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Article

Hyperplanes of Segre Geometries

Hendrik Van Maldeghem[1,*]

[1] Department of Mathematics: Algebra & Geometry, Ghent University, Krijgslaan 281, S25, 9000 Gent, Belgium

* **Correspondence:** hendrik.vanMaldeghem@ugent.be

Abstract: We classify the geometric hyperplanes of the Segre geometries, that is, direct products of two projective spaces. In order to do so, we use the concept of a *generalised duality*. We apply the classification to Segre varieties and determine precisely which geometric hyperplanes are induced by hyperplanes of the ambient projective space. As a consequence we find that all geometric hyperplanes are induced by hyperplanes of the ambient projective space if, and only if, the underlying field has order 2 or 3.

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1. Introduction

Recently, a very nice connection between the varieties of the first and the second row of the Freudenthal-Tits Magic Square was exhibited in [1] by De Schepper & Victoor. In the proof, they use the classification of geometric hyperplanes of the exceptional minuscule geometry of type E_6 (see [2]), which is the variety in the fourth cell of the second row. For the second and third cells, they exhibit properties of a certain geometric hyperplane that allows to make direct conclusions in the case of the third cell (where one deals with a line Grassmannian of projective 5-space). In case of the second cell, where one deals with a Segre variety, one has to work a little harder to arrive at the wanted conclusion. The reason for this extra work is the non-availability of a list of all geometric hyperplanes of that Segre variety. Reading the paper [1] for Mathematical Reviews, I realised that it might be possible to classify the geometric hyperplanes of the Segre variety in question. The result is written in the current paper. We do not restrict to the specific Segre variety of the second cell of the second row of the Freudenthal-Tits Magic Square, but we consider all Segre geometries obtained as the direct product of two projective spaces.

What we do is, we reduce the classification to the more combinatorial question of classifying *generalised dualities* between two projective spaces, a (to the best of our knowledge, new) notion that extends the notion of *generalised polarity*, which was introduced by Jacques Tits in [3] in order to describe embedded polar spaces. Remarkably, the geometric hyperplanes of line Grassmannian geometries, which correspond to the geometries in the third cell of the second row of the Freudenthal-Tits Magic square, are classified by generalised symplectic polarities of an appropriate projective space [4]. It makes one wonder whether something similar is true for the (three kinds of) hyperplanes of the exceptional geometry of type E_6 of the fourth cell of the second row.

So, we classify generalised dualities and then interpret the results back in the Segre geometries. We define the *grayscale index* of a geometric hyperplane and show that this notion naturally divides the hyperplanes in c classes, where c is one more than the (projective) dimension of the smallest generators of the Segre geometry. Moreover, we prove that there is a bijective correspondence between the grayscale indices and the orbits of hyperplanes under the automorphism group of a Segre variety. This is done by considering the dual Segre variety, which can be defined using the white geometric hyperplanes, that is, those with grayscale index 1.

It is worth noting that, in the special case of the Segre geometries of the second cell of the second row of the Freudenthal-Tits Magic Square, grayscale 0 hyperplanes correspond to black hyperplanes in the sense of Cooperstein & Shult [5], those with grayscale index $\frac{1}{2}$ with grey hyperplanes, and those with grayscale index 1 to the white hyperplanes, explaining and motivating the terminology of grayscale index.

Not all Segre geometries admit geometric hyperplanes of arbitrary admissible grayscale index. A sufficient and necessary condition is deduced, which is always satisfied for Segre varieties. Roughly, our main result reads as follows (see 3 for a more detailed version).

1.1. Main Result

Let H be a geometric hyperplane of the direct product geometry $\Pi_1 \times \Pi_2$ of two projective spaces of dimension d_1 and d_2, respectively. Then there exist complementary subspaces B_i and W_i of Π_i, $i \in \{1,2\}$, with $\dim B_1 = \dim B_2$ such that H contains $H_1 := W_1 \times \Pi_2 \cup \Pi_1 \times W_2$, H contains a hyperplane H_2 of $B_1 \times B_2$ determined by a (proper) duality from B_1 to B_2 (and hence B_1 and B_2 are dual to each other), and H contains lines having a point in each of H_1 and H_2.

We also explain te connection with the dual Segre geometry, and with the dual Segre variety. Finally, we restrict to the finite case and recover the embedded flag geometries of projective planes classified in [6]. We show that in the small cases $q = 2, 3$, all geometric hyperplanes of Segre varieties over $\mathbb{F}_q$ are induced by hyperplanes of the ambient projective spaces. Moreover, we also count the number of geometric hyperplanes with fixed grayscale index.

2. Background and Definitions

2.1. Point-line Geometries

A point-line geometry $\Delta = (X, \mathcal{L})$ consist of a points set X and a line set $\mathcal{L}$, for our purposes containing subsets of X. We now present the essentials for the current paper. More background may be found in the books [7] and [8].

All point-line geometries that we will encounter are *partial linear spaces*, that is, two distinct points are contained at most one common line—and points that are contained in a common line are called *collinear*; a point on a line is sometimes also called *incident with that line*. We will also always assume that each line has at least three points. In a general point-line geometry $\Delta = (X, \mathcal{L})$, one defines a *subspace* as a set of points with the property that it contains all points of each line having at least two points with it in common. It is called *singular* if each pair of points of it is collinear. It is called a *(geometric) hyperplane* if every line intersects it in at least one point—and then the line is either contained in it, or intersects it in exactly one point. We allow a hyperplane to coincide with the whole point set itself, but we call it *proper* if it doesn't.

Given a subset $S \subseteq X$ of points of a point-line geometry $\Delta = (X, \mathcal{L})$, we denote the intersection of all subspaces containing S by $\langle S \rangle$ and call it the subspace *generated by* S. A minimal generating set of a subspace will sometimes be called a *basis* of that subspaces (especially if the point-line geometry is a projective space, see Section 2.2 below).

We will sometimes view a subspace S of the point-line geometry $\Delta = (X, \mathcal{L})$ as a point-line geometry in the obvious way: the point set is $S \subseteq X$ and the lines are those of $\mathcal{L}$ entirely contained in

S. In this sense, S is a full subgeometry of Δ. In general, full subgeometries are defined as follows.

Let $\Delta' = (X', \mathcal{L}')$ be a point-line geometry with $X' \subseteq X$ and $\mathcal{L}' \subseteq \mathcal{L}$. Then we say that Δ' is a *full subgeometry* of Δ.

2.2. Segre Geometries

Let $\mathcal{L}$ be a skew field and V a right vector space over $\mathcal{L}$ of dimension at least 3. The corresponding projective geometry, with point set the set of 1-spaces of V, and lie set the 2-spaces, will be denoted by $\mathsf{PG}(V)$, or $\mathsf{PG}(d, \mathcal{L})$, if $\mathsf{dim} V = d + 1$.

Recall also that an *axiomatic projective plane* is a point-line geometry satisfying the following axioms.

(PP1) Each pair of distinct points is incident with exactly one line.
(PP2) Each pair of distinct lines is incident with exactly one point.
(PP3) There are four different points, no three of which are incident with the same line.

In this paper, a projective space of dimension d is one of the following.

- For $d \geq 3$, we have $\mathsf{PG}(d, \mathbb{L})$ for a certain skew field $\mathbb{L}$.
- For $d = 2$, it is an axiomatic projective plane.
- For $d = 1$, it is a set of at least three elements (a projective line).
- For $d = 0$, it is a singleton (a point).
- For $d = -1$, it is the empty set.

Let Π_1 and Π_2 be two projective spaces of non-negative dimension with respective points sets X_1 and X_2 and line sets $\mathcal{L}_1$ and $\mathcal{L}_2$. Then the *direct product space* $\Pi_1 \times \Pi_2$ is the point line geometry with point set $X_1 \times X_2$ and line set $\{\{x_1\} \times L_2 \mid x_1 \in X_1, L_2 \in \mathcal{L}_2\} \cup \{L_1 \times \{x_2\} \mid L_1 \in \mathcal{L}_1, x_2 \in X_2\}$. It is called a *Segre geometry (of type* $(\mathsf{dim}\Pi_1, \mathsf{dim}\Pi_2)$*)*. It is called *proper* if both the dimensions $\mathsf{dim}\Pi_1$ and $\mathsf{dim}\Pi_2$ are positive. For convenience, we will in this paper always assume that $\mathsf{dim}\Pi_1 \leq \mathsf{dim}\Pi_2$. If $\mathsf{dim}\Pi_1 = \mathsf{dim}\Pi_2 = 1$, then we call the Segre geometry a *grid*. It is easily seen that subsets of type $\{x_1\} \times \Pi_2$ and $\Pi_1 \times \{x_2\}$, with $x_1 \in \Pi_1$ and $x_2 \in \Pi_2$, are maximal singular subspaces and we call them *generators*.

The definition immediately makes clear that a Segre geometry of type (d_1, d_2) contains subspaces isomorphic to Segre geometries of type (ℓ_1, ℓ_2), for every $\ell_1 \leq d_1$ and $\ell_2 \leq d_2$.

Definition 1. *A point-line geometry isomorphic to a full subgeometry of some projective space is called* embeddable.

2.3. Segre Varieties

Since every proper Segre geometry contains a grid as a subspace, we deduce that Segre geometries are never full subgeometries of projective spaces over non-commutative skew fields. This in turn implies that a necessary condition for a Segre geometry $\Pi_1 \times \Pi_2$ to be embeddable is that Π_i, $i = 1, 2$, is defined over a commutative field whenever $\mathsf{dim}\Pi_i \geq 2$. Naturally, another necessary condition is that the planes of Π_1 and Π_2 are isomorphic, and if $\mathsf{dim}\Pi_1 = 1$, then the number of points of Π_1 is equal to the number of points on any line of Π_2.

That the above mentioned necessary conditions are also sufficient is proved by the existence of Segre varieties, which we now introduce.

In short, a Segre variety is the set of projective points corresponding to the pure tensors of a tensor product of two vector spaces over a common field $\mathbb{K}$. A more concrete definition is the following.

Definition 2.
- *Let* (d_1, d_2) *be a pair of natural numbers (hence nonzero) and set* $N = (d_1 + 1)(d_2 + 1) - 1 = d_1 d_2 + d_1 + d_2$. *Let* $\mathbb{K}$ *be a field. We consider a standard coordinatization of* $\mathsf{PG}(d_i, \mathbb{K})$, $i = 1, 2$, *and introduce the* Segre map

$$\sigma_{d_1,d_2} : \mathsf{PG}(d_1, \mathbb{K}) \times \mathsf{PG}(d_2, \mathbb{K}) \to \mathsf{PG}(N, \mathbb{K})$$

$$: ((x_0, x_1, \dots, x_{d_1}), (y_0, y_1, \dots, y_{d_2})) \mapsto (x_i y_j)_{0\le i\le d_1, 0\le j\le d_2}.$$

- *The image of* $\sigma_{d_1,\dots,d_n}$ *is called a* Segre variety (of type (d_1, d_2)) *and denoted by* $\mathcal{S})_{d_1,d_2}(\mathbb{K})$.

Denoting the coordinates of a generic point of $\mathsf{PG}(N, \mathbb{K})$ with $(x_{ij})_{0\le i\le d_1,0\le j\le d_2}$, the Segre variety $\mathcal{S})_{d_1,d_2}(\mathbb{K})$ consists precisely of the points $(x_{ij})_{0\le i\le d_1,0\le j\le d_2}$ such that, considered as a matrix, the rank of (x_{ij}) is equal to 1. For further reference, we call these the *matrix coordinates*.

This is all well known, see for instance [9]. We add here the following lemma. It is also well known, but we include a quick proof for convenience.

Lemma 1. *A Segre variety* $\mathcal{S})_{d_1,d_2}(\mathbb{K})$ *is determined by* $d_1 + 1$ *generators* $A_i := \{p_i\} \times \mathsf{PG}(d_2, \mathbb{K})$, $0 \le i \le d_1$, *with* $\{p_0, p_1, \dots, p_{d_1}\}$ *generating* $\mathsf{PG}(d_1, \mathbb{K})$, $d_2 + 1$ *generators* $B_j := \mathsf{PG}(d_1, \mathbb{K}) \times \{q_j\}$, $0 \le j \le d_2$, *with* $\{q_0, q_1, \dots, q_{d_2}\}$ *generating* $\mathsf{PG}(d_2, \mathbb{K})$, *and a point x of* $\mathcal{S})_{d_1,d_2}(\mathbb{K})$ *such that, if we write x as* $(p, q) \in \mathsf{PG}(d_1, \mathbb{K}) \times \mathsf{PG}(d_2,\ K)$, *then p is not contained in any hyperplane determined by* d_1 *points of* $\{p_0, p_1, \dots, p_{d_1}\}$, *and likewise for q.*

Proof. We claim that the generators through p are determined by the given generators and p itself. Indeed, our assumptions imply that A_k is complementary to $\overline{A}_k := \langle A_i \mid i \in \{0, 1, \dots, k-1, k+1, \dots, d_1\}\rangle$ (using a standard dimension argument). Hence $\langle p, \overline{A}_k\rangle$ intersects A_k in a unique point x_k. Now x_k is contained in the unique generator A through p intersecting all B_j. But the set of x_k, $0 \le k \le d_1$, generates A; hence A is determined. Likewise, the other generator B through p is determined.

Now the same argument, interchanging the roles of A with A_0, implies that each generator intersecting each A_k, $1 \le k \le d_1$ and A, is determined . But the union of all these generators is $\mathcal{S})_{d_1,d_2}(\mathbb{K})$. □

Remark 1. *The counterpart of 1 for Segre geometries is perhaps the following statement:* Let G_i be a basis of the projective space Π_i, $i = 1, 2$. Then $G_1 \times G_2$ generates the Segre geometry $\Pi_1 \times \Pi_2$. *Indeed, It is clear that* $\langle G_1 \times G_2\rangle$ *contains* $G_1 \times \Pi_2$. *Now every generator* $\Pi_1 \times \{x_2\}$, *with* $x_2 \in \Pi_2$, *intersects* $G_1 \times \Pi_2$ *in the set* $G_1 \times \{x_2\}$, *which is a basis of* $\Pi_1 \times \{x_2\}$. *Hence the latter is contained in* $\langle G_1 \times G_2\rangle$. *The assertion follows. In fact, we conjecture that* $G_1 \times G_2$ *is a basis of* $\Pi_1 \times \Pi_2$ *(which is certainly true if* $\Pi_1 \times \Pi_2$ *is embeddable).*

2.4. *Generalised Dualities*

Let again Π_1 and Π_2 be two projective spaces of non-negative dimension with respective points sets X_1 and X_2. We denote by Ω_i, $i = 1, 2$, the set of subspaces of Π_i, including the empty set and X_i itself. Extending the notion of a (generalised) polarity as defined by Tits [3, 8.3.2], a *generalised duality* between Π_1 and Π_2 is a relation $\bowtie$ between X_1 and X_2, such that, for each $x_1 \in X_1$, and each $x_2 \in X_2$, the sets $x_1^{\bowtie} := \{x_2 \in X_2 \mid x_1 \bowtie x_2\}$ and $x_2^{\bowtie} := \{x_1 \in X_1 \mid x_1 \bowtie x_2\}$ are (not necessarily proper) hyperplanes of Π_1 and Π_2, respectively. Set $\{i, j\} = \{1, 2\}$. We can extend $\bowtie$ to Ω_i by defining $S_i^{\bowtie} = \{x_j \in X_j \mid x_j \in x_i^{\bowtie}, \forall x_i \in S_i\}$. Then $S_i^{\bowtie}$ is an intersection of hyperplanes and hence a subspace. One now sees that $\bowtie$ defines a mapping from Ω_i to Ω_j reversing the inclusion relation. The *radical* $\mathsf{Rad}(\bowtie)$ of $\bowtie$ is by definition the pair $(X_2^{\bowtie}, X_1^{\bowtie})$. The generalised duality is called *nontrivial* if $\mathsf{Rad}(\bowtie) \ne (X_1, X_2)$. An *ordinary* duality is the mapping from Ω_1 to Ω_2 defined by $\bowtie$ in the case $\mathsf{Rad}(\bowtie) = (\emptyset, \emptyset)$. With these definitions we have the following lemma.

Lemma 2. *If* $\bowtie$ *is a nontrivial generalised duality between the projective spaces* Π_1 *and* Π_2 *with respective point sets* X_1 *and* X_2, *then*

$$1 \le \mathsf{dim}\Pi_1 - \mathsf{dim}X_2^{\bowtie} = \mathsf{dim}\Pi_2 - \mathsf{dim}X_1^{\bowtie} \le \min\{\mathsf{dim}\Pi_1, \mathsf{dim}\Pi_2\} + 1.$$

Proof. Set $\mathsf{dim}\Pi_i = d_i$ and $\mathsf{dim}X_i^{\bowtie} = \ell_i$, $i = 1, 2$. Let $\{p_0, \dots, p_{\ell_2}\}$ be a basis of $X_2^{\bowtie}$, and extend it to a basis $\{p_0, \dots, p_{d_1}\}$ of Π_1. We claim that $X_1^{\bowtie} = \{p_{\ell_2+1}, \dots, p_{d_1}\}^{\bowtie}$. Indeed, clearly $X_1^{\bowtie} \subseteq \{p_{\ell_2+1}, \dots, p_{d_1}\}^{\bowtie}$, so it remains to prove the reverse inclusion. Note that, if $x_2 \in X_2$ belongs to $x_1^{\bowtie} \cap y_1^{\bowtie}$, with $x_1, y_1 \in X_1$, then $\langle x_1, y_1 \rangle \subseteq x_2^{\bowtie}$, and hence x_2 belongs to $z_1^{\bowtie}$, for each $z_1 \in \langle x_1, y_1 \rangle$. It follows that, if $x_2 \in X_2$ belongs to $\{p_{\ell_2+1}, \ldots, p_{d_1}\}^{\bowtie}$, then, since it is also trivially contained in $\{p_0, \ldots, p_{\ell_2}\rangle^{\bowtie}$, we conclude $x_2 \in \langle p_0, \ldots, p_{d_1} \rangle^{\bowtie} = X_1^{\bowtie}$. The claim is proved.

Next we claim that the set $\{x_1^{\bowtie} \mid x_1 \in \langle p_{\ell_2+1}, \ldots, p_{\ell_2+e} \rangle\}$ is a dual subspace of Π_2, for $1 \le e \le d_1 - \ell_2$. Indeed, it suffices to prove that, given two point $y_1, z_1 \in X_1 \setminus X_2^{\bowtie}$, whenever $y_1^{\bowtie} \neq z_1^{\bowtie}$, then $\{x_1^{\bowtie} \mid x_1 \in \langle y_1, z_1 \rangle\}$ is a dual line. From the arguments in the previous paragraph it follows that it is contained in a dual line. Now let H_2 be any hyperplane of Π_2 containing $G_2 := y_1^{\bowtie} \cap z_1^{\bowtie}$. Select $x_2 \in H_2 \setminus G_2$. We may assume that $H_2 \notin \{y_1^{\bowtie}, z_1^{\bowtie}\}$. Then $x_2^{\bowtie} \cap \langle y_1, z_1 \rangle$ is a unique point u_1. It is now easy to see that $u_1^{\bowtie} = H_2$. The claim follows.

Next we claim that, for $1 \le e \le d_1 - \ell_2 - 1$, the hyperplane $p_{\ell_2+e+1}^{\bowtie}$ does not belong to the dual subspace $\{x_1^{\bowtie} \mid x_1 \in \langle p_{\ell_2+1}, \ldots, p_{\ell_2+e} \rangle\}$. Indeed, otherwise there exists $u_1 \in \langle p_{\ell_2+1}, \ldots, p_{\ell_2+e} \rangle$ with $u_1^{\bowtie} = p_{\ell_2+e+1}^{\bowtie}$. Considering $u_2^{\bowtie}$ for a point $u_2 \in X_2 \setminus u_1^{\bowtie}$, we see that some point of $\langle u_1, p_{\ell_2+e+1} \rangle$ belongs to $X_2^{\bowtie}$, a contradiction. The claim is proved.

It now follows from the previous claim that the dimension of $\{p_{\ell_2+1}, \ldots, p_{d_1}\}^{\bowtie}$, and hence of $X_1^{\bowtie}$ by our first claim, is equal to $d_2 - (d_1 - \ell_2)$.

The lemma is proved. □

By 2 the following definition makes sense. The *grayscale index* of a nontrivial generalised duality between the projective space Π_1 and Π_2 both of dimension at least 1, is the fractional number

$$\frac{\min\{\mathsf{dim}\Pi_1, \mathsf{dim}\Pi_2\} + 1 - \mathsf{dim}\Pi_1 + \mathsf{dim}X_2^{\bowtie}}{\min\{\mathsf{dim}\Pi_1, \mathsf{dim}\Pi_2\}},$$

which lies in the unit interval $[0, 1]$, and which is 1 if, and only if, $X_2^{\bowtie}$ is a hyperplane of Π_1 (or, equivalently, $X_1^{\bowtie}$ is a hyperplane of Π_2); it is 0 if, and only if, at least one of $X_1^{\bowtie}$ or $X_2^{\bowtie}$ is empty.

Generalised dualities with grayscale index 0 are called *black*, those with grayscale index 1 *white*, and those with grayscale index

$$1 - \frac{1}{\min\{\mathsf{dim}\Pi_1, \mathsf{dim}\Pi_2\}}$$

are called *silver*. Note that black and silver coincide when $\min\{\mathsf{dim}\Pi_1, \mathsf{dim}\Pi_2\} = 1$.

3. The Basic Observation

We observe the following connection between geometric hyperplanes of Segre geometries and generalised dualities.

Observation 1. *Let Π_1 and Π_2 be two projective spaces of positive dimension with respective points sets X_1 and X_2. Let $H \subseteq X_1 \times X_2$ be a set of points of the Segre geometry $\Pi_1 \times \Pi_2$. Then H is a (proper) geometric hyperplane if, and only if, the relation $\bowtie$ between X_1 and X_2 defined by $p_1 \bowtie p_2$, $p_i \in X_i$, $i = 1, 2$, if $(p_1, p_2) \in H$, is a nontrivial generalised duality.*

Proof. If H is a proper hyperplane, then it is clear that, for $i \in \{1, 2\}$, and for each $p_i \in X_i$, the set $p_i^{\bowtie}$ is either a hyperplane, or the whole set X_j, where $\{i, j\} = \{1, 2\}$, and at least once it is a (proper) hyperplane. Hence $\bowtie$ is a generalised duality.

Conversely, suppose $\bowtie$ is a generalised duality between Π_1 and Π_2. Set $H = \{(p_1, p_2) \mid p_1 \bowtie p_2\}$. Then clearly every generator intersects H in a hyperplane of the full generator, and at least once it is a proper hyperplane. Since every line is contained in some generator, H is hence a proper hyperplane. □

Geometric hyperplanes of Segre geometries corresponding to white, silver and black generalised dualities will be called *white, silver* and *black* themselves, respectively.

4. Classification and Description of all Generalised Dualities

In this section we prove the following classification result.

Theorem 2. *Let Π_1 and Π_2 be two projective spaces of nonnegative dimensions d_1, d_2, with $d_1 \leq d_2$ and with point sets X_1, X_2, respectively. Let $\bowtie$ be a nontrivial generalised duality between Π_1 and Π_2. Then it arises from one of the following constructions.*

(*i*) *The generalised duality is white, there exist hyperplanes $H_1 \subseteq X_1$ and $H_2 \subseteq X_2$ such that $x_1 \bowtie x_2$ if, and only if, either $x_1 \in H_1$ or $x_2 \in H_2$ (or both).*

(*ii*) *There are subspaces $S_1 \subseteq X_1$ and $S_2 \subseteq X_2$ of dimensions $d_1 - 2$ and $d_2 - 2$, respectively, and an arbitrary bijection β from the set of proper hyperplanes of X_1 containing S_1 to the set of hyperplanes of X_2 containing S_2 such that, for arbitrary $x_1 \in X_1$ and $x_2 \in X_2$, we have $x_1 \bowtie x_2$ if, and only if, either $x_1 \in S_1$, or $x_2 \in S_2$, or $x_1 \notin S_1$, $x_2 \notin S_2$ and $\beta(\langle x_1, S_1\rangle) = \langle x_2, S_2\rangle$. This generalised duality is silver.*

(*iii*) *There exist subspaces $S_1 \subseteq X_1$ and $S_2 \subseteq X_2$ of dimensions $d_1 - d$ and $d_2 - d$, respectively, with $3 \leq d \leq d_1 + 1$, also subspaces T_1, T_2 complementary to S_1, S_2, respectively (hence both of dimension $d - 1$), and an arbitrary ordinary duality $\beta : T_1 \to T_2$. such that, for arbitrary $x_1 \in X_1$ and $x_2 \in X_2$, we have $x_1 \bowtie x_2$ if, and only if, either $x_1 \in S_1$, or $x_1 \notin S_1$ and $x_2 \in \langle S_2, \beta(\langle x_1, S_1\rangle \cap T_1)\rangle$. If $d = d_1 + 1$, then the generalised duality is black.*

Proof. We first check that the given constructions indeed yield nontrivial generalised dualities. For (*i*) this is trivial. For (*ii*), this follows from the observation that $x_1^{\bowtie} = X_2$ if $x_1 \in S_1$, and $x_1^{\bowtie} = \beta(\langle x_1, S_1\rangle)$ if $x_1 \in X_1 \setminus S_1$; likewise for $x_2^{\bowtie}$ with $x_2 \in X_2$. Finally, a similar argument in case (*iii*) is valid for $x_1 \in X_1$. Now let $x_2 \in X_2$. Since S_2 appears to be contained in $x_1^{\bowtie}$, for each $x_1 \in X_1$, we have $x_2^{\bowtie} = X_1$, for each $x_2 \in S_2$. Now let $x_2 \in X_2 \setminus S_2$. Then the definition in (*iii*) implies that (only) all points x_1 of $\langle S_1, \beta^{-1}(\langle S_2, x_2\rangle \cap T_2\rangle$ satisfy $x_1 \bowtie x_2$, and these form a hyperplane of X_1. So, we indeed have generalised dualities.

Now let $\bowtie$ be a nontrivial generalised duality between Π_1 and Π_2. By 2, we have

$$1 \leq d := \mathsf{dim}\Pi_1 - \mathsf{dim}X_2^{\bowtie} = \mathsf{dim}\Pi_2 - \mathsf{dim}X_1^{\bowtie} \leq d_1 + 1.$$

First suppose $d = 1$. Setting $H_1 = X_2^{\bowtie}$ and $H_2 = X_1^{\bowtie}$, we see that we are in case (*i*).

Now suppose $d \geq 2$. Then choose subspaces T_1 and T_2 in Π_1 and Π_2, respectively, complementary to $S_1 := X_2^{\bowtie}$ and $S_2 := X_1^{\bowtie}$, respectively. Then $\mathsf{dim}T_1 = \mathsf{dim}T_2 = d - 1$. The mapping $\beta : T_1 \to T_2^*$, with T_2^* the dual of T_2 (hence, T_2^* consists of the hyperplanes of T_2), mapping each point $x_1 \in T_1$ to the hyperplane $x_1^{\bowtie} \cap T_2$ (which is indeed a hyperplane, as $x_1 \notin X_2^{\bowtie}$ and $S_1 = X_1^{\bowtie} \subseteq x_1^{\bowtie}$). Now if $d = 2$, then β is bijective since each point $x_2 \in T_2$ is the image under β of the point $T_1 \cap x_2^{\bowtie}$. Considering β as a bijection from the set of proper hyperplanes of Π_1 containing S_1 to the set of proper hyperplanes of Π_2 containing S_2 mapping $\langle S_1, x_1\rangle$ to $\langle S_2, \beta(x_1)\rangle$, $x_1 \in T_1$, we obtain the situation as in case (*ii*).

Now let $d \geq 3$. Then the argument in the second paragraph (the second claim) of the proof of 2 shows that the image under β of a subspace U_1 of T_1 is a subspace of T_2^* (a dual subspace of T_2), properly containing the image under β of a proper subspace of U_1. The Fundamental Theorem of Projective Geometry now implies that β is an isomorphism from T_1 to T_2^*, hence an ordinary duality from T_1 to T_2. □

There are some interesting consequences.

Corollary 1. (*i*) *Every Segre geometry contains white geometric hyperplanes.*

(*ii*) *A (proper) Segre geometry $\Pi_1 \times \Pi_2$, with $\mathsf{dim}\Pi_1 \leq \mathsf{dim}\Pi_2$, admits silver geometric hyperplanes if, and only if, the cardinalities of the line(s) of Π_1 and Π_2 are the same.*

(*iii*) *A (proper) Segre geometry $\Pi_1 \times \Pi_2$, with $\mathsf{dim}\Pi_1 \leq \mathsf{dim}\Pi_2$, admits a geometric hyperplane which is neither white nor silver if, and only if, $\mathsf{dim}\Pi_1 \geq 2$ and $\Pi_1 \times \Pi_2$ admits a black geometric* *hyperplane if, and only if,* $\mathsf{dim}\Pi_1 \geq 2$ *and every plane of* Π_1 *is isomorphic to the dual of every plane of* Π_2 *(the latter is equivalent to* Π_1 *being isomorphic to the dual of a subspace of* Π_2*).*

5. The Structure of Hyperplanes of Segre Geometries

5.1. Black Hyperplanes of Symmetric Segre Geomeries

A Segre geometry $\Pi_1 \times \Pi_2$ is called *symmetric* if Π_1 is isomorphic to the dual Π_2^* of Π_2 (the dual of a projective line being the same projective line).

A point-line geometry $\Delta = (X, \mathcal{L})$ is called a *long root subgroup geometry of type* A_n if there is a projective space Π of dimension n such that X can be identified with the set of incident point-hyperplane pairs of Π, and a typical member of $\mathcal{L}$ is given by the set of point-hyperplane pairs such that the point runs through all points of a given line, or the hyperplane runs through all hyperplanes containing a common subspace of dimension $n-2$.

Lemma 3. *Each black geometric hyperplane of a symmetric Segre geometry* $\Pi \times \Pi^*$ *carries the structure of a long root subgroup geometry of type* A_d*, where* $2 \leq d = \mathsf{dim}\Pi$*. Conversely, each full subgeometry of* $\Pi \times \Pi^*$ *abstractly isomorphic to a long root subgroup geometry of type* A_d *is a geometric hyperplane of* $\Pi \times \Pi^*$.

Proof. A black geometric hyperplane H of $\Pi \times \Pi^*$ is determined by an ordinary duality β between Π and Π^*. Identifying the hyperplanes of Π with their image under β, we see that H precisely consists of all incident point-hyperplane pairs of Π. The fact that lines of H correspond to lines of the long root subgroup geometry of type A_d follows from β being defined on all subspaces of Π (in particular on lines), and its inverse on all subspaces of Π^*.

Now suppose that X is the point set of a full subgeometry of $\Pi \times \Pi^*$ isomorphic to a long root subgroup geometry Δ of type A_d. First we assume $d \geq 3$.

Notice the following properties of Δ. We may assume that X is the set of incident point-hyperplane pairs of the projective space $\mathsf{PG}(d, \mathcal{L})$, for some skew field $\mathcal{L}$.

(*i*) There are two natural families of maximal singular subspaces (and they both have dimension $d-1$). One family corresponds to the set of incident point-hyperplane pairs of $\mathsf{PG}(d, \mathcal{L})$ with fixed hyperplane, and the other with fixed point. Hence all subspaces of one family are isomorphic to each other, and two such subspaces of distinct systems are dual to each other.

(*ii*) Every point is contained in exactly two singular subspaces of dimension $d-1$, one of each natural type, and those two subspaces intersect exactly in that given point.

(*iii*) Given two singular subspaces M and M' of dimension $d-1$ and of different natural systems. Then for each point $x \in M$ there exists at least one point $x' \in M'$ such that x and x' are collinear to a (unique) common point.

Let $x = (x_1, x_2) \in X$. Let M and M' be the two singular subspaces of Δ containing x (see (*ii*) above). Since $M \cap M' = \{x\}$ (by (*i*) above), the subspaces M and M' are contained in distinct maximal singular subspaces of $\Pi \times \Pi^*$. Without loss of generality we may assume that M is contained in $\{x_1\} \times \Pi^*$ and M' in $\Pi \times \{x_2\}$. By connectivity, it follows that every maximal singular subspace of Δ in the same system as M is contained in a singular subspace of the form $\{z_1\} \times \Pi^*$, with $z_1 \in \Pi_1$, and likewise for the other system. We claim that every subspace $\Pi \times \{y_2\}$, $y_2 \in \Pi^*$, contains a maximal singular subspace of Δ. Indeed, consider an arbitrary maximal singular subspace M'' of Δ disjoint from M and belonging to the same natural system as M'. Then there exists $x_2' \in \Pi^*$ such that $\Pi \times \{x_2'\}$ contains M''. Let If $x_2' = y_2$, then we are done, so assume $x_2' \neq y_2$. Then the line L joining (x_1, x_2') and (x_1, y_2) intersects M in some point (x_1, z_2). By (*iii*) above, there exist points $(x_1', z_2) \in X$ and $(x_1', x_2') \in M''$. Then the line L' joining (x_1', z_2) and (x_1', x_2') is contained in X and contains the point (x_1', y_2). Hence there is some maximal singular subspace of Δ containing (x_1', y_2) in $\Pi \times \{y_2\}$ and the claim follows.

Similarly, one shows that every subspace $\{y_1\} \times \Pi^*$ contains a subspace of dimension $d-1$ of Δ. It now follows that X is a geometric hyperplane of $\Pi \times \Pi^*$.

There remains the case $d = 2$. If we can show that, for each point $x \in X$, the lines of Δ through x are contained in different generators of $\Pi \times \Pi^*$, then the rest of the proof above is valid and the lemma would follow.

First suppose that two disjoint lines L, L' of Δ from different systems are contained in disjoint planes π, π', respectively, of $\Pi \times \Pi^*$. Call lines of Δ contained in planes disjoint from, or equal to, π *horizontal*, and the others *vertical* (then L and L' are horizontal; this means that lines of the form $\{*\} \times L^*$, with L^* a line of Π^*, are horizontal and those of the form $L \times \{*\}$, with L a line of Π vertical, or vice versa). Now, either (1) there are unique points $p \in L$ and $p' \in L'$ collinear in $\Pi \times \Pi^*$, or (2) each point of L is collinear to some point of L'. Suppose first the former. For each point $q \in L$ there exists a unique point $q' \in L'$ and a unique point $p_q \in X$ collinear to both q and q'. The lines qp_q and $q'p_q$ can only both be vertical if $q = p$ and $q' = p'$, and they are never both horizontal. Hence, without loss of generality, for at least $\left\lceil \frac{|mathbbK|}{2} \right\rceil$ points q, the line qp_q is horizontal. But two such distinct lines never meet in Δ, hence they should never meet in $\Pi \times \Pi^*$ either. Consequently $\left\lceil \frac{|mathbbK|}{2} \right\rceil \leq 1$, implying $|mathbbK| = 2$. In that case, we similarly see that the points p and p' correspond to each other, that is, there is a point $p'' \in X$ with pp'' and $p'p''$ lines of Δ (and they are vertical). We set $L = \{p, q, r\}$, and then we may assume qp_q and $r'p_r$ are vertical. Let s be the "third" point on the vertical line qp_q, and s' the third point on the vertical line $p'p''$. Then there is a point $s'' \in X$ such that ss'' and $s's''$ are lines of Δ. Let t be the third point on the line $r'p_r$. Since the points q and p_r are collinear in $\Pi \times \Pi^*$, and the same holds for p_q and r', the points s and t are also collinear in $\Pi \times \Pi^*$. So, the line ss'' is horizontal, as well as the line, say M, joining t with the third point on the line ss''. Now, since $p'p''$ and pp'' are both vertical, the line $s's''$ is horizontal (as otherwise pp'' and $s's''$ would intersect). But now ss'', $s's''$ and M are horizontal, implying that M and $s's''$ intersect, a contradiction.

Hence we may assume (2) that each point q of L is collinear to some point q'' of L' in $\Pi \times \Pi^*$. We also use the same notation for q' and p_q as in the previous paragraph. If $q' \neq q''$, then either $p_q \in L$ or $p_q \in L'$, both contradictions. Hence $q = q'$ and all lines qp_q and $q'p_q$ are vertical. For two distinct choices of q, say q and r, we now see that qp_q and $r'p_r$ are vertical and the same reasoning as in (1), interchanging vertical with horizontal, yields again a contradiction.

Now suppose for a contradiction that for some point $x \in X$ the two lines L, L' of Δ through x are contained in the same generator. Then for arbitrary points $p \in L \setminus \{x\}$ and $p' \in L' \setminus \{x\}$, the lines M and M' of Δ through p and p', respectively, distinct from L and L', respectively, are both vertical and belong to a different natural system. This contradicts the previous paragraphs. □

In the case $\mathsf{dim}\Pi = \mathsf{dim}\Pi^* = 1$, things are much easier. Recall that an *ovoid* of a grid is a set of points intersecting every generator in exactly one point. Obviously, an ovoid is a geometric hyperplane of a grid. Hence we have the following lemma, the proof of which is trivial.

Lemma 4. *If* $\mathsf{dim}\Pi = \mathsf{dim}\Pi^* = 1$*, then a black geometric hyperplane H of* $\Pi \times' \Pi^*$ *is equivalent to an ovoid. Also H defines a bijection* $\beta : \Pi \to \Pi^*$ *such that* $H = \{(x, \beta(x)) \mid x \in \Pi\}$ *and every such bijection defines a black geometric hyperplane of* $\Pi \times \Pi^*$. □

5.2. The General Case

We can now describe all geometric hyperplanes of an arbitrary Segre geometry. We may restrict ourselves to the proper case.

Theorem 3. *Let* $\Pi_1 \times \Pi_2$ *be a proper Segre geometry with* $\mathsf{dim}\Pi_1 = d_1 \leq d_2 = \mathsf{dim}\Pi_2$ *and let H be a proper geometric hyperplane.*

(*i*) *If H is white, then there exist hyperplanes W_1 and W_2 of* Π_1 *and* Π_2*, respectively, such that* $H = W_1 \times \Pi_2 \cup \Pi_1 \times W_2$. *Conversely, every such set is a white hyperplane.*

(*ii*) *If H is silver, then there exist subhyperplanes W_1 and W_2 and complementary lines L_1, L_2 of Π_1 and Π_2, respectively, and a bijection $\beta : L_1 \to L_2$, such that H is the union over all point $p \in L$ of $\langle W_1, p\rangle \times \langle W_2, \beta(p)\rangle$. Conversely, every such bijection defines via this expression a silver geometric hyperplane of $\Pi_1 \times \Pi_2$.*

(*iii*) *If H has grayscale index $1 - \frac{k}{d_1}$, with $2 \le k \le d_1$, then it arises from the following construction. There are subspaces W_1 and W_2 of Π_1 and Π_2, respectively, of dimension $d_1 - k - 1$ and $d_2 - k - 1$, respectively. There are also subspaces B_1 and B_2 of $\Pi_1 \times \Pi_2$, respectively, complementary to W_1, W_2, respectively, hence of dimension k, with B_2 dual to B_1. We set $B_1 = \mathsf{PG}(\mathcal{L}^k)$ and $B_2 = \mathsf{PG}(\mathcal{L}^k)^*$ (and use coordinate tuples to describe the points of both projective spaces). There is a skew field isomorphism $\theta : \mathcal{L} \to \mathcal{L}$, and H is the union over all points $(x_0, \dots, x_k)$ of B_1 of*

$$\langle W_1, (x_0, \dots, x_k)\rangle \times \langle W_2, (a_0, \dots, a_k)\rangle, \quad a_0^\theta x_0 + \cdots + a_k^\theta x_k = 0.$$

Conversely, every such set defines a hyperplane of $\Pi_1 \times \Pi_2$ of grayscale index $1 - \frac{k}{d_1}$. Geometrically, H is the union of all full subgrids having a point in the Segre subgeometry $W_1 \times W_2$ and one in a full long root subgroup subgeometry of type A_d in $B_1 \times B_2$.

Proof. This theorem follows from 1, 2, 3 and 4, taking into account that for every isomorphism between two projective spaces, coordinates can be chosen such that the isomorphism acts coordinatewise as a (common) skew field isomorphism. □

The letters "*W*" and "*B*" stand for the "white" and the "black", respectively. In particular, the intersection of $B_1 \times B_2$ with H is a (maximal) black geometric hyperplane of a full Segre subgeometry, which we call a *black part of H*. If H is silver, then a black part is given by the ovoid $H \cap L_1 \times L_2$, with the notation of 3(*ii*). The black part of a white hyperplane is empty.

6. Dual Segre Geometries

Just like the hyperplanes of a projective space define the point set of the dual projective space, the white hyperplanes of a Segre geometry $\Pi_1 \times \Pi_2$ define the point set of a dual Segre geometry $\Pi_1^* \times \Pi_2^*$ (with the obvious identification). Here we show that one can recognise the lines of the dual Segre geometry from considering intersections of white hyperplanes. The reason why we do this will become clear in 7.

Proposition 1. *Let H and H' be two white hyperplanes of the Segre geometry $\Pi_1 \times \Pi_2$. Then H and H' define collinear points in the dual Segre geometry if, and only if, there exists a white geometric hyperplane $H'' \notin \{H, H'\}$ of $\Pi_1 \times \Pi_2$ containing $H \cap H'$. In such a case, the dual line through H and H' consists of all white hyperplanes containing $H \cap H'$.*

Proof. Let $H = H_1 \times \Pi_2 \cup \Pi_1 \times H_2$ and $H' = H_1' \times \Pi_2 \cup \Pi_1 \times H_2'$. If either $H_1 = H_1'$ or $H_2 = H_2'$, say $H_1 = H_1'$, then it is obvious that a geometric hyperplane $H_1'' \times \Pi_2 \cup \Pi_1 \times H_2''$ contains $H \cap H'$ if, and only if, $H_1'' = H_1$. In this case we obtain a line of the dual Segre geometry, exactly as wanted by the assertion.

Now assume $H_1 \neq H_1'$ and $H_2 \neq H_2'$. Let $p_i \in H_i \setminus H_i'$ and $p_i' \in H_i' \setminus H_i$, $i \in \{1, 2\}$. Then both (p_1, p_2') and (p_1', p_2) belong to $H \cap H'$. Clearly, a white geometric hyperplane containing $H \cap H'$ must be of the form $H'' := H_1'' \times \Pi_2 \cup \Pi_1 \times H_2''$, with $H_i \cap H_i' \subseteq H_i''$, $i \in \{1, 2\}$. Then $(p_1, p_2') \in H''$ if, and only if, either $p_1 \in H_1''$ (that is, $H_1'' = H_1$) or $p_2' \in H_2''$ (that is, $H_2'' = H_2'$). Similarly, using $(p_1', p_2) \in H''$ implies that either $H_1'' = H_1'$ or $H_2'' = H_2$. If follows that $H'' \in \{H, H'\}$. □

7. Segre Varieties

The necessary conditions mentioned in the first paragraph of 2.3 show that Segre varieties of type (d_2, d_2), with $d_1 \le d_2$, admit geometric hyperplanes of any grayscale index $\frac{k}{d_1}$, $0 \le k \le d_1$. We now determine which geometric hyperplanes arise from hyperplanes of the ambient projective space.

It is clear from the description of the hyperplanes in 3 that, with the notation of 3(*iii*), H is generated, as a subspace, by $W_1 \times \Pi_1 \cup \Pi_1 \times W_2$ and any black part $\mathcal{B}$. In $\mathcal{S})_{d_1,d_2}(\mathbb{K})$, this generates a subspace of dimension

$$((d_1 - k)(d_2 + 1) - 1 + (d_2 - k)(d_1 + 1) - 1 - (d_1 - k)(d_2 - k) + 1) + \dim\langle\mathcal{B}\rangle + 1.$$

Since $\dim\langle\mathcal{B}\rangle \in \{k^2 + 2k, k^2 + 2k - 1\}$, we calculate that

$$\dim\langle H\rangle \in \{d_1d_2 + d_1 + d_2 - 1, d_1d_2 + d_1 + d_2\}.$$

We have shown:

Lemma 5. *A geometric hyperplane of a Segre variety is induced by a hyperplane of the ambient projective space if and only if any black part of it is induced by a hyperplane of the appropriate ambient projective subspace.* □

As in 3(*iii*), a black hyperplane H of $\mathcal{S})_{k,k}(\mathbb{K})$, $k \geq 2$, can be represented in coordinates by the set of points $((x_0, \ldots, x_k), (a_0, \ldots, a_k))$, with $a_0^\theta x_0 + \cdots + a_k^\theta x_k = 0$, where $\theta : \mathbb{K} \to \mathbb{K}$ is a field automorphism. It is clear that, if θ is the identity, then the equations represents a hyperplane of $\mathsf{PG}(k^2 + 2k, \mathbb{K})$; if θ is not the identity, then the points with coordinates in the fixed field satisfying this equation generate a hyperplane H' of $\mathsf{PG}(k^2 + 2k, \mathbb{K})$. Picking an element $a \in \mathbb{K}$ not fixed under θ, the point $((1, -a^\theta, 0, \ldots, 0), (a, 1, 0, \ldots, 0))$ belongs to H but not to H'.

If $k = 1$, then the black hyperplane H can be chosen to contain the points $((1, 0), (1, 0))$, $((0, 1), (0, 1))$ and $((1, 1), (1, 1))$, which span the hyperplane H' of $\mathsf{PG}(3, \mathbb{K})$ with equation $x_{12} = x_{21}$ (in matrix coordinates, see 2.3). Then

$$H = \{((1, x), (1, \beta(x))) \mid x \in \mathbb{K}\} \cup \{((0, 1), (0, 1))\},$$

with $\beta : \mathbb{K} \to \mathbb{K}$ a permutation of $\mathbb{K}$ fixing 0 and 1. Clearly the point $((1, x), (1, \beta(x)))$ belongs to H' if and only if $x = \beta(x)$, implying that H is induced by a hyperplane of $\mathsf{PG}(3, \mathbb{K})$ if and only if the permutation β is the identity.

The last two paragraphs now easily imply the following proposition.

Proposition 2. *A black geometric hyperplane of a Segre variety $\mathcal{S})_{k,k}(\mathbb{K})$ is induced by a hyperplane of the ambient projective space if, and only if, the involved duality of* $\mathsf{PG}(k, \mathbb{K})$ *(if $k \geq 2$), or permutation of* $\mathsf{PG}(1, \mathbb{K})$ *(if $k = 1$) stems from a linear transformation.* □

Combining 5 and 2, we obtain a precise description of all hyperplanes of any Segre variety induced by hyperplanes of the ambient projective space.

Now note that all white hyperplanes arise from hyperplanes of the ambient projective space. In fact, these form the dual Segre variety.

Proposition 3. *The hyperplanes of the ambient projective space* $\mathsf{PG}(V)$ *of a Segre variety* $\mathcal{S})_{d_1,d_2}(\mathbb{K})$ *inducing a white geometric hyperplane in* $\mathcal{S})_{d_1,d_2}(\mathbb{K})$ *form the point set in the dual* $\mathsf{PG}(V)^*$ *of* $\mathsf{PG}(V)$ *of an isomorphic Segre variety* $\mathcal{S})_{d_1,d_2}(\mathbb{K})^*$*. In dual matrix coordinates, that Segre variety is also given by the rank* 1 *matrices.*

Proof. We first claim that the intersection of two white hyperplanes of $\mathcal{S})_{d_1,d_2}(\mathbb{K})$ generates a subspace of dimension $N - 2$ of $\mathsf{PG}(N, \mathbb{K})$, with $N = d_1d_2 + d_1 + d_2$. Indeed, choosing appropriate coordinates we may assume that the two hyperplanes have respective equation $a_{11} = 0$ and $a_{ij} = 0$ in matrix coordinates. The claim follows now from the fact that every base point belongs to $\mathcal{S})_{d_1,d_2}(\mathbb{K})$. The first assertion of proposition now follows from 1.

Now the second assertion. The hyperplanes with respective equations $a_{ij} = 0$ and $a_{\ell k} = 0$ share the same generators from one system if, and only if, either $i - \ell$ or $j = k$. That means that the hyperplanes with dual matrix coordinates all zero except for the elements in one particular row or column, belong to the dual Segre variety. Now we claim that the hyperplane with dual coordinates all 1 induces a white geometric hyperplane H in $\mathcal{S})_{d_1,d_2}(\mathbb{K})$. Indeed, let W_1 be the hyperplane of $'\mathsf{PG}(d_1,\mathbb{K})$ with equation $x_0 + \cdots + x_{d_1} = 0$ and select arbitrarily a point $p \in W_1$ and an arbitrary point q in $\mathsf{PG}(d_2,\mathbb{K})$. Then clearly the matrix coordinates of $\sigma_{d_1,d_2}(p,q)$ have all rows identical (to the coordinates of q), hence the matrix has rank 1. Hence $W_1 \times \mathsf{PG}(d_2,\mathbb{K})$ belongs to H. Similarly, also $\mathsf{PG}(d_1,\mathbb{K}) \times W_2$, with W_2 the hyperplane of $'\mathsf{PG}(d_2,\mathbb{K})$ with equation $x_0 + \cdots + x_{d_2} = 0$, is contained in H. The claim follows.

Now the second assertion follows from 1. □

Corollary 2. *All geometric hyperplanes of a Segre variety arising from hyperplanes of the ambient projective space and with fixed grayscale index are projectively equivalent.*

Proof. This follows from 5 and 2 by noticing that a linear bijective map between two vector spaces can always be written with the identity matrix by choosing the bases appropriately. □

3 allows us to talk about black, silver and white points , and, more generally, about points with a certain grayscale index. Indeed, the grayscale index of a point is the grayscale index of the geometric hyperplane determined by that point in the dual Segre variety (in the dual projective space, that point is a hyperplane).

Proposition 4. *Let t be the grayscale index of a point p of $\mathsf{PG}(d_1d_2 + d_1 + d_2,\mathbb{K})$ with respect to $\mathcal{S})_{d_1,d_2}(\mathbb{K})$, with $d_1 \leq d_2$. Let r be the rank of the matrix coordinates of p. Then $d_1 + 1 = r + td_1$.*

Proof. First notice that, since the rank of the matrix of the coordinates of a point is an invariant for the automorphism group of the Segre variety, it suffices to show the given equality for one example of each rank. Dualising, we consider the hyperplane H with equation $x_{00} + x_{11} + \cdots + x_{r-1,r-1} = 0$. If B_1 and B_2 are the subspaces of $\mathsf{PG}(d_1,\mathbb{K})$ and $\mathsf{PG}(d_2,\mathbb{K})$, respectively, generated by the first r base points, and W_1 and W_2 those generated by the remaining base points, then we see that H contains in $W_1 \times \mathsf{PG}(d_2,\mathbb{K}) \cup \mathsf{PG}(d_1,\mathbb{K}) \times W_2$ and intersects $B_1 \times B_2$ in a black hyperplane. This implies, using 3, that the grayscale index t of H is equal to (and we apply the definition)

$$t = \frac{d_1 + 1 - r}{d_1},$$

from which we derive $td_1 + r = d_1 + 1$. □

8. The Finite Case

In the finite case, following a well-established habit, we denote the Segre variety $\mathcal{S})_{d_1,d_2}(\mathbb{F}_q)$ by $\mathcal{S})_{d_1,d_2}(q)$. There is a beautiful connection between black geometric hyperplanes of $\mathcal{S})_{2,2}(q)$ and embedded flag geometries of projective planes (the latter are the long root subgroup geometries of $\mathsf{PG}(2,q)$). Indeed, it follows from 3 that a black geometric hyperplane of $\mathcal{S})_{2,2}(q)$ can be presented as the set of points $(a_ix_j)_{0\leq i,j\leq 2}$, with $a_0^\theta x_0 + a_1^\theta x_1 + a_2^\theta x_2 = 0$, $a_i, x_j \in \mathbb{F}_q$, with θ a field automorphism. This is exactly the description of an arbitrary embedding of the long root subgroup geometry of $\mathsf{PG}(2,q)$ in either $\mathsf{PG}(7,q)$ or $\mathsf{PG}(8,q)$. So, we have shown the following result.

Proposition 5. *There is a bijective correspondence between the embedded long root subgroup geometries of $\mathsf{PG}(2,q)$ generating $\mathsf{PG}(7,q)$ or $\mathsf{PG}(8,q)$ and the point sets of $\mathsf{PG}(8,q)$ which are a black geometric hyperplane of some Segre variety of type $(2,2)$ of $\mathsf{PG}(8,q)$.*

Without any doubt, this proposition is also true for the long root subgroup geometries of $\mathsf{PG}(n,q)$, but there are no results available in the literature that combine with the current paper into a proof. And without any doubt, this proposition is wrong in the general case (over arbitrary fields), because there are fields $\mathbb{K}$ for which the hyperplane section of $\mathcal{S})_{2,2}(\mathbb{K})$ producing a black geometric hyperplane is *not* relatively universal, which means that also some embedding in $\mathsf{PG}(8,\mathbb{K})$ exists projecting down onto that hyperplane section. That embedding has very low probability of being contained in $\mathcal{S})_{2,2}(\mathbb{K})$ (but we have no proof of that, it is speculation at this point).

Noticing that $\mathsf{PG}L_2(q)$ is isomorphic to $\mathsf{Sym}(q+1)$, the symmetric group on $q+1$ letters, if, and only if, $q \in \{2,3\}$, and that $\mathsf{Aut}\mathbb{F}_q = \{\mathrm{id}\}$ if, and only if, q is a prime, we deduce from 3 the following result.

Proposition 6. *Every geometric hyperplane of $\mathcal{S})_{d_1,d_2}(q)$ is induced by a hyperplane of the ambient projective space if, and only if, $q \in \{2,3\}$.*

Let $\binom{m}{r}_q$ be the q-binomial coefficient, that is,

$$\binom{m}{r}_q = \frac{(q^m-1)(q^{m-1}-1)\cdots(q^{m-r+1}-1)}{(q^r-1)(q^{r-1}-1)\cdots(q-1)}$$

is the number of subspaces of dimension r in a vector space of dimension m over $\mathbb{F}_q$. Setting $q = p^e$, with p prime and e a natural number, the order of the automorphism group of $\mathbb{F}_q$ is e, and the next result follows now directly from 3.

Proposition 7. *Let $1 \le d_1 \le d_2$ be natural numbers.*

(*i*) *The number of white hyperplanes of $\mathcal{S})_{d_1,d_2}(q)$ is equal to*

$$\frac{q^{d_1+1}-1}{q-1}\cdot\frac{q^{d_2+1}-1}{q-1}.$$

(*ii*) *The number of silver hyperplanes of $\mathcal{S})_{d_1,d_2}(q)$ is equal to*

$$(q+1)!\binom{d_1+1}{d_1-1}_q\binom{d_2+1}{d_2-1}_q.$$

(*iii*) *The number of geometric hyperplanes of $\mathcal{S})_{d_1,d_2}(q)$ of grayscale index $1-\frac{k}{d_1}$, $k \ge 2$, is equal to*

$$e\cdot\binom{d_1+1}{d_1-k}_q\binom{d_2+1}{d_2-k}_q\cdot|\mathsf{PG}L_{k+1}(q)|.$$

(*iv*) *The number of geometric hyperplanes of $\mathcal{S})_{d_1,d_2}(q)$ of grayscale index $1-\frac{k}{d_1}$, $k \ge 0$, arising from hyperplanes of the ambient projectie space, is equal to*

$$\binom{d_1+1}{d_1-k}_q\binom{d_2+1}{d_2-k}_q\cdot|\mathsf{PG}L_{k+1}(q)|.$$

As an immediate consequence of 4 and 7, we have the following well-known result, due to Landsberg [10].

Corollary 3. *The number of $m \times n$ matrices over $\mathbb{F}_q$ of rank r is equal to*

$$\binom{m}{r}_q\binom{n}{r}_q\cdot|\mathrm{GL}_r(q)|.$$

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Declaration of Competing Interest

The author declares no conflict of interest to this work.

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