\}$, $H_i \,|\, (K_4 \setminus \{e\}) \circ \overline{K}_n$. Let the edge $e = ad$. Let $(L, *)$ be a quasigroup of order n, where $L = \{0, 1, 2, \ldots, n-1\}$. We have $V((K_4 \setminus \{e\}) \circ \overline{K}_n) = \bigcup_{j=0}^{n-1} \{(a, j), (b, j), (c, j), (d, j)\}$. Then the graphs in $\{(((a,\ell),(b,k),(c,\ell * k)); (b,k)(d,\ell),(c,\ell * k)(d,\ell + 1)) \,|\, \forall\, \ell, k \in L\}$, each one of them is isomorphic to H_1, decompose the graph $(K_4 \setminus \{e\}) \circ \overline{K}_n$, the graphs in $\{(((a,\ell),(b,k),(c,\ell * k)); (c,\ell * k)(d,\ell),(d,\ell)(b,k+1)) \,|\, \forall\, \ell, k \in L\}$, each one of them is isomorphic to H_3, decompose the graph $(K_4 \setminus \{e\}) \circ \overline{K}_n$ and the graphs in $\{(((a,\ell),(b,k),(d,\ell * k),(c,k)); (c,k)(b,k+1)) \,|\, \forall\, \ell, k \in L\}$, each one of them is isomorphic to H_4, decompose the graph $(K_4 \setminus \{e\}) \circ \overline{K}_n$. □

For the rest of the paper, we fix the *layers* of the graph $G \circ \overline{K}_m$ as follows: let $V(G) = \{a, b, c, d, \ldots, w, x\}$ and let $V(\overline{K}_m) = \{0, 1, 2, \ldots, m-1\}$. Then $V(G \circ \overline{K}_m) = V(G) \times V(\overline{K}_m) = \{a \times V(\overline{K}_m)\} \cup \{b \times V(\overline{K}_m)\} \cup \{c \times V(\overline{K}_m)\} \cup \cdots \cup \{x \times V(\overline{K}_m)\}$. For convenience, we write $A = a \times V(\overline{K}_m) = \{(a, 0), (a, 1), (a, 2), \ldots, (a, m-1)\} = \{a_0, a_1, a_2, \ldots, a_{m-1}\}$, where for all i such that $0 \leq i \leq m-1$, a_i, denotes the vertex (a, i). Similarly, $B, C, \ldots, X$ are defined. $A, B, C, \ldots, X$ are the layers of $G \circ \overline{K}_m$, see Figure 1.

Lemma 5. *For $n \geq 2$, the graph $(K_4 \circ \overline{K}_n)(5)$ has an H_2-decomposition.*

Proof. We complete the proof in two cases.

Case 1. n is odd.

Let $V(K_4) = \{a, b, c, d\}$ and let $V(\overline{K}_n) = \{0, 1, 2, \ldots, n-1\}$. By Lemma 3, $K_4 \setminus \{e\} \,|\, K_4(5)$ and hence it is enough to to prove that $H_2 \,|\, (K_4 \setminus \{e\}) \circ \overline{K}_n$. Let the edge $e = ad$. Let σ be the cyclic permutation $(0, 1, 2, 3, \ldots, n-1)$ on $\{0, 1, 2, \ldots, n-1\}$.

Subcase 1.1. $n = 3$.

Let $H_2^1 = ((a_0, b_0, c_0); c_0a_2, c_0d_1)$, $H_2^2 = ((a_0, b_1, c_2); b_1a_2, b_1d_0)$ and $H_2^3 = ((b_0, c_2, d_1); d_1c_1, d_1b_1)$ be three edge-disjoint copies of H_2 in $(K_4 \setminus \{e\}) \circ \overline{K}_3$. Then the graphs in $\{\sigma^0(H_2^j) = H_2^j, \sigma^1(H_2^j), \sigma^2(H_2^j) \,|\, 1 \le j \le 3\}$, each one of them is isomorphic to H_2, decompose the graph $(K_4 \setminus \{e\}) \circ \overline{K}_3$, where σ^i acts on the subscripts of the vertices of H_2^j.

Subcase 1.2. $n \ge 5$.

For all i such that $0 \le i \le (n-3)/2$, let $H_2^i = ((a_0, b_i, c_{2i}); c_{2i}a_{n-1}, c_{2i}d_{3i+1})$; for $i = (n-1)/2$, let $H_2^i = ((a_0, b_{\frac{n-1}{2}}, c_{n-1}); b_{\frac{n-1}{2}}a_{\frac{n+1}{2}}, b_{\frac{n-1}{2}}d_{\frac{n-3}{2}})$; for $i = (n+1)/2$, let $H_2^i = ((b_0, c_{\frac{n+1}{2}}, d_1); d_1c_1, d_1b_1)$ and for all i such that $(n+3)/2 \le i \le n-1$, let $H_2^i = ((b_0, c_i, d_{2i}); b_0a_{i-\frac{n-1}{2}}, b_0d_{2i-n-1})$, where the subscripts are taken modulo n, see Figure 3. Then the graphs in $\{\sigma^i(H_2^j) \,|\, 0 \le i, j \le n-1\}$, each one of them is isomorphic to H_2, decompose the graph $(K_4 \setminus \{e\}) \circ \overline{K}_n$. The (base) graphs H_2^i, $0 \le i \le n-1$ are described in Figure 3.

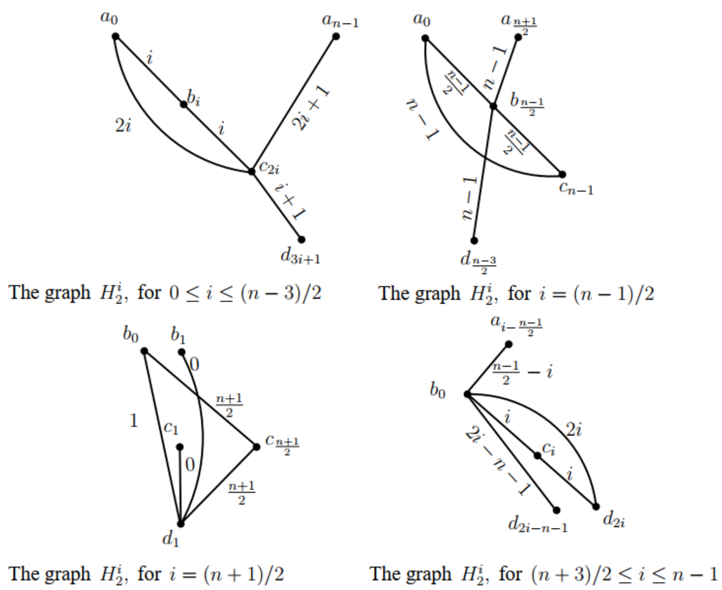

Figure 3. The Labels on the Edges of the Graphs Denote the Distances of the Respective Edges in the Bipartite Subgraphs $\langle A \cup B \rangle$, $\langle B \cup C \rangle$, $\langle A \cup C \rangle$, $\langle B \cup D \rangle$ and $\langle C \cup D \rangle$ of $(K_4 \setminus \{e\}) \circ \overline{K}_n$; in Each of the Graphs, the Distances Are Computed from A to B, B to C, A to C, B to D and C to D. From the Union of These Graphs, It Is Clear That the Edges of Distance i, $0 \le i \le n-1$, from A to B, B to C, A to C, B to D and C to D are all Present Exactly Once. Consequently, When We Apply the Permutation σ^i, to the Above Bipartite Graphs Yield a Required Decomposition

Case 2. n is even.

The graph $(K_4 \circ \overline{K}_n)(5) = ((K_4 \circ \overline{K}_2) \circ \overline{K}_{\frac{n}{2}})(5) \cong (K_4 \circ \overline{K}_2)(5) \circ \overline{K}_{\frac{n}{2}}$. We shall prove that $H_2 \,|\, (K_4 \circ \overline{K}_2)(5)$ and apply Lemma 1. Let $V(K_4) = \{a, b, c, d\}$ and let $V(\overline{K}_2) = \{0, 1\}$. Let $\sigma = (0, 1)$ be a permutation on $\{0, 1\}$. Let $H_2^1 = ((a_0, b_0, c_1); b_0c_0, b_0d_1)$, $H_2^2 = ((b_0, c_0, d_1); b_0a_0, b_0a_1)$, $H_2^3 = ((a_1, c_0, d_0); d_0a_0, d_0b_1)$, $H_2^4 = ((a_0, b_1, d_0); a_0c_1, a_0d_1)$, $H_2^5 = ((a_0, c_0, d_1);\ d_1a_1, d_1b_1)$, $H_2^6 = ((a_1, b_0, d_0); a_1c_0, a_1d_1)$, $H_2^7 = ((a_1, b_0, c_1); c_1a_0, c_1d_0)$, $H_2^8 = ((b_0, c_1, d_0); c_1a_1, c_1d_1)$, $H_2^9 = ((b_1, c_0, d_1); b_1a_1, b_1c_1)$, $H_2^{10} = ((b_0, c_0, d_0);\ d_0a_1, d_0b_1)$, $H_2^{11} = ((a_0, b_0, c_0); c_0b_1, c_0d_0)$ and $H_2^{12} =$

$((a_0, c_0, d_0); a_0b_0, a_0b_1)$ be twelve edge-disjoint copies of H_2 in $(K_4 \circ \overline{K}_2)(5)$. Then the graphs in $\{\sigma^0(H_2^j) = H_2^j, \sigma^1(H_2^j) \,|\, 1 \leq j \leq 12\}$, each one of them is isomorphic to H_2, decompose the graph $(K_4 \circ \overline{K}_2)(5)$. □

Lemma 6. *For $n \geq 2$, the graph $(K_3 \circ \overline{K}_n)(5)$ has an H_1-decomposition.*

Proof. We complete the proof in two cases.
Case 1. n is odd.

Let $V(K_3) = \{a, b, c\}$ and let $V(\overline{K}_n) = \{0, 1, 2, 3, \ldots, n-1\}$. Let $\sigma = (0, 1, 2, 3, \ldots, n-1)$ be a permutation on $\{0, 1, 2, 3, \ldots, n-1\}$. For $0 \leq i \leq n-1$, let $H_1^i = ((a_0, b_i, c_{2i}); a_0b_{i+1}, b_ia_{n-2})$; for $0 \leq i \leq n-1$, let $H_1^{n+i} = ((a_0, b_i, c_{2i}); b_ic_{2i+1}, c_{2i}b_{(n+i)-2})$ and for $0 \leq i \leq n-1$, let $H_1^{2n+i} = ((a_0, b_i, c_{2i}); a_0c_{2i+2}, c_{2i}a_{n-1})$, where the subscripts are taken modulo n, see Figure 4. Then the graphs in $\{\sigma^i(H_1^j) \,|\, 0 \leq i \leq n-1, 0 \leq j \leq 3n-1\}$, each one of them is isomorphic to H_1, decompose the graph $(K_3 \circ \overline{K}_n)(5)$, where σ^i acts on the subscripts of the vertices of H_1^j. The (base) graphs H_1^i, totally $3n$ in number, with the distances of their edges are shown in the graph of Figure 4.

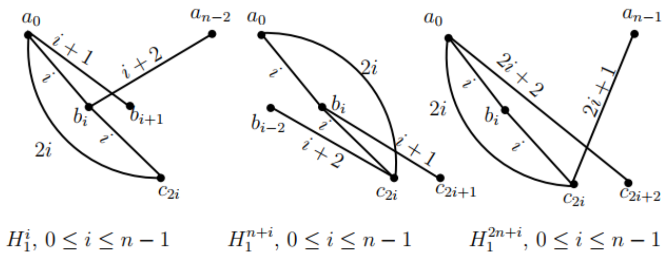

Figure 4. In the Union of the Above Graphs Each Edge of Distance i, $0 \leq i \leq n-1$, from A to B, B to C and A to C Occurs Exactly Five Times

Case 2. n is even.

The graph $(K_3 \circ \overline{K}_n)(5) = ((K_3 \circ \overline{K}_2) \circ \overline{K}_{\frac{n}{2}})(5) \cong (K_3 \circ \overline{K}_2)(5) \circ \overline{K}_{\frac{n}{2}}$. We shall prove that $H_1 \,|\, (K_3 \circ \overline{K}_2)(5)$ and apply Lemma 1. Let $V(K_3) = \{a, b, c\}$ and let $V(\overline{K}_2) = \{0, 1\}$. Let $\sigma = (0, 1)$ be a permutation on $\{0, 1\}$. Let $H_1^1 = ((a_0, b_0, c_1); b_0c_0, a_0b_1)$, $H_1^2 = ((a_1, b_0, c_0); c_0a_0, b_0c_1)$, $H_1^3 = ((a_0, b_1, c_0); a_0c_1, b_1a_1)$, $H_1^4 = ((a_0, b_0, c_0); b_0a_1, a_0b_1)$, $H_1^5 = ((a_0, b_0, c_0); c_0b_1, b_0c_1)$ and $H_1^6 = ((a_0, b_0, c_0); a_0c_1, c_0a_1)$ be six edge-disjoint copies of H_1 in $(K_3 \circ \overline{K}_2)(5)$. Then the graphs in $\{\sigma^i(H_1^j) \,|\, 0 \leq i \leq 1, 1 \leq j \leq 6\}$, each one of them is isomorphic to H_1, decompose the graph $(K_3 \circ \overline{K}_2)(5)$. □

Lemma 7. *For $n \geq 2$, the graph $(K_3 \circ \overline{K}_n)(5)$ has an H_2-decomposition.*

Proof. We complete the proof in two cases.
Case 1. n is odd.

Let $V(K_3) = \{a, b, c\}$ and let $V(\overline{K}_n) = \{0, 1, 2, 3, \ldots, n-1\}$. Let $\sigma = (0, 1, 2, 3, \ldots, n-1)$ be a permutation on $\{0, 1, 2, 3, \ldots, n-1\}$. For $0 \leq i \leq n-1$, let $H_2^i = ((a_0, b_i, c_{2i}); a_0b_{i+1}, a_0b_{i+2})$; for $0 \leq i \leq n-1$, let $H_2^{n+i} = ((a_0, b_i, c_{2i}); b_ic_{2i+1}, b_ic_{2i+2})$ and for $0 \leq i \leq n-1$, let $H_2^{2n+i} = ((a_0, b_i, c_{2i}); c_{2i}a_{n-2}, c_{2i}a_{n-1})$, where the subscripts are taken modulo n, see Figure 5. Then the graphs in $\{\sigma^i(H_2^j) \,|\, 0 \leq i \leq n-1, 0 \leq j \leq 3n-1\}$, each one of them is isomorphic to H_2, decompose the graph $(K_3 \circ \overline{K}_n)(5)$, where σ^i acts on the subscripts of the vertices of H_2^j. The (base) graphs H_2^i, totally $3n$ in number, with the distances of their edges are shown in the graph of Figure 5.
Case 2. n is even.

The graph $(K_3 \circ \overline{K}_n)(5) = ((K_3 \circ \overline{K}_2) \circ \overline{K}_{\frac{n}{2}})(5) \cong (K_3 \circ \overline{K}_2)(5) \circ \overline{K}_{\frac{n}{2}}$. We shall prove that $H_2 \,|\, (K_3 \circ \overline{K}_2)(5)$ and apply Lemma 1. Let $V(K_3) = \{a, b, c\}$ and let $V(\overline{K}_2) = \{0, 1\}$. Let $\sigma = (0, 1)$ be a permutation on $\{0, 1\}$. Let $H_2^1 = ((a_0, b_0, c_1); a_0c_0, a_0b_1)$, $H_2^2 = ((a_1, b_0, c_0); b_0a_0, b_0c_1)$, $H_2^3 = ((a_0, b_1, c_0); c_0b_0, c_0a_1)$, $H_2^4 = ((a_0, b_0, c_0); a_0b_1, a_0c_1)$, $H_2^5 = ((a_0, b_0, c_0); b_0a_1, b_0c_1)$ and

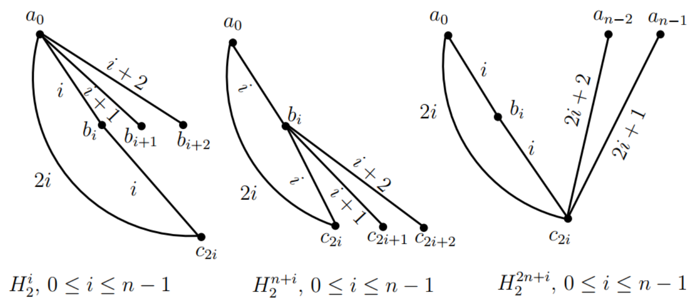

Figure 5. In the Union of the Above Graphs Each Edge of Distance i, $0 \le i \le n-1$, from A to B, B to C and A to C Occurs Exactly Five Times

$H_2^6 = ((a_0, b_0, c_0); c_0a_1, c_0b_1)$ be six edge-disjoint copies of H_2 in $(K_3 \circ \overline{K}_2)(5)$. Then the graphs in $\{\sigma^i(H_2^j) \,|\, 0 \le i \le 1,\ 1 \le j \le 6\}$, each one of them is isomorphic to H_2, decompose the graph $(K_3 \circ \overline{K}_2)(5)$. □

Lemma 8. *For $n \ge 2$, the graph $(K_3 \circ \overline{K}_n)(5)$ has an H_3-decomposition.*

Proof. We complete the proof in two cases.
Case 1. n is odd.

Let $V(K_3) = \{a, b, c\}$ and let $V(\overline{K}_n) = \{0, 1, 2, 3, \ldots, n-1\}$. Let $\sigma = (0, 1, 2, 3, \ldots, n-1)$ be a permutation on $\{0, 1, 2, 3, \ldots, n-1\}$. For $0 \le i \le n-1$, let $H_3^i = ((a_0, b_i, c_{2i}); a_0b_{i+1}, b_{i+1}a_{n-1})$; for $0 \le i \le n-1$, let $H_3^{n+i} = ((a_0, b_i, c_{2i}); b_ic_{2i+1}, c_{2i+1}b_{i-1})$ and for $0 \le i \le n-1$, let $H_3^{2n+i} = ((a_0, b_i, c_{2i}); c_{2i}a_{n-1}, a_{n-1}c_{2i+1})$, where the subscripts are taken modulo n, see Figure 6. Then the graphs in $\{\sigma^i(H_3^j) \,|\, 0 \le i \le n-1,\ 0 \le j \le 3n-1\}$, each one of them is isomorphic to H_3, decompose the graph $(K_3 \circ \overline{K}_n)(5)$, where σ^i acts on the subscripts of the vertices of H_3^j. The (base) graphs H_3^i, totally $3n$ in number, with the distances of their edges are shown in the graph of Figure 6.

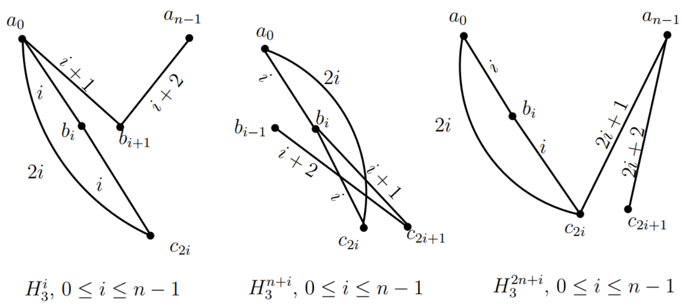

Figure 6. In the Union of the Above Graphs Each Edge of Distance i, $0 \le i \le n-1$, from A to B, B to C and A to C Occurs Exactly Five Times

Case 2. n is even.

The graph $(K_3 \circ \overline{K}_n)(5) = ((K_3 \circ \overline{K}_2) \circ \overline{K}_{\frac{n}{2}})(5) \cong (K_3 \circ \overline{K}_2)(5) \circ \overline{K}_{\frac{n}{2}}$. We shall prove that $H_3 \,|\, (K_3 \circ \overline{K}_2)(5)$ and apply Lemma 1. Let $V(K_3) = \{a, b, c\}$ and let $V(\overline{K}_2) = \{0, 1\}$. Let $\sigma = (0, 1)$ be a permutation on $\{0, 1\}$. Let $H_3^1 = ((a_0, b_0, c_1); a_0b_1, b_1c_0)$, $H_3^2 = ((a_1, b_0, c_0); b_0c_1, c_1a_0)$, $H_3^3 = ((a_0, b_1, c_0); c_0a_1, a_1b_0)$, $H_3^4 = ((a_0, b_0, c_0); a_0b_1, b_1a_1)$, $H_3^5 = ((a_0, b_0, c_0); b_0c_1, c_1b_1)$ and $H_3^4 = ((a_0, b_0, c_0); c_0a_1, a_1c_1)$ be six edge-disjoint copies of H_3 in $(K_3 \circ \overline{K}_2)(5)$. Then the graphs in $\{\sigma^i(H_3^j) \,|\, 0 \le i \le 1,\ 1 \le j \le 6\}$, each one of them is isomorphic to H_3, decompose the graph $(K_3 \circ \overline{K}_2)(5)$. □

Lemma 9. *For $n \ge 2$, the graph $(K_3 \circ \overline{K}_n)(5)$ has an H_4-decomposition.*

Proof. We complete the proof in two cases.

Case 1. n is odd.

Let $V(K_3) = \{a, b, c\}$ and let $V(\overline{K}_n) = \{0, 1, 2, 3, \ldots, n-1\}$.

Let $\sigma = (0, 1, 2, 3, \ldots, n-1)$ be a permutation on $\{0, 1, 2, 3, \ldots, n-1\}$.

Subcase 1.1. $n \equiv 3 \pmod 4$.

For all i such that $2 \leq i \leq (n-1)/2$, let $H_4^{6(i-2)} = ((b_i, c_0, b_{n-i}, a_0); a_0b_x)$, $H_4^{6(i-2)+1} = ((b_i, c_0, b_{n-i}, a_0); a_0b_y)$, $H_4^{6(i-2)+2} = ((c_{n-i}, a_0, c_i, b_0); b_0c_x)$, $H_4^{6(i-2)+3} = ((c_{n-i}, a_0, c_i, b_0); b_0c_y)$, $H_4^{6(i-2)+4} = ((a_{n-i}, b_0, a_i, c_0); c_0a_x)$ and $H_4^{6(i-2)+5} = ((a_{n-i}, b_0, a_i, c_0); c_0a_y)$, where

$$x = \begin{cases} i+1 & \text{if } i \neq (n-1)/2, \\ 2 & \text{if } i = (n-1)/2. \end{cases}, \quad y = \begin{cases} n-i-1 & \text{if } i \neq (n-1)/2, \\ n-2 & \text{if } i = (n-1)/2. \end{cases}$$

and the subscripts are taken modulo n; let $H_4^{3n-9} = ((a_0, b_1, a_1, b_0); b_0c_1)$, $H_4^{3n-8} = ((a_0, b_1, a_1, b_0); b_0c_{n-1})$, $H_4^{3n-7} = ((b_0, c_1, b_1, c_0); c_0a_1)$, $H_4^{3n-6} = ((b_0, c_1, b_1, c_0); c_0a_{n-1})$, $H_4^{3n-5} = ((c_0, a_1, c_1, a_0); a_0b_1)$, $H_4^{3n-4} = ((c_0, a_1, c_1, a_0); a_0b_{n-1})$, $H_4^{3n-3} = ((b_1, a_0, b_{n-1}, c_0); c_0b_0)$, $H_4^{3n-2} = ((b_0, c_{n-1}, a_0, c_1); c_1a_1)$ and $H_4^{3n-1} = ((a_1, c_0, a_{n-1}, b_0); b_0a_0)$. Then the graphs in $\{\sigma^i(H_4^j) \,|\, 0 \leq i \leq n-1,\ 0 \leq j \leq 3n-1\}$, each one of them is isomorphic to H_4, decompose the graph $(K_3 \circ \overline{K}_n)(5)$, where σ^i acts on the subscripts of the vertices of H_4^j.

The (base) graphs H_4^i, totally $3n$ in number, with the distances of their edges are shown in the graphs of Figure 7 and Figure 8 of the appendix. In the union of the graphs of Figure 8, the edges with distances in $\{2, 3, 4, \ldots, n-3, n-2\}$ from A to B, B to C and A to C appear exactly five times. In the union of the graphs of Figure 7, the edges with distances $0, 1$ and $n-1$ from A to B, B to C and A to C appear exactly five times.

Subcase 1.2. $n \equiv 1 \pmod 4$.

For all i such that $3 \leq i \leq (n-1)/2$, let $H_4^{6(i-3)} = ((b_i, c_0, b_{n-i}, a_0); a_0b_x)$, $H_4^{6(i-3)+1} = ((b_i, c_0, b_{n-i}, a_0); a_0b_y)$, $H_4^{6(i-3)+2} = ((c_{n-i}, a_0, c_i, b_0); b_0c_x)$, $H_4^{6(i-3)+3} = ((c_{n-i}, a_0, c_i, b_0); b_0c_y)$, $H_4^{6(i-3)+4} = ((a_{n-i}, b_0, a_i, c_0); c_0a_x)$ and $H_4^{6(i-3)+5} = ((a_{n-i}, b_0, a_i, c_0); c_0a_y)$, where

$$x = \begin{cases} i+1 & \text{if } i \neq (n-1)/2, \\ 3 & \text{if } i = (n-1)/2. \end{cases}, \quad y = \begin{cases} n-i-1 & \text{if } i \neq (n-1)/2, \\ n-3 & \text{if } i = (n-1)/2. \end{cases}$$

and the subscripts are taken modulo n; let $H_4^{3n-15} = ((b_0, a_1, b_1, a_0); a_0b_2)$, $H_4^{3n-14} = ((b_0, a_1, b_1, a_0); a_0b_{n-2})$, $H_4^{3n-13} = ((c_0, b_1, c_1, b_0); b_0c_2)$, $H_4^{3n-12} = ((c_0, b_1, c_1, b_0); b_0c_{n-2})$, $H_4^{3n-11} = ((a_0, c_0, a_1, c_1); c_1a_{n-1})$, $H_4^{3n-10} = ((a_0, c_0, a_1, c_1); c_1a_3)$, $H_4^{3n-9} = ((b_1, a_0, b_{n-1}, c_0); c_0b_0)$, $H_4^{3n-8} = ((b_0, c_{n-1}, a_0, c_1); c_1a_1)$, $H_4^{3n-7} = ((a_1, c_0, a_{n-1}, b_0); b_0a_0)$, $H_4^{3n-6} = ((b_2, c_0, b_{n-2}, a_0); a_0b_1)$, $H_4^{3n-5} = ((b_2, c_0, b_{n-2}, a_0); a_0b_{n-1})$, $H_4^{3n-4} = ((c_{n-2}, a_0, c_2, b_0); b_0c_1)$, $H_4^{3n-3} = ((c_{n-2}, a_0, c_2, b_0); b_0c_{n-1})$, $H_4^{3n-2} = ((a_{n-2}, b_0, a_2, c_0); c_0a_1)$ and $H_4^{3n-1} = ((a_{n-2}, b_0, a_2, c_0); c_0a_{n-1})$. Then the graphs in $\{\sigma^i(H_4^j) \,|\, 0 \leq i \leq n-1,\ 0 \leq j \leq 3n-1\}$, each one of them is isomorphic to H_4, decompose the graph $(K_3 \circ \overline{K}_n)(5)$.

The (base) graphs H_4^i, totally $3n$ in number, with the distances of their edges are shown in the graphs of Figure 9 and Figure 10 of the appendix. In the union of the graphs of Figure 9, the edges with distances in $\{3, 4, \ldots, n-4, n-3\}$ from A to B, B to C and A to C appear exactly five times. In the union of the graphs of Figure 10, the edges with distances $0, 1, 2, n-1$ and $n-2$ from A to B, B to C and A to C appear exactly five times.

Case 2. n is even.

The graph $(K_3 \circ \overline{K}_n)(5) = ((K_3 \circ \overline{K}_2) \circ \overline{K}_{\frac{n}{2}})(5) \cong (K_3 \circ \overline{K}_2)(5) \circ \overline{K}_{\frac{n}{2}}$. We shall prove that $H_4 \,|\, (K_3 \circ \overline{K}_2)(5)$ and apply Lemma 2. Let $V(K_3) = \{a, b, c\}$ and let $V(\overline{K}_2) = \{0, 1\}$. Let $\sigma = (0, 1)$ be a permutation on $\{0, 1\}$. Let $H_4^1 = ((b_0, a_1, b_1, a_0); a_0c_0)$, $H_4^2 = ((b_0, a_0, b_1, a_1); a_1c_0)$, $H_4^3 = ((c_0, b_1, c_1, b_0); b_0a_0)$, $H_4^4 = ((c_0, b_0, c_1, b_1); b_1a_0)$, $H_4^5 = ((a_0, c_1, a_1, c_0); c_0b_0)$ and $H_4^6 = ((a_1, c_0, a_0, c_1); c_1b_0)$ be six edge-disjoint copies of H_4 in $(K_3 \circ \overline{K}_2)(5)$. Then the graphs in $\{\sigma^i(H_4^j) \,|\, 0 \leq i \leq 1,\ 1 \leq j \leq 6\}$, each one of them is isomorphic to H_4, decompose the graph $(K_3 \circ \overline{K}_2)(5)$. □

We use the following two theorems and a lemma in the proof of Theorem 1.

Theorem 2. (see [2]). *For $n \geq 3$, K_n can be decomposed into K_3, K_4, K_5, K_6 and K_8.*

Theorem 3. [16]. *If m and n are at least 3, then for $1 \leq i \leq 4$, $H_i \mid K_m \circ \overline{K}_n$ if and only if $5 \mid mn(m-1)$.*

Lemma 10. [17] *For $1 \leq i \leq 4$, $H_i \mid K_8(5)$.*

Observation 4. *It is clear that if $\lambda_1 \mid \lambda$ and $G(\lambda_1)$ has an H-decomposition, then $G(\lambda)$ also has an H-decomposition.*

Proof of Theorem 1. The proof of the necessity is obvious and we prove the sufficiency in two cases.
Case 1. $g.c.d(\lambda, 5) = 1$.

The result follows by Theorem 3 and Observation 4.

Case 2. $g.c.d(\lambda, 5) = 5$.

First we prove this case for $\lambda = 5$. By Theorem 2 and the tensor product is distributive over edge-disjoint union of subgraphs, it is enough to prove that for $m \in \{3, 4, 5, 6, 8\}$ and for $1 \leq i \leq 4$, $H_i \mid (K_m \circ \overline{K}_n)(5)$. If $m \in \{3, 4\}$, then the result follows by Lemmas 4, 5, 6, 7, 8 and 9 and if $m \in \{5, 6\}$, then the result follows by Theorem 3. If $m = 8$, then $(K_8 \circ \overline{K}_n)(5) = K_8(5) \circ \overline{K}_n = H_i \circ \overline{K}_n \oplus H_i \circ \overline{K}_n \oplus \ldots \oplus H_i \circ \overline{K}_n$, by Lemma 10 and for $1 \leq i \leq 4$, $H_i \mid H_i \circ \overline{K}_n$, by Lemmas 1 and 2. If $\lambda > 5$, the result follows by Observation 4. □

Theorem 5. [12]. *If m and n are at least 3, then $C_5 \mid (K_m \circ \overline{K}_n)(\lambda)$ if and only if $\lambda(m-1)n$ is even and $5 \mid \lambda m(m-1)n$.*

Combining Theorems 1 and 5, we obtain a complete solution to the decomposition of the λ-fold complete equipartite graphs into any simple connected unicyclic graph of order five.

Declaration of Competing Interest

There is no conflict of interest related to this work.

Acknowledgment

The author is grateful to Professor P. Paulraja for his valuable suggestion. The author would like to thank the referees for their careful reading and suggestions which improved the presentation of the paper.

References

1. Lindner, C. C. and Rodger, C. A., 2009. *Design Theory* (2nd ed.). CRC Press.
2. Abel, R. J. R., Bennett, F. E. and Greig, M., 2007. PBD-closure. In C. J. Colbourn & J. H. Dinitz (Eds.), *CRC Handbook of Combinatorial Designs* (2nd ed., pp. 247–255). Chapman & Hall/CRC.
3. Hanani, H., 1961. The existence and construction of balanced incomplete block designs. *Annals of Mathematical Statistics*, 32, pp.361–386.
4. Huang, Q., 1991. On the decomposition of K_n into complete m-partite graphs. *Journal of Graph Theory*, 15, pp.1–6.
5. Ruiz, S., 1985. Isomorphic decomposition of complete graphs into linear forests. *Journal of Graph Theory*, 9, pp.189–191.
6. Tverberg, H., 1982. On the decomposition of K_n into complete bipartite graphs. *Journal of Graph Theory*, 6, pp.493–494.

7. Brouwer, A. E., Schrijver, A. and Hanani, H., 2006. Group divisible designs with block-size four. *Discrete Mathematics*, 306, pp.939–947.

8. Hanani, H., 1975. Balanced incomplete block designs and related designs. *Discrete Mathematics*, 11, pp.255–369.

9. Alspach, B. and Gavlas, H., 2001. Cycle decompositions of K_n and $K_n - I$. *Journal of Combinatorial Theory Series B*, 81, pp.77–99.

10. Alspach, B., Gavlas, H., Šajna, M. and Verrall, H., 2003. Cycle decompositions IV: Complete directed graphs and fixed-length directed cycles. *Journal of Combinatorial Theory Series A*, 103, pp.165–208.

11. Asplund, A., Chaffee, J. and Hammer, J. M., 2017. Decomposition of a complete bipartite multigraph into arbitrary cycle sizes. *Graphs and Combinatorics*, 33, pp.715–728.

12. Billington, E. J., Hoffman, D. G. and Maenhaut, B. H., 1999. Group divisible pentagon systems. *Utilitas Mathematica*, 55, pp.211–219.

13. Lindner, C. C. and Rodger, C. A., 1992. Decompositions into cycles II: Cycle systems. In J. H. Dinitz & D. R. Stinson (Eds.), *Contemporary Design Theory: A Collection of Surveys* (pp. 325–369). John Wiley & Sons.

14. Bermond, J.-C., Huang, C., Rosa, A. and Sotteau, D., 1980. Decomposition of complete graphs into isomorphic subgraphs with five vertices. *Ars Combinatoria*, 10, pp.211–254.

15. Bermond, J.C. and Schönheim, J., 1977. G-decomposition of K_n, where G has four vertices or less. *Discrete Mathematics, 19*(2), pp.113-120.

16. Huang, M. H., 2015. Decomposing complete equipartite graphs into connected unicyclic graphs of size five. *Utilitas Mathematica, 97*, pp.109–117.

17. Paulraja, P. and Sivakaran, T. Decompositions of some regular graphs into unicyclic graphs with five vertices (submitted).

Appendix

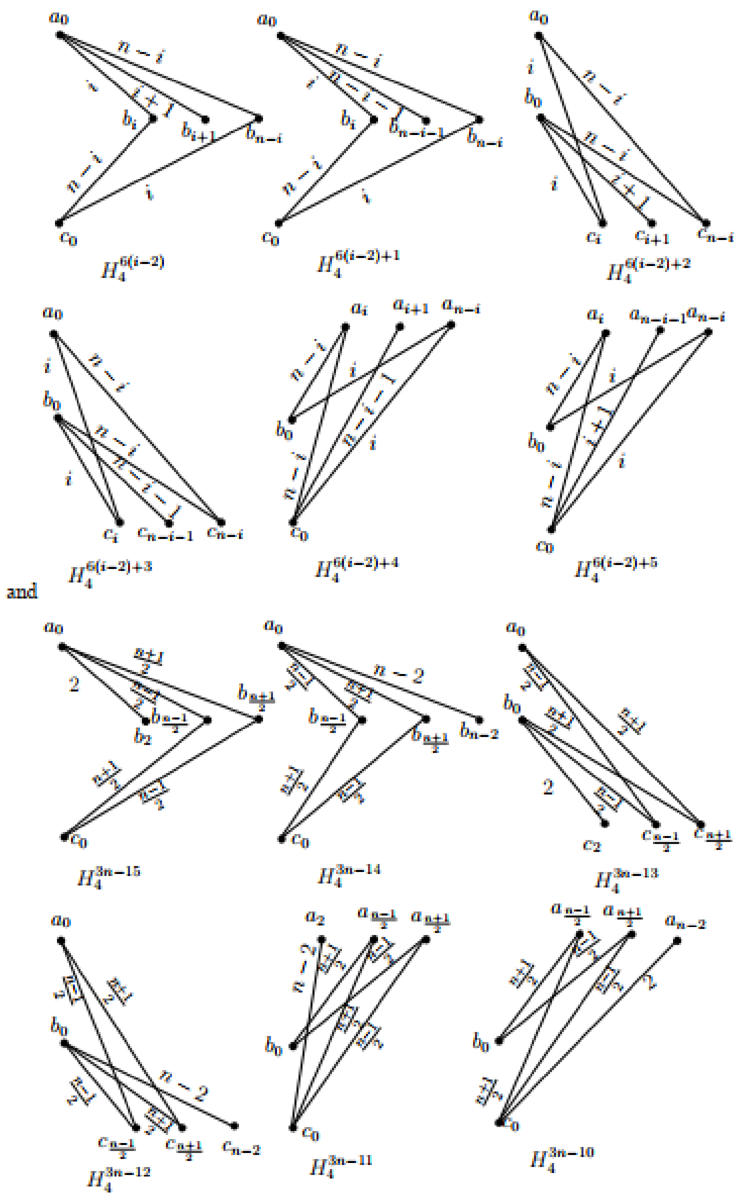

Figure 7. In the Union of the Above Graphs Each Edge of Distance in $\{0, 1, n-1\}$ from A to B, B to C and A to C Occurs Exactly Five Times

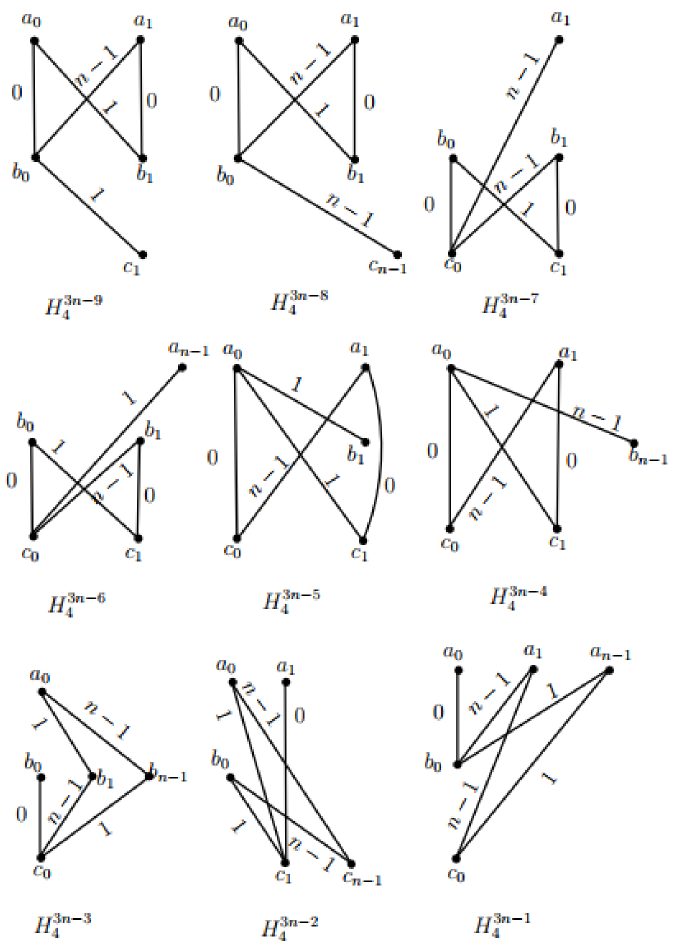

Figure 8. In the Union of the Above Graphs Each Edge of Distance in $\{2, 3, 4, \ldots, n-3, n-2\}$ from A to B, B to C and A to C Occurs Exactly Five Times

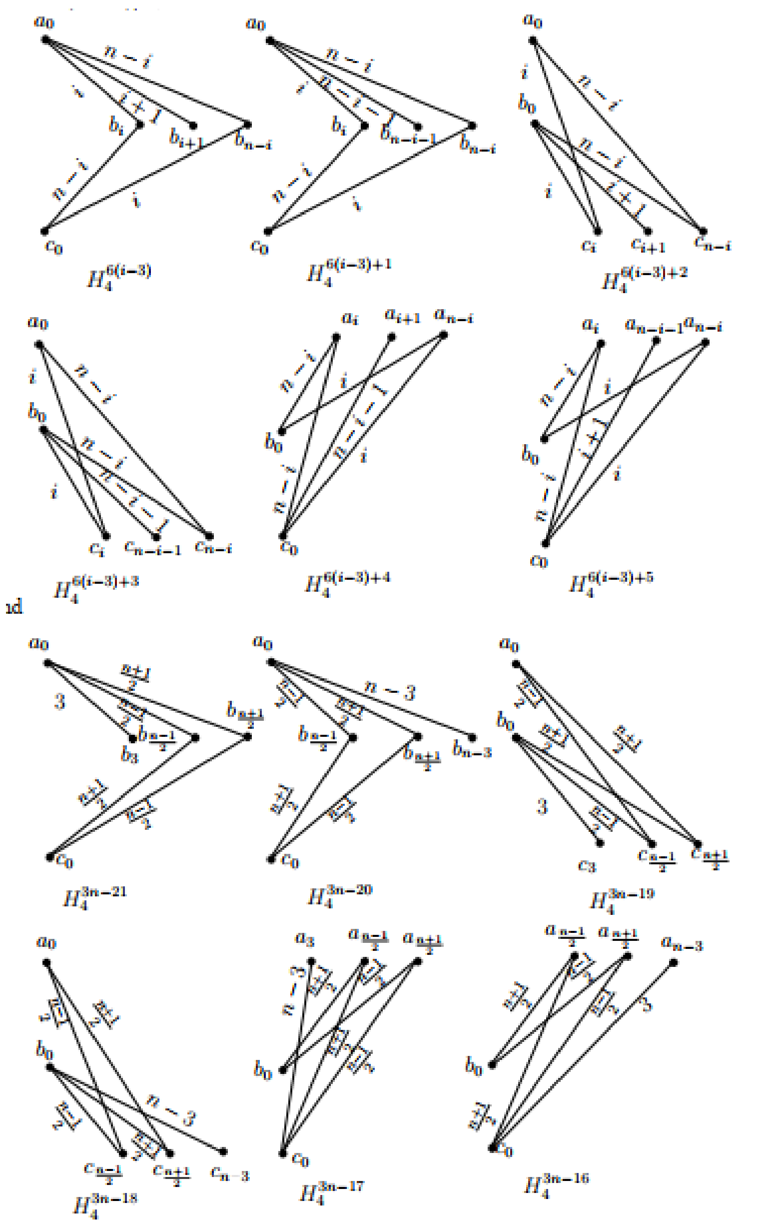

Figure 9. In the Union of the Above Graphs Each Edge of Distance in $\{3, 4, \ldots, n-4, n-3\}$ from A to B, B to C and A to C Occurs Exactly Five Times

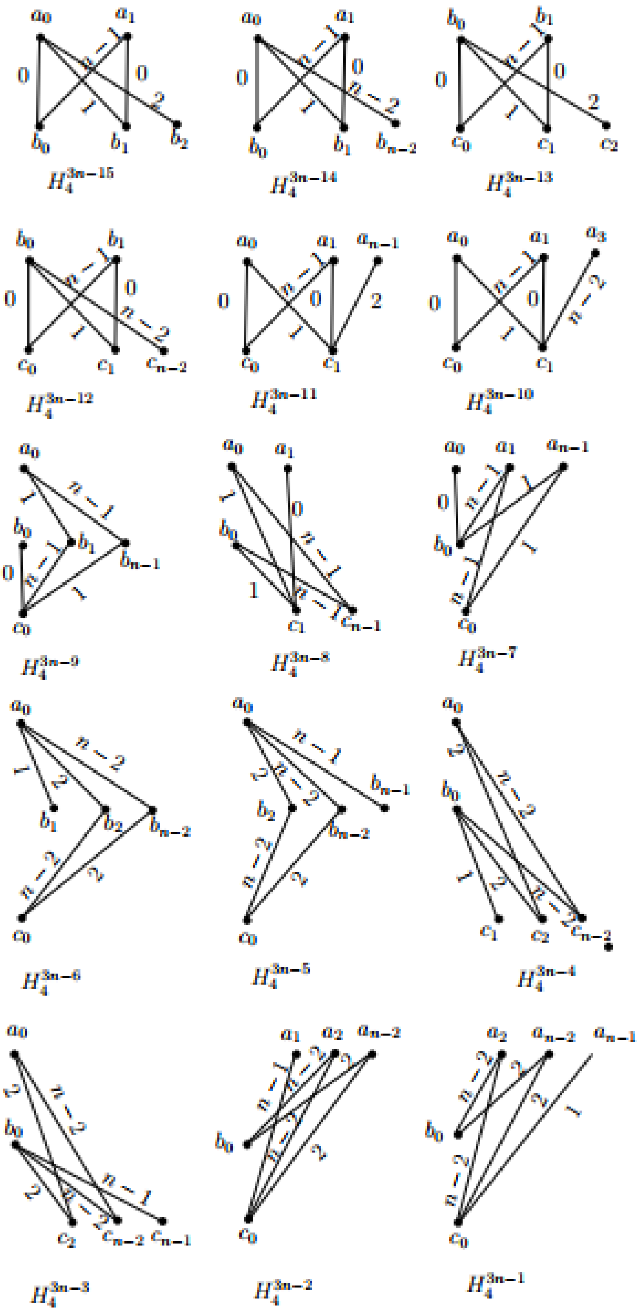

Figure 10. In the Union of the Above Graphs Each Edge of Distance in $\{0, 1, 2, n-1, n-2\}$ from A to B, B to C and A to C Occurs Exactly Five Times