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Article

Hub Cover Pebbling Number

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Abstract: The hub cover pebbling number, $h^*(G)$, of a graph G, is the least non-negative integer such that from all distributions of $h^*(G)$ pebbles over the vertices of G, it is possible to place at least one pebble each on every vertex of a set of vertices of a hub set for G using a sequence of pebbling move operations, each pebbling move operation removes two pebbles from a vertex and places one pebble on an adjacent vertex. Here we compute the hub cover pebbling number for wheel related graphs.

Keywords: Pebbling number, Cover pebbling number, Hub set
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1. Introduction

Pebbling was introduced in the field of graph theory by Chung [1]. Hulbert published details of graph pebbling in his survey paper [2]. At this stage there are many papers in this area contributed by many authors. Over the vertices of a graph G we distribute non negative number of pebbles. So a distribution of pebbles is a function from $V(G)$ to $N \cup \{0\}$. Here we consider simple connected graphs for our discussion. All basic concepts in graph are taken from the book entitled Graph Theory by Harary [3]. A pebbling move operation removes two pebbles from a vertex and the places one pebble on an adjacent vertex. For pebbling related concepts the readers can refer [1].

In cover pebbling it is require to put at least one pebble on every vertex of the vertex cover at the end of the pebbling move operation. The least number of pebbles having the property that from all distributions of $\gamma(G)$ pebbles, it is possible to move a pebble to every vertex simultaneously using a sequence of pebbling moves is called the cover pebbling number. In [4] Crull et al., studied the cover pebbling number for complete graphs, paths and trees. Covering cover pebbling number [5] and domination cover pebbling number [6] are few other variations which come from the definition of cover pebbling. We introduce a new variation in the next section, named as 'Hub cover pebbling number', by combining the two concepts hub set and cover pebbling number, like they did in [5] and [6].

2. Hub Cover Pebbling Number

A *hub set* in a graph G is a set U of vertices in G such that any two vertices outside U are connected by a path whose internal vertices lie in U. It was introduced by Walsh [7]. Adjacent vertices are joined by a path with no internal vertices, so the condition holds vacuously for them. The hub number of G, denoted $h(G)$, is the minimum size of a hub set in G. Placing transmitters at the vertices of a hub set would facilitate communication among all vertices outside the set; Hence we seek a small hub set from which we introduce hub cover pebbling number (HCPN) of a graph G.

Definition 1. *The hub cover pebbling number, $h^*(G)$, of a graph G, is the least non-negative integer such that from all distributions of $h^*(G)$ pebbles over the vertices of G, it is possible to place at least one pebble each on every vertex of a set of vertices of a hub set for G using a sequence of pebbling move operations, each pebbling move operation removes two pebbles from a vertex and places one pebble on an adjacent vertex.*

Let G_1 and G_2 be two graphs of order n_1 and n_2 and size m_1 and m_2 respectively. The *Union* of G_1 and G_2 is the graph denoted by $G_1 \cup G_2$ [3] having the vertex set $V_1 \cup V_2$ and the edge set $E_1 \cup E_2$. The *join* $G_1 + G_2$ of G_1 and G_2 is the graph obtained from $G_1 \cup G_2$ by joining each vertex of G_1 with every vertex of G_2 by an edge.

Definition 2. [8] *The fan graph F_n for $n \geq 4$ is defined as the join of K_1 and P_{n-1}, a path graph on $n-1$ vertices.*

Definition 3. [8] *The wheel graph $W_n = K_1 + C_{n-1}$ is a graph where the vertex of degree $n-1$ is called the central vertex and all other vertices on the cycle graph C_{n-1} are called rim vertices.*

Definition 4. [8] *The helm graph H_n is the graph obtained from a wheel W_n by attaching pendant edge to each rim vertex of the wheel graph.*

Definition 5. [8] *The flower Fl_n is the graph obtained from a helm H_n by joining each pendant vertex to the central vertex of the helm graph.*

Definition 6. [8] *The friendship graph FR_n is a collection of n triangles, all having a common vertex.*

Remark 1. *$p(v)$ denotes the number of pebbles on the vertex v of G and $p(G)$ denotes the total number of pebbles placed on the vertices of the graph G. In a distribution of pebbles over the vertices of a graph G, if $p(v) \geq 1$ then the vertex v is called occupied vertex. Otherwise v is an unoccupied vertex.*

Theorem 1. *For a fan graph F_n $(n \geq 4)$, $h^*(F_n) = n - 3$.*

Proof. Consider the following labeling for F_n $(n \geq 4)$: label the vertices of P_{n-1} as $a_1, a_2, \ldots, a_{n-1}$, and label the vertex of K_1 as a_n.

For F_4, if we place one pebble on any vertex that would suffice to form a hub set and hence $h^*(F_4) = 1$. Assume that $n \geq 5$.

Consider the distribution: $p(a_i) = 1$ for $1 \leq i \leq n-4$ and $p(a_j) = 0$ for all $j \neq i$. Let $S = \{a_i : 1 \leq i \leq n-4\}$. We can not find a path between the vertices a_{n-3} and a_{n-1} such that every internal vertex is a member of S and hence S is not a hub set for F_n. Thus $h^*(F_n) \geq n-3$.

Let $p(F_n) = n-3$. If $p(a_n) \geq 1$ then for any two unoccupied vertices a_k and a_l, we can find the path $a_k a_n a_l$. Thus $\{a_n\}$ forms a hub set for F_n. If $p(a_i) \geq 2$ for some i where $1 \leq i \leq n-1$ then we move a pebble to a_n and then for any two unoccupied vertices a_k and a_l, we can find the path $a_k a_n a_l$ as $\{a_n\}$ forms a hub set for F_n. So $p(a_n) = 0$ and $p(a_i) \leq 1$ for all $1 \leq i \leq n-1$. Since $p(F_n) = n-3$ and $p(a_n) = 0$, we have two more unoccupied vertices a_k and a_l where $k < l$. Let $T = V(F_n) - \{a_k, a_l, a_n\}$. As both a_k and a_l are adjacent to a_n, it is enough to find a path between a_k and a_l such that every internal vertex is a member of T. If a_k and a_l are adjacent then we are done. Otherwise, there exists a path $a_k a_{k+1} \ldots a_{l-1} a_l$ and hence T is a hub set of F_n. Thus $h^*(F_n) \leq n-3$. □

Theorem 2. *For a wheel graph W_n ($n \geq 5$), $h^*(W_n) = n - 4$.*

Proof. For W_5, if we place one pebble on any vertex that would suffice to form a hub set and hence $h^*(W_5) = 1$. Assume that $n \geq 6$.

Consider the distribution: $p(a_i) = 1$ for $1 \leq i \leq n-5$ and $p(a_j) = 0$ for all $j \neq i$. Let $S = \{a_i : 1 \leq i \leq n-5\}$. We can not find a path between the vertices a_{n-3} and a_{n-1} such that every internal vertex is a member of S and hence S is not a hub set for W_n. Thus $h^*(W_n) \geq n - 4$.

Let $p(W_n) = n - 4$. If $p(a_n) \geq 1$ then for any two unoccupied vertices a_k and a_l, we can find the path $a_k a_n a_l$. Thus $\{a_n\}$ forms a hub set for W_n. If $p(a_i) \geq 2$ for some i ($1 \leq i \leq n-1$) then we move one pebble to a_n and then for any two unoccupied vertices a_k and a_l, we can find the path $a_k a_n a_l$ so that $\{a_n\}$ forms a hub set for W_n. So $p(a_n) = 0$ and $p(a_i) \leq 1$ for $1 \leq i \leq n-1$. Since $p(W_n) = n - 4$ and $p(a_n) = 0$, we have three more unoccupied vertices a_k, a_l and a_m where $k < l < m$. Let $T = V(W_n) - \{a_k, a_l, a_m, a_n\}$. As a_k, a_l and a_m are adjacent to a_n, it is enough to find a path between a_k & a_l, a_k & a_m and a_l & a_m such that every internal vertex is a member of T for each path. If they are not adjacent to each other, then there exist following paths: the path $a_k a_{k+1} \dots a_{l-1} a_l$ between the vertices a_k & a_l, the path $a_l a_{l+1} \dots a_{m-1} a_m$ between the vertices a_l & a_m and the path $a_m a_{m+1} \dots a_{n-1} a_1 \dots a_k$ between the vertices a_m & a_k. Hence T forms a hub set for W_n and therefore $h^*(W_n) \leq n - 4$. □

Theorem 3. *For a flower graph Fl_n ($n \geq 4$), $h^*(Fl_n) = 2n - 4$.*

Proof. Consider the following labeling for Fl_n ($n \geq 4$): label the vertices of the wheel graph as: $a_1, a_2, \dots, a_{n-1}, a_n$, where a_n is the center vertex and is of degree $2(n-1)$ and label the vertex of degree two which is adjacent to a_i ($1 \leq i \leq n-1$) as x_i.

For Fl_n ($n \geq 4$), let $S = V(Fl_n) - \{x_{n-1}, a_n, a_{n-1}, a_{n-2}\}$. We place one pebble each on every vertex of S. Then we can not find a path between the vertices x_{n-1} and a_{n-2} such that every internal vertex is a member of S and hence S is not a hub set for Fl_n. Thus $h^*(Fl_n) \geq 2n - 4$.

Consider the distribution of $2n-4$ pebbles on the vertices of Fl_n. Clearly we are done if $p(a_n) \geq 1$ or $p(a_i) \geq 2$ or $p(x_i) \geq 2$ for some i, where $1 \leq i \leq n-1$. So, we assume $p(a_n) = 0$, $p(a_i) \leq 1$ and $p(x_i) \leq 1$ for all i, where $1 \leq i \leq n-1$. Since $p(Fl_n) = 2n - 4$, we get two more unoccupied vertices. Since a_n is adjacent to these unoccupied vertices, we look for adjacency between these two unoccupied vertices. If both of these unoccupied vertices are adjacent, then we are done. Otherwise, we consider the following cases:

Case 1. Let $p(a_k) = 0$ and $p(a_l) = 0$ for some k and l ($1 \leq k < l \leq n-1$).

Let $T = V(Fl_n) - \{a_k, a_l, a_n\}$. We get the path $a_k a_{k+1} \dots a_l$ such that every internal vertex is a member of T.

Case 2. Let $p(x_k) = 0$ and $p(x_l) = 0$ for some k and l ($1 \leq k < l \leq n-1$).

Let $T = V(Fl_n) - \{x_k, x_l, a_n\}$. We get the path $x_k a_k a_{k+1} \dots a_l x_l$ such that every internal vertex is a member of T.

Case 3. $p(a_k) = 0$ and $p(x_l) = 0$ for some k and l, where $1 \leq k \leq n-1$ and $1 \leq l \leq n-1$.

Let $T = V(Fl_n) - \{a_k, x_l, a_n\}$. We get the path $a_k a_{k+1} \dots a_l x_l$ such that every internal vertex is a member of T.

From the above cases, we can conclude that T is a hub set for Fl_n and hence $h^*(Fl_n) \leq 2n - 4$ □

Theorem 4. *For a friendship graph FR_n ($n \geq 2$), $h^*FR_n) = 2n - 1$.*

Proof. The vertices of a graphs are $a_1, a_2, \dots, a_{2n-1}, a_{2n}$ such that only a_{2i-1} and a_{2i} are adjacent ($1 \leq i \leq n$) and label the common vertex which is of degree $2n$ as a_{2n+1}.

Let $S = V(FR_n) - \{a_{2n-2}, a_{2n}, a_{2n+1}\}$. Place one pebble each on the vertices of S. Clearly, we can not find a path between the vertices a_{2n} and a_{2n-2} such that every internal vertex is a member of S and hence S is not a hub set for FR_n. Thus $h^*(FR_n) \geq 2n - 1$.

Consider the distribution of $2n-1$ pebbles on the vertices of FR_n. Clearly we are done if $p(a_{2n+1}) \geq 1$ or $p(a_i) \geq 2$ for some i, where $1 \leq i \leq 2n$. So, we assume $p(a_{2n+1}) = 0$, $p(a_i) \leq 1$ for all i, where $1 \leq i \leq 2n$. Since $p(FR_n) = 2n - 1$, we get an additional unoccupied vertex, namely a_k, for some k $(1 \leq k \leq 2n)$. But a_k is adjacent to a_{2n+1}. Thus, $T = V(FR_n) - \{a_k, a_{2n+1}\}$ forms a hub set for FR_n and hence $h^*(FR_n) \leq 2n - 1$. □

Theorem 5. *For a helm graph H_n $(n \geq 4)$, $h^*(H_n) = 8(n-3) + 1$.*

Proof. Label the vertices of the wheel graph as: $a_1, a_2, \ldots, a_{n-1}, a_n$, where a_n is the centre vertex and is of degree $n-1$ and label the vertex of degree one which is adjacent to a_i $(1 \leq i \leq n-1)$ as x_i.

Place $8n - 24$ pebbles on x_1. We have to move one pebble each to every a_i so that we can form a path between any two x_i's through a_i's. While doing so, we burn all the pebbles from the vertex x_1 and place one pebble each to every a_i except a_1 and a_n. Clearly, every path between x_1 and a_n should go through a_1 and there is no pebble on a_1. Thus, $h^*(H_n) \geq 8(n-3) + 1$.

Let $p(H_n) = 8n - 23$. Let $S = \{a_i : p(a_i) \geq 1 (i \neq n)\}$, $T = \{a_j : p(a_j) = 0 (j \neq n)\}$ and assume $|S| = x \geq 0$. If $p(a_n) \geq 2(n - 1 - x)$ then clearly we are done by moving one pebble each to the set of vertices of T. Let $\sum_{i=1}^{n} p(a_i) \geq 4n - 11$. Assume all the pebbles are placed on a_i $(i \neq n)$. From the vertex a_i, there are $n-3$ vertices that are at distance two and two vertices are at distance one. To put one pebble each on those vertices, we burn $4(n-4) + 2(2) = 4n - 12$ pebbles from the vertex a_i and then we have at least one pebble on a_i. Hence, the set $\{a_i : i \neq n\}$ forms a hub set for H_n. If $|S| \geq 2$, then it is easy to move one pebble each to every vertex of T. To complete this process, we consider the following cases:

Case 1. $p(a_k) \geq 1$ is odd, where $a_k \in S$.

Let $p(a_k) \geq 3$. First, we retain one pebble on a_k. If the adjacent vertices of a_k are in S then we move $\frac{p(a_k)-1}{2}$ pebbles to a_n. If one of the adjacent vertices of a_k is in T, then we move a pebble to that vertex and then $\frac{p(a_k)-3}{2}$ pebbles to a_n. If all the adjacent vertices of a_k are in T and $p(a_k) \geq 5$, then we move one pebble each to those vertices and then $\frac{p(a_k)-5}{2}$ pebbles to a_n. If all the adjacent vertices of a_k are in T and $p(a_k) = 3$, then we move one pebble to one of its adjacent vertex of a_k in T. If $p(a_k) = 1$ then we just retain that pebble on a_k itself.

Case 2. $p(a_l) \geq 2$ is even, where $a_l \in S$.

Let $p(a_l) \geq 4$. First, we retain two pebbles on a_l. If the adjacent vertices of a_l is in S then we move $\frac{p(a_l)-2}{2}$ pebbles to a_n. If one of its adjacent vertex of a_l is in T, then we move one pebble to that vertex and then $\frac{p(a_l)-4}{2}$ pebbles to a_n. If all the adjacent vertices of a_l are in T and $p(a_l) \geq 6$, then we move one pebble each to those adjacent vertices and then $\frac{p(a_l)-6}{2}$ pebbles to a_n. If both the adjacent vertices of a_l are in T and $p(a_l) = 4$, then we move a pebble to one of its adjacent vertices of a_l in T. If $p(a_l) = 2$ then we just retain that pebble on a_l itself.

From the above cases, we note that certain number of vertices from T, (say $y \geq 0$ vertices), are also pebbled using pebbling moves and also we could have moved some amount of pebbles to the vertex a_n. Thus we can place one pebble each to the remaining unpebbled vertices of T using the pebbles from a_n, since the vertex a_n contains at least $2(n - 1 - (x + y))$ pebbles. Hence we have pebbled all the vertices of the set $\{a_i : i \neq n\}$ and hence we are done.

Assume $\sum_{i=1}^{n-1} p(x_i) \geq 4n - 11$. Let $S = \{a_i : p(a_i) \geq 1 (i \neq n)\}$, $T = \{a_j : p(a_j) = 0 (j \neq n)\}$, $U = \{x_i : p(x_i) \geq 1\}$, and $V = \{x_j : p(x_j) = 0\}$. Consider $\sum p(a_i) + p(a_n) = p_a$ where $a_i \in S$ and $p_a \geq 0$. This implies that $\sum p(x_i) \geq 8n - 23 - p_a$ where $x_i \in U$. If $|S| \geq 1$, we do the same procedure as we discussed in the previous cases for the number of pebbles on a_i. Suppose some of the vertices of T, say z vertices, are not pebbled by these cases. To pebble those z vertices of T, we use the pebbles on the vertices of U.

Case 3. $p(x_m) \geq 1$ is odd, where $x_m \in U$.

We retain one pebble on x_m and then we move $\frac{p(x_m)-1}{2}$ pebbles to a_m. Now, we undertake the operations as in Case 1 or Case 2 with the number of pebbles on the vertex a_m again, excluding the number of pebbles placed on the vertex a_m.

Case 4. $p(x_m) \geq 2$ is even, where $x_m \in U$.

We do not retain any pebble on x_m. We move $\frac{p(x_m)}{2}$ pebbles to a_m. Now, we undertake the operations as in Case 1 or Case 2 with the number of pebbles on the vertex a_m again, excluding the number of pebbles placed on the vertex a_m.

So, the vertex a_n receives some amount of pebbles from the vertex x_m through a_m, that is, the vertex a_n receives at least $2z$ pebbles which is enough to put one pebble each to the z unpebbled vertices of T. Hence we have pebbled all the vertices of the set $\{a_i : i \neq n\}$ and hence we are done.

Let $|S| = 0$. This implies that $\sum p(x_i) = 8n - 23$ where $x_i \in U$. Assume $|U| \geq 3$. Now, we undertake the operations as in Case 3 and Case 4 with the number of pebbles on the vertex x_i. We can easily placed one pebble each on every vertex a_i, $i \neq n$ which is adjacent to an unpebbled vertex x_j and hence we are done. Assume $|U| = 1$ or 2. Now, we undertake the operations as in Case 3 and Case 4 with the number of pebbles on the vertex x_i. We can easily placed one pebble each on every vertex a_i, $i \neq n$ which is adjacent to an unpebbled vertex x_j and hence we are done. There is an exception case where we cannot place a pebble on the vertex a_k of the set $\{a_i\}(i \neq n)$. This case exists only when $p(x_k) = 1$ and $p(x_l) = 8n - 24$ or $p(x_k) = 8n - 23$. But in this case also, we can find a path between any two unpebbled vertices of H_n in which all the internal vertices are members of $S - \{a_k\} \cup \{x_k\}$ and hence we are done. Thus $h^*(H_n) \leq 8n - 23$. □

Declaration of Competing Interest

The authors declare no conflicts of interest to this work.

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