Ars Combinatoria, 160: 07–12
DOI:10.61091/ars-160-02
http://www.combinatorialpress.com/ars
Received 15 March 2020, Accepted 28 May 2020, Published 30 September 2024

Article

Nilpotent Covers of Dihedral Groups

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Abstract: Let G be a group, and let $c \in \mathbb{Z}^+ \cup \{\infty\}$. We let $\sigma_c(G)$ be the maximal size of a subset X of G such that, for any distinct $x_1, x_2 \in X$, the group $\langle x_1, x_2 \rangle$ is not c-nilpotent; similarly we let $\Sigma_c(G)$ be the smallest number of c-nilpotent subgroups of G whose union is equal to G. In this note we study D_{2k}, the dihedral group of order $2k$. We calculate $\sigma_c(D_{2k})$ and $\Sigma_c(D_{2k})$, and we show that these two numbers coincide for any given c and k.

Keywords: Dihedral group, Nilpotent cover, Non-nilpotent subset
2010 Mathematics Subject Classification: 05C70

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1. Introduction

In this paper we are interested in how best to cover a group with nilpotent subgroups. In order to make this precise, we need a definition. Note, first, that throughout this paper G is a finite group, and c is either a positive integer or the symbol ∞. We will say that a finite group N is *c-nilpotent* if it is nilpotent with nilpotency class at most c; in particular any nilpotent group is ∞-nilpotent. Now the definition that we want is as follows.

Definition 1. *Let c be a positive integer. A* c-nilpotent cover *of G is a family $\mathcal{M}$ of c-nilpotent subgroups of G whose union equals G. A c-nilpotent cover is called* minimal *if it is a c-nilpotent cover of minimal size.*

Note that a 1-nilpotent cover is nothing more than an *abelian cover* of G; similarly an ∞-nilpotent cover is a *nilpotent cover* of G. In what follows we write $\Sigma_c(G)$ for the size of a minimal c-nilpotent cover of G.

It turns out that there is a nice connection between c-nilpotent covers and "non-c-nilpotent subsets"; let us define this latter concept:

Definition 2. *A* non-c-nilpotent subset *is a subset X of G such that for any two distinct elements $x, y \in X$ the subgroup $\langle x, y \rangle$ they generate is not c-nilpotent.*

Note that a non-1-nilpotent subset is nothing more than a *non-commuting subset* of G; similarly a non-∞-nilpotent subset is a *non-nilpotent subset* of G. In what follows we write $\sigma_c(G)$ for the size of a maximal non-c-nilpotent subset of G. Now let us state the nice connection that we alluded to above.

Lemma 1. $\sigma_c(G) \leq \Sigma_c(G)$.

Proof. Let $\mathcal{A}$ be a c-nilpotent cover of G of size $\Sigma_c(G)$, and let X be a non-c-nilpotent subset of G. The result follows by observing that every element of X lies in an element of $\mathcal{A}$, and no element of $\mathcal{A}$ contains more than one element of X. □

In this note we are interested in the situation where $G = D_{2k}$, the dihedral group of order $2k$. For $c \in \mathbf{Z}^+ \cup \{\infty\}$ and $k \in \mathbb{Z}^+$ with $k \geq 2$, we are interested in calculating $\sigma_c(G)$, $\Sigma_c(G)$, and ascertaining for which values of c and k these two values coincide.

1.1. Main results

Our main result is the following.

Theorem 1. *Let $G = D_{2k}$ with $k \geq 2$, and let c be a positive integer.*

$$\sigma_c(G) = \Sigma_c(G) = \begin{cases} \frac{k}{\gcd(k,2^c)} + 1, & \text{if } k \notin \{2^1, 2^2, \dots, 2^c\}. \\ 1, & \text{if } k \in \{2^1, 2^2, \dots, 2^c\}. \end{cases} \tag{1}$$

In particular, writing $|k|_{2'}$ for the largest odd factor dividing k, we have

$$\sigma_\infty(G) = \Sigma_\infty(G) = \begin{cases} |k|_{2'} + 1, & \text{if } k \text{ is not a power of } 2. \\ 1, & \text{if } k \text{ is a power of } 2\text{ , i.e } G \text{ is nilpotent.} \end{cases}$$

Note that Theorem 1 implies that $\sigma_c(D_{2k}) = \Sigma_c(D_{2k})$ for all $c \in \mathbf{Z}^+ \cup \{\infty\}$ and all $k \in \mathbf{Z}^+$ with $k \geq 2$.

1.2. Connection to the literature

The value of $\Sigma_\infty(G)$ has been studied for various groups; it has usually been denoted $\omega(\mathcal{N}_G)$. Endimioni has proved that if a finite group G satisfies $\Sigma_\infty(G) \leq 3$ then G is nilpotent, while if $\Sigma_\infty(G) \leq 20$ then G is solvable; furthermore these bounds cannot be improved [1]. Tomkinson has shown that if G is a finitely generated solvable group such that $\Sigma_\infty(G) = n$, then $|G/Z^*(G)| \leq n^{n^4}$, where $Z^*(G)$ is the hypercentre of G [2].

The computation of $\sigma_\infty(G)$ for particular classes of groups G has recently started to garner attention. For instance, lower bounds for $\sigma_\infty(G)$ [3] when $G = \mathrm{GL}_n(q)$. The second author, along with Azad and Britnell, extended this result to deal with finite simple groups of Lie type – they showed that in such a group G, the size of a non-c-nilpotent subset (for any c) is approximately equal to the number of maximal tori in G [4].

The particular statistics $\sigma_1(G)$ and $\Sigma_1(G)$, have attracted considerable recent attention over many years, and have been calculated for various groups G; much of this work has concentrated on the case of almost simple groups G [5–10].

2. Background on Dihedral Groups

The set of symmetries of a regular k-gon form a group, the dihedral group of order $2k$, which we shall denote D_{2k}. We shall allow $k = 2$ here, in which case, D_{2k} is just the Klein 4-group.

We will refer to elements of D_{2k} in the usual way, as rotations and reflections; let us collect some basic facts that are all well-known.

Lemma 2. *1. The group D_{2k} has presentation $\{x, y : x^k = y^2 = 1, y^{-1}xy = x^{-1}\}$.*

2. *The set of rotations forms a cyclic subgroup, C_k, of order k; all elements in $D_{2k} \backslash C_k$ are reflections and so, in particular, have order 2.*

3. *Let x_1 be a rotation, y_1 a reflection; then $y_1^{-1}x_1y_1 = x_1^{-1}$.*

4. *Every subgroup of D_{2k} is either dihedral or cyclic.*

5. *Let y_1, y_2 be reflections in G with lines of reflection ℓ_1 and ℓ_2, respectively. Let θ be the angle between ℓ_1 and ℓ_2 and write $\theta = \frac{\pi a}{b}$ where a and b are co-prime integers. Then y_1y_2 is a rotation by 2θ and $\langle y_1, y_2\rangle \cong D_{2b}$.*

6. *If $\ell > 1$ is a divisor of k, then D_{2k} contains exactly k/ℓ subgroups isomorphic to $D_{2\ell}$. Every reflection of D_{2k} lies in a unique subgroup isomorphic to $D_{2\ell}$, and if ℓ_1, ℓ_2 are lines of reflection for two reflections in such a group, then the angle between ℓ_1 and ℓ_2 is an integer multiple of π/ℓ.*

Note that in (2) we implicitly took the identity element to be a rotation; we shall do this throughout.

Proof. Items (1) and (2) are standard facts about dihedral groups; see, for instance, [11]. Throughout this proof we shall fix x to be the anti-clockwise rotation by $2\pi/k$ so, in particular, x has order k.

Let us consider (3): Since the set of rotations forms a normal subgroup of D_{2k}, it is clear that $y_1^{-1}x_1y_1$ is a rotation and hence so is the element $y_1^{-1}x_1y_1x_1$. Now let P be a point on the edge of the k-gon that lies on the mirror line of y_1: one can check directly that the element $y_1^{-1}x_1y_1x_1$ fixes the point P. Since the only point fixed by a non-identity rotations is the center of the k-gon, we conclude that $x_1^{-1}y_1^{-1}x_1^{-1}y_1 = 1$. Thus $x_1^{-1} = y_1x_1y_1^{-1}$ as required.

For (4), let $H \leq D_{2k}$ and let C_k be the group of rotations in D_{2k}. There are two possibilities for $H \cap C_k$. Suppose $H = H \cap C_k$: in this case $H \leq C_k$ and so H is cyclic. Suppose $|H : H \cap C_k| = 2$: then $H \cap C_k$ is cyclic, and so $H \cap C_k = \langle x_1\rangle$ for some rotation x_1. Now let y be an element of $H \setminus (H \cap C_k)$; so y is a reflection. From (3) we have that $y^{-1}x_1y = x_1^{-1}$. Therefore $H = \langle x_1, y\rangle$ is a group of order 2ℓ such that $x_1^\ell = y^2 = 1$ and $y^{-1}x_1y = x_1^{-1}$. If $\ell > 1$, (1) implies that H is dihedral. If $\ell = 1$ then $|H| = 2$ and H is cyclic.

For (5), notice that $y_1y_2 = x_1$ is a rotation. Let P be a point on ℓ_2 on the edge of the k-gon. Let $P' = y_1y_2(P) = y_1(P)$and observe that $\angle POP' = 2\theta$. Hence y_1y_2 is a rotation by $2\theta = \frac{2\pi a}{b}$. Clearly y_1y_2 is a rotation of order b. Furthermore $\langle y_1, y_2\rangle = \langle x_1, y_2\rangle$. But now (3) implies that $y_2^{-1}x_1y_2 = x_1^{-1}$ and (1) implies that $\langle x_1, y_2\rangle = D_{2b}$ as required.

Finally, the second part of (6) follows directly from (5). For the first part, write $\ell = k/t$ and observe that x^t is of order ℓ. Then (1) implies that, for each $i = 0, \dots, t-1$, the group $H_i = \langle x^t, x^iy\rangle$ is dihedral of order $2(k/t) = 2l$. Direct calculation implies that the reflections in H_i consist of the set

$$\{x^iy, x^{i+t}y, x^{i+2t}y, \dots, x^{i+(\ell-1)t}y\}.$$

We conclude, first, that the groups are all distinct and every reflection is contained in such a group. Next observe that, if two reflections are in distinct groups H_i and H_j, then the angle between their mirror lines is not a multiple of π/ℓ, and so (5) implies that they do not lie *together* in a dihedral subgroup isomorphic to $D_{2\ell}$. In particular, $H_0, \dots, H_{t-1}$ are all of the subgroups isomorphic to $D_{2\ell}$, and every reflection lies in a unique one of these groups. □

The next three lemmas concern the nilpotency, or otherwise, of the dihedral group D_{2k}.

Lemma 3. *Let $G = D_{2k} = \{x, y : x^k = y^2 = 1, y^{-1}xy = x^{-1}\}$. Then*

$$Z(D_{2k}) = \begin{cases} \{1\}, & \text{if } k \text{ odd;} \\ \{1, x^{\frac{k}{2}}\}, & \text{if } k \text{ is even and } k > 2; \\ D_{2k}, & \text{if } k = 2. \end{cases}$$

In the case where k is even and $k > 2$, $D_{2k}/Z(D_{2k}) \cong D_k$.

Proof. If $k = 2$, the result is obvious. Assume, then, that $k > 2$. Lemma 2 (3) says that if x_1 is a rotation and y is a reflection, then $x_1y = yx_1^{-1}$. This implies, first, that since $k > 2$, no reflection lies in $Z(G)$; it implies, second, that a rotation lies in $Z(G)$ if and only if it is of order 2. The result follows. □

Lemma 4. *The group $G = D_{2k}$ is nilpotent if and only if k is a power of 2.*

Proof. If k is a power of 2, then $|G|$ is a power of 2, and hence G is nilpotent. If k is not a power of 2, then there exists a non-trivial rotation, x, of odd order. But now, if y is a reflection (and hence of even order), then Lemma 2 (3) implies that x and y do not commute. Thus G is not nilpotent. □

We investigate the situation where k is a power of 2 in more detail: We know that G is nilpotent, but what is its nilpotency class?

Lemma 5. *Let $G = D_{2^{c+1}}$ with $c \geq 1$. Then the nilpotency class of G is equal to c.*

Proof. If $c = 1$, then G is abelian and therefore has nilpotency class equal to 1, and the result holds. Now assume that the result is true for c and prove that it is true for $c + 1$. So let $G = D_{2(2^{c+1})}$. The upper central series of G is, by definition,

$$1 \lhd Z(G) \cdots \lhd G.$$

In particular, the number of terms between $Z(G)$ and G in the upper central series for G is equal to the number of terms in the upper central series for $G/Z(G)$. But Lemma 3 implies that $Z(D_{2k}) \cong C_2$ and $G/Z(G) \cong D_{2(2^c)}$. By assumption $D_{2(2^c)}$ has nilpotency class equal to c. Therefore the nilpotency class of G is $c + 1$, as required. □

3. Main Results

Our aim in this section is to prove Theorem 1. We will prove Theorem 1 with two lemmas.

Lemma 6. *Let $G = D_{2k}$ with $k \geq 2$ and let c be a positive integer. Then there exists a c-nilpotent cover of G of size t where*

$$t = \begin{cases} \frac{k}{\gcd(k,2^c)} + 1, & \text{if } k \notin \{2^1, 2^2, \ldots, 2^c\}. \\ 1, & \text{if } k \in \{2^1, 2^2, \ldots, 2^c\}. \end{cases}$$

Proof. If $k \in \{2^1, 2^2, \ldots, 2^c\}$, then Lemma 5 implies that G is nilpotent of class $\leq c$, and so $\{G\}$ is a c-nilpotent cover of G, and the result holds. Assume, then, that $k \notin \{2^1, 2^2, \ldots, 2^c\}$.

Write $\ell = \gcd(k, 2^c)$. Observe, first, that there is a cyclic subgroup containing all rotations. We need to show, therefore, that we can find $\frac{k}{\ell}$ c-nilpotent subgroups containing all reflections. But now Lemma 2(6) implies that every reflection lies in one of $\frac{k}{\ell}$ subgroups isomoprhic to $D_{2\ell}$. Lemma 5 says that $D_{2\ell}$ is nilpotent of class $\leq c$ and we are done. □

Lemma 7. *Let $G = D_{2k}$ with $k \geq 2$ and let c be a positive integer. Then there exists a non-c-nilpotent subset of G of size t where*

$$t = \begin{cases} \frac{k}{\gcd(k,2^c)} + 1, & \text{if } k \notin \{2^1, 2^2, \ldots, 2^c\}. \\ 1, & \text{if } k \in \{2^1, 2^2, \ldots, 2^c\}. \end{cases}$$

Proof. If $k \in \{2^1, 2^2, \ldots, 2^c\}$, then we can take any element $x \in G$ and $\{x\}$ will be (vacuously) a non-c-nilpotent subset of G, and the result holds. Assume, then, that $k \notin \{2^1, 2^2, \ldots, 2^c\}$.

Write $\ell = \gcd(k, 2^c)$. Let x be a rotation of order k, and y any reflection. Since k does not divide 2^c, x does not lie in a nilpotent dihedral subgroup of nilpotency class $\leq c$. We need to show, therefore, that we can find $\frac{k}{\ell}$ reflections, any pair of which does not generate a c-nilpotent subgroup.

Consider the set $y, xy, x^2y, \ldots, x^{k/\ell-1}y$. Any pair of these reflections have mirror lines that differ by an angle *strictly less* than π/ℓ. Lemma 2 (5) implies that they generate a dihedral group of order strictly greater than ℓ. Then, Lemma 5 implies that this group is not nilpotent of class at most c. The result follows. □

We have all the components that we need to prove our main result.

Proof of Theorem 1. Write t for the quantity on the right hand side of (1). Lemma 6 implies that $\Sigma_c(G) \leq t$; Lemma 7 implies that $t \leq \sigma_c(G)$; finally Lemma 1 implies that $\sigma_c(G) \leq \Sigma_c(G)$. We conclude that

$$t \leq \sigma_c(G) \leq \Sigma_c(G) \leq t.$$

We conclude immediately that $\sigma_c(G) = \Sigma_c(G) = t$, as required. □

Declaration of Competing Interest

The authors declare no conflicts of interest to this work.

Acknowledgments

The main result of this paper is from the PhD thesis of the first author. The second author is one of the supervisors of this thesis, and this supervisory arrangement has been made possible through a grant from the London Mathematical Society's "Mentoring African Research Mathematicians" scheme. Both authors would like to thank the LMS for their invaluable support. Both authors would like to thank Dr Ian Short for helpful discussions and support.

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