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Article

Edge Domination and Secure Edge Domination in Mycielski Graph of Trees

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Abstract: The secure edge dominating set of a graph G is an edge dominating set F with the property that for each edge $e \in E - F$, there exists $f \in F$ adjacent to e such that $(F - \{f\}) \cup \{e\}$ is an edge dominating set. In this paper, we obtained upper bounds for edge domination and secure edge domination number for Mycielski of a tree.

Keywords: Edge domination, Secure edge domination, Mycielski graph, Trees, Graph theory, Domination in graphs

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1. Introduction

The concept, domination of a graph G was introduced by Claude Berge in 1958. Later, Cockayne and Stephen Hedetniemi introduced edge domination of graphs. The concept of secure domination was introduced by Cockayne [1], which is applied to protect a system, using guards that can defend each vertex of G from an attack. In this paper, we initiate the study of secure edge domination in Mycielski of trees.

Let G(V,E) be a simple graph, with a finite nonempty set V(G)= $\{v_1, v_2....v_n\}$ called the *vertex-set* of G and E(G) called the *edge-set* of G. *Degree* of a vertex $v \in$ V(G) is the number of edges adjacent to *v*. A *vertex induced subgraph of G* induced by a vertex set S denoted by G[S], is a graph with vertex set S and edge set consists of edges *uv*, whenever $u, v \in$ S and $uv \in$ E(G). The *open neighborhood* of a vertex $v \in$ V(G), denoted by $N(v)$, is the set of all vertices of G which are adjacent to *v*. $N[v] = N(v) \cup \{v\}$ is called *closed neighborhood* of v in the graph G. An *edge covering* of G is a subset C of E(G) such that every vertex of G is incident to some edges of C and the size of a minimum edge covering of G is $\beta'(G)$.

A connected acyclic graph is called *Tree*. *Leaf* of a tree is a vertex having degree one. *Remote vertex* of a tree is the vertex having more than one leaf.

The *Mycielski graph of G*, $\mu(G)$, is a graph with vertex set V(G) $\cup \{v'_1, v'_2, ...v'_n\} \cup \{z\}$ and edge set E(G) $\cup \{v_i v'_j, v'_i v_j : v_i v_j$ is an edge in G$\} \cup \{v'_i z : 1 \le i \le n\}$ [2].

A set of edges F in graph G is an *edge dominating set* if every edge $e \in E - F$ is adjacent to at least one edge in F. The *edge domination number*, $\gamma'(G)$ is the minimum cardinality of an edge dominating set of G [3–5]. *Secure edge dominating set* of G is an edge dominating set F with property that for each edge $e \in E - F$, there exists $f \in F$ adjacent to e such that $(F - \{f\}) \cup \{e\}$ is an edge dominating set.

The *secure edge dominating number*, $\gamma'_s(G)$ is the minimum cardinality of a secure edge dominating set.

In 1977, Mitchell and Hedetniemi in [6] constructed an algorithm for edge domination number of a tree. Later many studies have been done in the edge domination number of many graphs. In 2016, Kulli, [7] presented lower bound of secure edge domination of graph G and found secure edge domination number of many classes of graphs.

Related to this ideas, we have initiated our study in secure edge domination of Mycielski graphs. And found upper bounds for edge domination number and secure edge domination number of Mycielski graphs in [8].

In this paper, we obtained an upper bound for the edge domination and secure edge domination number in Mycielski of trees.

2. Edge Domination in Mycielski of Trees

2.1. Construction of edge dominating set in Mycielski of trees

Let T_0 be the given tree. Now consider the following set of vertices; U={Remote vertices of T_0} and L={leaf adjacent to the remote vertices of T_0 }.

Step 1.

Case a. T_0 have atleast one remote vertex.

Consider the remote vertices and its neighbors in T_0. Since U={remote vertices of T_0}, we denote the number of remote vertices in T_0 as $|U|$. Now construct two graphs, T_1 and T_2 as follows:

$T_1 = G[N[U]-L]$, the graph induced by remote vertices and its neighbors, except the leafs incident to the remote vertices.

$T_2 = T_0 - N[U]$, obtained by removing all remote vertices and their neighbors from T_0.

Case b. T_0 have no remote vertex .

In this case consider $|U| = 1$ and proceed to Step2.

Step 2.

Case a. T_0 have atleast one remote vertex

Find the minimum edge cover of T_1 and T_2 and denote the size of minimum edge covers as n_1 and n_2 respectively. While computing cardinality of edge cover, the isolated vertices are counted once. Here the graphs T_1 and T_2, may be connected or disconnected.

Case a(1). Both T_1 and T_2 are connected.

Then, $n_1 = \beta'(T_1)$ and $n_2 = \beta'(T_2)$.

Case a(2). Both T_1 and T_2 are disconnected.

Then,

$n_1 = \sum_{i=1}^{r} \beta'_i$, where β'_i is the edge cover of components, $C_1, C_2, ..., C_r$ of T_1.

$n_2 = \sum_{i=1}^{s} \beta'_i$, where β'_i is the edge cover of components, $C_1, C_2, ..., C_s$ of T_2.

Case a(3). If T_1 is connected and T_2 is disconnected.

Then, $n_1 = \beta'(T_1)$ and $n_2 = \sum_{i=1}^{s} \beta'_i$, where β'_i is the edge cover of components, $C_1, C_2, ..., C_s$ of T_2. (If T_1 is disconnected and T_2 is connected, then $n_1 = \sum_{i=1}^{r} \beta'_i$, where β'_i is the edge cover of components, $C_1, C_2, ..., C_r$ of T_1 and $n_2 = \beta'(T_2)$.)

Case b. T_0 have no remote vertices

In this case, we denote the minimum cardinality of edge cover of T_0 as n_1 and $n_2 = 0$.

Theorem 1. *The edge domination number of Mycielski of tree*

$$\gamma'(\mu(T)) \leq n_1 + n_2 + |U|.$$

Proof. Let T_0 be the given tree and $\mu(T_0)$ be its Myceilski graph. Let U={$u_i, u_j, u_q, u_k....$} be the set of all remote vertices of T_0 and L={$u_{i1}, u_{i2}, ... , u_{j1}, u_{j2}, ..., u_{q1}, u_{q2}, ..., u_{k1}, u_{k2}...$} be the leaves adjacent to remote vertices in U. Consider the neighbors of u_i except leaves adjacent to it, $N[U] -$

$L=\{u_y, u_i, u_m, u_n, u_j, u_q, u_s, u_t, u_k, u_r, ...\}$. Then let T_1 be the graph induced by vertex set $N[U] - L$ and D_1 be the edge cover of T_1. The edge cover D_1 will consists of all the vertices of $N[U] - L$ as the end vertices of edges in D_1.
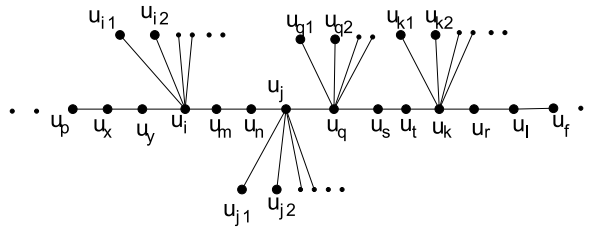

Figure 1. Tree T_0

If T_1 is disconnected, then let $C_1, C_2, ..., C_r$ are components of T_1. Consider the edges which cover each component C_i of the graph T_1. Let $Ai=\{u_y u_i, u_m u_n, u_j u_q, u_s u_t\ u_k u_r, ...\}$ be the edge cover of a component C_i. Then vertex set of A_i consists of every vertices of C_i as the end vertex of edges in A_i. Thus edge cover of T_1 is $D_1 = \cup A_i$ for i=1,2,3...,r corresponding to each component in C_i in T_1. If any of the component is an isolated vertex then consider an edge incident to it from T_0 in D_1.
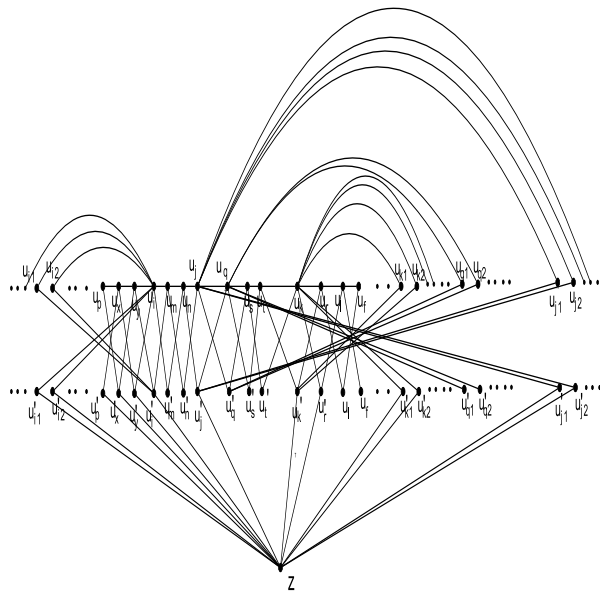

Figure 2. Myceilski of Tree T_0

Thus we constructed a set of edges D_1 which consists of every vertices in $N[U] - L$. Hence D_1 can dominate every edges incident to remote vertices and its neighbors except the edges adjacent to leafs of remote vertices in $\mu(T_0)$. That is, any edge $u_y u_i$ in D_1 can dominate edges $\{u_x u_y, u'_x u_y, u_i u_m, u_i u'_y, ... u_i u_{i1}, u_i u_{i2}, ..\}$.

Now remove all the remote vertices and neighbors from T_0 to get the graph T_2. To construct a dominating set, find the edge cover D_2 of T_2. If T_2 is disconnected then, let $B_i=\{u_p u_x, u_l u_f, ...\}$ be the edge cover of a component C_i of T_2, which can dominate the edges of C_i and edges $\{u'_p u_x, u_p u'_x, u'_l u_f, u_l u'_f, ...\}$ in $\mu(T_0)$. Thus any edge $u_p u_x$ in edge cover $D_2 = \cup B_i$ for i=1,2,3,...,s dominates edges $u'_p u_x, u_p u'_x$ adjacent to T_2 in $\mu(T_0)$. Thus $D_1 \cup D_2$ can dominate all the edges uv, uv' and $u'v$ except the edges $u_{i1} u'_i, u_{i2} u'_i, ..$ for each remote vertex u_i.

Now let us consider the edges of type $u'_i z$ corresponding to each remote vertex u_i which can dominates $u_{i1} u'_i, u_{i2} u'_i, ...$and edges $v'z$ for v be any vertex in tree T_0. Thus corresponding to each remote vertex, consider $D_3=\{u'_i z, u'_j z, u'_q z, u'_k z, ...\}$, where $|D_3|=|U|$. If T_0 is a tree without remote vertex then to dominate edges $v'z$, consider $D_3=\{v'z\}$.

Thus D= $D_1 \cup D_2 \cup D_3$ is an edge dominating set of $\mu(T_0)$, Where $|D_1| = n_1$, $|D_2| = n_2$ and $|D_3| = |U|$. Therefore, $\gamma'(\mu(T)) \leq n_1 + n_2 + |U|$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3. Secure Edge Domination of Mycielski of Trees

To construct a secure edge dominating set, we have to create an edge dominating set S such that, for every e$\in E(T) - S$ there exists a $f \in S$ such that $(S - \{f\}) \cup \{e\}$ is an edge dominating set. That is, we have to replace each edge e not in S with neighbors of e in S, provided that S still dominates $\mu(T)$. But the edge dominating set $D = D_1 \cup D_2 \cup D_3$ is not a secure edge dominating set. Since, for edge $u_y u_i' \in E(T) - D$, $(D - \{u_y u_i\}) \cup u_y u_i'$ is not an edge dominating set. Hence, to make D a secure edge dominating set, we have to add some guards corresponding to each edges in D.

3.1. Construction of secure edge dominating set in Mycielski graph of tree

To safeguard each edges of $D = D_1 \cup D_2 \cup D_3$, consider D_1, D_2 and D_3 separately, see Figure 2.

Step 1. Guards for the edges $\{u_y u_i, u_m u_n, u_j u_q.....\}$ in D_1.

Consider the remote vertex set U=$\{u_i, u_j, ...\}$, then D_1 consists of edges of the form X=$\{u_y u_i, u_j u_q, u_k u_r, ...\}$ corresponding to each remote vertices $\{u_i, u_j, u_q, u_k, ...\} \in$ U and Y=$\{u_m u_n, u_s u_t, ...\}$ where $\{u_m, u_n, u_s, u_t, ...\}$ are neighbors of remote vertices. The number of edges in X is at most $|U|$ and consider the number of edges in Y as n_3. Now set $\{u_i u_y', u_q u_j', u_r u_k', ...\}$ as the guard of each edge in Y and $\{u_n u_m', u_t u_s', ...\}$ as the guard for each edge in X. Thus number of guards of Y is same as the number of edges in Y, which is equal to $|U|$. Similarly number of guards for edges in X is n_3. Thus, minimum number of guards required to secure $D_1 = |U| + n_3$.

Step 2. Guards for the edges $\{u_p u_x, u_l u_f, ...\}$ in D_2.

For each edge $u_p u_x$ in D_2, set $u_x u_p'$ as guards. Therefore guards for edges of D_2 are $\{u_x u_p', u_f u_l', ...\}$. Thus number of guards for D_2 is same as the number of edges in D_2 which is equal to n_2. Thus, Minimum number of guards required to secure $D_2 = n_2$.

Step 3. Guards of edges in D_3.

To secure each edges $\{u_i' z, u_j' z, u_q' z, u_k' z, ...\}$ in D_3, set the edge $v_k' z$ as guard. Thus, Minimum number of guards required to secure $D_3 = 1$. Thus total number of guards required to secure D= $|U|+n_3+n_2+1$.

Theorem 2. *Secure edge domination number of Mycielski graph of a tree T is,*

$$\gamma_s'(\mu(T)) \leq 2|U| + n_1 + 2n_2 + n_3 + 1.$$

Proof. By Theorem 1, we know that, edge domination number of Mycielski of tree is $\gamma'(\mu(T)) \leq n_1 + n_2 + |U|$, where D= $D_1 \cup D_2 \cup D_3$ is the edge dominating set of $\mu(T)$. Now to safe guard D, for each edge in $D_1 \cup D_2 \cup D_3$, we have to find a guard. Let S=$S_1 \cup S_2 \cup S_3$ be the secure edge dominating set, where $S_1 = \{u_i u_y', u_j u_q', ...\} \cup \{u_n u_m', u_t u_s', ...\} \cup D_1$, $S_2 = \{u_x u_p', u_f u_l', ...\} \cup D_2$ and $S_3 = \{v'z\} \cup D_3$. Since S contains $D_1 \cup D_2 \cup D_3$, clearly S is an edge dominating set. Now we prove that S is a secure edge dominating set.

For, we have to replace each edge f not in S with adjacent edge e in S and prove that F= $(S - \{e\}) \cap \{f\}$ is an edge dominating set.

Case 1. Consider the edges $u_x u_y \notin S$ which is adjacent to $u_y u_i$ in S_1.

Case 1.1. Consider edges $u_x u_y, u_x' u_y \notin S$ which is adjacent to vertex u_y where $u_y u_i \in S_1$.

Consider edge $u_x u_y \notin S$ and replace $u_x u_y$ instead of $u_y u_i$ in S_1. That is F= $(S - \{u_y u_i\}) \cap \{u_x u_y\}$. Now to check that F is an edge dominating set, it is enough to verify that the adjacency of deleted edge is still preserved by F. The edges incident to vertex u_y is dominated by $u_x u_y$ and edges adjacent to vertex u_i is dominated by the guard $u_i u_y'$ in S_1. Thus S_1 act as a secure edge dominating set. Similar argument also hold if we replace $u_x' u_y$ instead of $u_y u_i$.

Case 1.2. Consider edges $u_i u_q', u_i u_{i1}, u_i u_{i2}, ..., u_i u_{i1}', u_i u_{i2}', ... \notin S$ which is adjacent to vertex u_i where $u_y u_i \in S_1$. Here each edge $u_i u_q', u_i u_{i1}, u_i u_{i2}, ..., u_i u_{i1}', u_i u_{i2}', ... \notin S_1$ is not only adjacent to $u_y u_i$, but also

adjacent to $u_i u'_y$, the guard of $u_y u_i$. Consider the edge $u_i u'_q$ adjacent to vertex u_i of edge $u_y u_i$ and F= $(S - \{u_i u'_y\}) \cap \{u_i u'_q\}$. Then F act as a dominating set, since edges adjacent to u_i is dominated by $u_y u_i$ and $u_i u'_q$. Remaining is to prove that F also dominates edges adjacent to u'_y. If there exists no edge adjacent to u'_y other than $u_i u'_y$ then proof is trivial. If there exists an $u_x u_y$, then $u_x u'_y$ will be adjacent to u'_y. Since S consists of $D_1 \cup D_2 \cup D_3$, which cover every vertices of tree T except the leafs. There will exists an edge $u_p u_x$ in S which is adjacent to $u_x u'_y$. Hence F dominates $\mu(T)$. Similar conditions hold if we replace $u_m u_n$ with adjacent edges not in S_1.

Case 2. Consider edges $u_r u_l \notin S$ which is adjacent to $u_l u_f \in S_2$.

Similar to Case 1, for each edge adjacent to u_l, remove $u_l u_f$ from S_2 and replace the edge adjacent to u_l. And for edges adjacent to u_f, remove guard $u_f u'_l$ from S. Then both F= $(S - \{u_l u_f\}) \cup \{u_r u_l\}$ and F= $(S - \{u_f u'_l\}) \cap \{u_f u_e\}$ dominates $\mu(T)$ with similar arguments of Case 1.

Case 3. Consider edges $u'_r z$ and $u_{i1} u'_i \notin S$ which is adjacent to $\{u'_i z\} \in S_3$.

For each edge adjacent to $u_{i1} u'_i$, remove $u'_i z$ from S. Then F= $(S - \{u'_i z\}) \cap \{u_{i1} u'_i\}$ act as an edge dominating set, since adjacency of u'_i is preserved by $u_{i1} u'_i \in F$ and adjacency of z by $v'z$. Hence F dominates $\mu(T)$. For edge $\{u'_r z\} \notin S$, remove $\{v'z\}$ from S. Then F= $(S - \{v'z\}) \cup \{u'_r z\}$ is an edge dominating set , since adjacency of z is preserved by $\{u'_r z\}$. Now it is only to check whether adjacency of u'_r is dominated. Since S consists of all vertices of tree T except leafs, for an edge uv' adjacent to $v'z$ will get dominated by edge vu in S. Thus F is an edge dominating set.
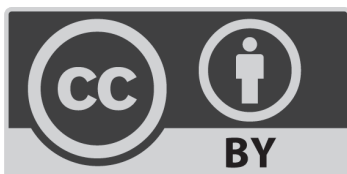Hence S secure $\mu(T)$, where $S = |S_1 \cup S_2 \cup S_3| = 2|U| + n_1 + 2n_2 + n_3 + 1$.   □

Competing Interest

There is no conflict of interest related to this work.

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