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Article

The Monophonic-triangular Number of a Graph

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Abstract: A path $x_1, x_2, \ldots, x_n$ in a connected graph G has no edge $x_i x_j$ $(j \geq i + 3)$ is called a *monophonic-triangular path* or *mt*-path. A non-empty subset M of $V(G)$ is a *monophonic-triangular set* or *mt-set* of G if every member in $V(G)$ exists in a *mt-path* joining some pair of members in M. The *monophonic-triangular number* or *mt-number* is the lowest cardinality of an *mt-set* of G and it is symbolized by $mt(G)$. The general properties satisfied by *mt*-sets are discussed. Also, we establish *mt*-number boundaries and discover similar results for a few common graphs. Graphs G of order p with $mt(G) = p$, $p - 1$ or $p - 2$ are characterized.

Keywords: Monophonic path, Monophonic-triangular path, Monophonic-triangular set, Monophonic-triangular number
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1. Introduction

In this paper, a graph $G = (V, E)$ refers to a finite undirected connected graph that lacks loops and multiple edges. The order and size of a graph are represented by p and q, respectively. For basic definitions and terminologies, we refer to [1–3]. A vertex v of a graph G is said to be *simplicial* or *extreme* if the subgraph induced by its neighbors in complete. If the degree of a vertex v is one, then v is called an *end-vertex*.

In graph theory, there are many types of paths of interest joining any two vertices in a graph. The important paths joining any two vertices in a graph are geodesic path (a shortest path), detour path (a longest path), monophonic path (a chordless path), detour monophonic path (a longest chordless path) and triangle free detour path (a longest path having no triangle). In the year 2021, Santhakumaran and Titus [4] introduced a new path named as monophonic-triangular path or *mt*-path by not allowing a cycle of order more than 3 (i.e., a triangle) in it. Hence, if an edge e of a monophonic path P is an edge of a triangle in G, then we enlarge the path P by replacing the edge e by the remaining two edges of the triangle. Let Q be the revised path. Then the paths P and Q are known as monophonic-triangular path or *mt*-path. Clearly, the monophonic-triangular path covers more number of vertices than the usual monophonic path. Therefore, the monophonic-triangular path is a more powerful tool for covering the vertices of a graph.

The distance $d(a, b)$ between the vertices a and b in a graph G is the length of one of the shortest

$a-b$ paths in G. An *$a-b$ geodesic* is an $a-b$ path with length $d(a,b)$. A non-empty subset S of $V(G)$ is a *geodetic set* of G if every member in $V(G)$ lies on a geodesic path joining some pair of members in S. The *geodetic number* is the smallest cardinality of a geodetic set of G and it is denoted by $g(G)$. The geodetic number of a graph was introduced in [5] and further investigated in [6–8].

The *detour distance* $D(a,b)$ between the vertices a and b in a graph G is the length of one of the longest $a-b$ paths in G. In [9], the detour distance was first introduced. An *$a-b$ detour* is an $a-b$ path with length $D(a,b)$. A non-empty subset S of $V(G)$ is a *detour set* of G if every member in $V(G)$ lies on a detour path joining some pair of members in S. The *detour number* of a graph is the smallest cardinality of a detour set of G and it is denoted by $dn(G)$. The detour number of a graph was introduced in [10] and expanded upon in [11, 12].

An $a-b$ path P is said to be an *$a-b$ monophonic path* in G if P has no chords. The *monophonic distance* $d_m(a,b)$ between the vertices a and b in a graph G is the length of one of the longest $a-b$ monophonic paths in G. The monophonic distance was introduced by Santhakumaran and Titus in [13] and further studied in [14]. A non-empty subset S of $V(G)$ is a *monophonic set* of G if every member in $V(G)$ lies on a monophonic path joining some pair of members in S. The *monophonic number* is the smallest cardinality of a monophonic set of G and it is denoted by $m(G)$. The concept of monophonic number was studied in [15–17].

An *$a-b$ detour monophonic path* is one of the longest $a-b$ monophonic paths. A non-empty subset S of $V(G)$ is a *detour monophonic set* of G if every member in $V(G)$ lies on a detour monophonic path joining some pair of members in S. The *detour monophonic number* is the lowest cardinality of a *detour monophonic set* of G and it is symbolized by $dm(G)$. The idea of detour monophonic number of a graph was presented in [18], which was expanded upon in [19].

An $a-b$ path P is said to be an *$a-b$ triangle free path* in G if no three vertices of P produce a cycle C_3 in G. The *triangle free detour distance* $D_{\triangle f}(a,b)$ between the vertices a and b in a graph G is the length of one of the longest $a-b$ triangle free path in G. An *$a-b$ triangular free detour* is an $a-b$ triangle free path with length $D_{\triangle f}(a,b)$. The concept was studied in [20] and further studied in [21]. A non-empty subset S of $V(G)$ is a *triangle free detour set* of G if every member in $V(G)$ exists in a triangle free path joining some pair of members in S. The *triangle free detour number* is the lowest cardinality of a triangle free detour set of G and it is symbolized by $dn_{\triangle f}(G)$.

A path $x_1, x_2, \ldots, x_n$ in a connected graph G with no edge $x_i x_j$ $(j \geq i+3)$ is called a *monophonic-triangular path* or *mt*-path. The *monophonic-triangular distance* or *mt-distance* $d_{mt}(a,b)$ from a to b is defined as the length of one of the longest $a-b$ *mt*-paths in G. The *mt-eccentricity* $e_{mt}(v)$ of a vertex v in G is defined as the maximum *mt*-distance between v and other vertices in G. The *mt-radius* $rad_{mt}(G)$ is defined as the minimum *mt*-eccentricity among the vertices of G and the *mt-diameter* $diam_{mt}(G)$ is defined as the maximum *mt*-eccentricity among the vertices of G. The concept of monophonic-triangular distance in graphs introduced in [4]. This new distance motivates us to introduce a new parameter *"monophonic-triangular number"*.

The following theorems are used in the sequel.

Theorem 1. [4] *Let G be a connected graph of order $p \geq 2$. Then $diam_{mt}(G) = 1$ if and only if $G = K_2$.*

Theorem 2. [16] *Every extreme vertex of a connected graph G belongs to every monophonic set of G.*

Theorem 3. [16] *Let G be a connected graph of order $p \geq 3$. Then $m(G) = p-1$ if and only if $G = K_1 + \cup m_j K_j$, where $\sum m_j \geq 2$.*

Theorem 4. [5] *Each extreme vertex of a connected graph G belongs to every geodetic set of G.*

2. Monophonic-triangular Number of a Graph

Definition 1. *A non-empty subset M of $V(G)$ is a monophonic-triangular set or mt-set of G if every member in $V(G)$ exists in an mt-path joining some pair of members in M. The monophonic-triangular number or mt-number is the smallest cardinality of an mt-set of G and it is symbolized by $mt(G)$.*

Example 1. *(i) For the graph G given in Figure 1, $M_1 = \{u, x\}$ and $M_2 = \{z, x\}$ are the minimum mt-sets of G. Hence $mt(G) = 2$.*

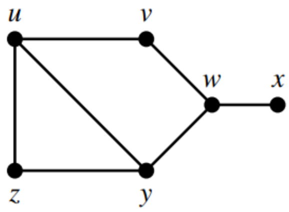

Figure 1. A Graph G with $mt(G) = 2$

(ii) Consider the graph G given in Figure 2. It can be easily verified that $M_1 = \{a_1, a_3, a_5, b_2, b_3, c_2, c_4, d_1, d_2\}$ is a minimum geodetic set, $M_2 = \{a_1, b_1, c_1, d_1\}$ is a minimum detour set, $M_3 = \{a_1, a_5, b_2, c_2, c_4, d_1, d_2\}$ is a minimum monophonic set, $M_4 = \{a_1, a_5, b_2, b_3, c_2, c_4, d_1, d_2\}$ is a minimum detour monophonic set, $M_5 = \{a_1, a_5, b_1, c_1, d_1, d_2\}$ is a minimum triangle free detour set and $M_6 = \{a_3, b_2, c_2, c_4, d_1\}$ is a minimum mt-set of G and so $d(G) = 9$, $D(G) = 4$, $m(G) = 7$, $dm(G) = 8$, $dn_{\triangle}f(G) = 6$ and $mt(G) = 5$. Hence the parameters based on the paths geodesic, detour, monophonic, detour monophonic, triangle free detour and monophonic-triangular are different.

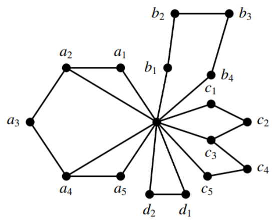

Figure 2. A Graph G with $d(G) = 9$, $D(G) = 4$, $m(G) = 7$, $dm(G) = 8$, $dn_{\triangle}f(G) = 6$ and $mt(G) = 5$

Theorem 5. *(i) Each end-vertex of G is a member of every mt-set of G.*

(ii) No cut-vertex of G is a member of any minimum mt-set of G.

Proof. (i) Let v be an end-vertex of G. Then v is either the initial vertex or the terminal vertex of any monophonic-triangular path containing the vertex v. Hence v is not an internal vertex of any monophonic-triangular path so that v is a member of every mt-set of G.

(ii) Suppose v is a cut-vertex of G and let M be a minimum mt-set of G that contains v. Now, claim that each component of $G - v$ contains an element of M. If not, then there is a component B of $G - v$ such that B contains no element of M. Let u be any vertex in B. Since M is an mt-set, there exist two vertices x and y in M such that v exists in some $x - y$ monophonic-triangular path, say P. Then $u \neq x, y$. Since v is a cut-vertex of G, the $x - u$ subpath P_1 of P and the $u - y$ subpath P_2 of P both contain v, it results that P is not a path, it contradicts our assumption. Thus each component of $G - v$ contains an element of M. Let V_1 and V_2 be two different components of $G - v$ and let $u \in V_1$ and $w \in V_2$. Then v is an internal vertex of an $u - w$ monophonic-triangular path in G. Let $M' = M - \{v\}$. Then each vertex that lies on an $u - v$ monophonic-triangular path also exists on an $u - w$ monophonic-triangular path and hence M' is an mt-set of G, it contradicts to M a minimum mt-set of G.

□

Corollary 1. *If G is a tree graph with some k end-vertices, then* $mt(G) = k$.

Corollary 2. *If a graph G with* $k \geq 2$ *end-blocks, then* $mt(G) \geq k$.

Corollary 3. *In a graph G, if k is the largest number of blocks to which a vertex in G belongs, then* $mt(G) \geq k$.

Theorem 6. *1. For the graph* K_p $(p \geq 2)$, $mt(K_p) = 2$.

2. *For the graph* C_p $(p \geq 3)$, $mt(C_p) = 2$.

3. *For the graph* $W_p = K_1 + C_{p-1}$ $(p \geq 5)$, $mt(W_p) = 2$.

4. *For the graph* $K_{m,n}$ $(m, n \geq 2)$, $mt(K_{m,n}) = min\{4, m, n\}$.

Proof. 1. For any two vertices x and y in K_p, any vertex $v \neq x, y$ is a member on the x, v, y monophonic-triangular path. Hence $M = \{x, y\}$ is an mt-set of K_p and so $mt(K_p) = 2$.

2. For any two non-adjacent vertices x and y in C_p, every vertex of C_p exists on an $x-y$ monophonic-triangular path. Hence $M = \{x, y\}$ is an mt-set of C_p and so $mt(C_p) = 2$.

3. Let $M = \{x, y\}$, where x and y are any two non-adjacent vertices in W_p. It is obvious that each vertex of W_p exists on an $x - y$ monophonic-triangular path. Then M is an mt-set of W_p and so $mt(W_p) = 2$.

4. Let the partite sets of $K_{m,n}$ $(2 \leq m \leq n)$ be $V_1 = \{x_1, x_2, \ldots, x_m\}$ and $V_2 = \{y_1, y_2, \ldots, y_n\}$. When $m = 2$ or 3, $M = V_1$ is a minimum mt-set of $K_{m,n}$ and so $mt(K_{m,n}) = |M| = m$. When $m \geq 4$, Let $M = \{x_1, x_2, y_1, y_2\}$. It is evident that every vertex in V_1 is a member on a $y_1 - y_2$ monophonic-triangular path and every vertex in V_2 is a member on an $x_1 - x_2$ monophonic-triangular path. As a result, M is an mt-set of $K_{m,n}$ and hence $mt(K_{m,n}) \leq |M| = 4$. It is evident that neither two members nor three members subset of $V(K_{m,n})$ will form an mt-set of $K_{m,n}$, we have $mt(K_{m,n}) = 4$.

□

Proposition 1. *If G is any connected graph, then* $2 \leq mt(G) \leq m(G) \leq p$.

Proof. To form an mt-set, we need minimum two vertices and so $mt(G) \geq 2$. Since every monophonic set is a monophonic-triangular set, $mt(G) \leq m(G)$. Also, since $V(G)$ is a monophonic set, we have $m(G) \leq p$. Thus $2 \leq mt(G) \leq m(G) \leq p$. □

Remark 1. *For the cycle* C_p, $mt(C_p) = 2$ *and for the complete graph* K_2, $mt(K_2) = 2 = p$. *Therefore the bounds for* $mt(G)$ *in Proposition 1 are sharp.*

We provide a better upper bound for the mt-number of a graph in the subsequent theorem.

Theorem 7. *For any connected graph G with p vertices,* $mt(G) \leq p - diam_{mt}(G) + 1$.

Proof. Let $P : a = a_0, a_1, \ldots, a_d = b$ be an $a - b$ mt-path of length $d = diam_{mt}(G)$. Then $S = V(G) - \{a_1, a_2, \ldots, a_{d-1}\}$ is an mt-set of G and so $mt(G) \leq |S| = p - d + 1$. Hence $mt(G) \leq p - diam_{mt}(G) + 1$.

□

Remark 2. *In* K_3, $diam_{mt}(K_3) = 2$ *and* $mt(K_3) = 2 = p - diam_{mt}(K_3) + 1$. *Therefore the bound in Theorem 7 is sharp. The inequality in Theorem 7 can also be strict. For the cycle* C_p $(p \geq 4)$, $diam_{mt}(C_p) = p - 2$ *and* $mt(C_p) = 2$. *Thus* $mt(C_p) < p - diam_{mt}(C_p) + 1$. *Also, since* $diam_m(G) \leq diam_{mt}(G)$, *we have* $mt(G) \leq p - diam_m(G) + 1$.

Theorem 8. *Let G be a graph with order $p \geq 2$. Then $G = K_2$ if and only if $mt(G) = p$.*

Proof. Supposing that $G = K_2$, subsequently $mt(G) = 2 = p$. Conversely, let $mt(G) = p$. If $G \neq K_2$, then G contains minimum 3 vertices. Since G is connected with at least 3 vertices, $diam_{mt}(G) \geq 2$. Then by Theorem 7, $mt(G) \leq p - diam_{mt}(G) + 1 \leq p - 1$, it contradicts our assumption. Hence $G = K_2$. □

Theorem 9. *Let G be a graph with order $p \geq 3$. Then G is either K_3 or $K_{1,p-1}$ if and only if $mt(G) = p - 1$.*

Proof. Supposing that $G = K_3$ or $K_{1,p-1}$, subsequently by Theorem 6(i) and Corollary 1, $mt(G) = p-1$. Conversely, suppose that $mt(G) = p - 1$. Then by Proposition 1, we have $m(G) = p - 1$ or p. If $m(G) = p$, then $G = K_p$ and so by Theorem 6 (i), $mt(G) = 2 = p - 1$ only for $p = 3$. Hence $G = K_3$. If $m(G) = p - 1$, then by Theorem 3, $G = K_1 + \cup m_j K_j$, where $\sum m_j \geq 2$. Now, we claim that G is a star. i.e., $j = 1$. If not, then G has at least one clique K_j of order more than one. Let S be a collection of exactly one vertex from each clique of $G = K_1 + \cup m_j K_j$. Clearly, S is an mt-set of G and so $mt(G) \leq |S| \leq p - 2$, it contradicts our assumption. Hence G is a star. □

Theorem 10. *Let G be a graph with order $p \geq 5$. Then G is either a double star or $K_{1,p-1} + e$ if and only if $mt(G) = p - 2$.*

Proof. If G is a double star, then by Corollary 1, $mt(G) = p - 2$. If $G = K_{1,p-1} + e$, then G contains exactly one cut-vertex, say x; two simplicial vertices of degree two, say y_1 and y_2; and $p - 3$ end-vertices. Let M be the end-vertices set of G. By Theorem 5(i), every member of M is included in each mt-set of G. Let $M' = M \cup \{y_1\}$. As a result, a minimum mt-set of G is M' and hence $mt(G) = p - 2$.

Conversely, let G be a graph of order $p \geq 5$ such that $mt(G) = p - 2$. Then by Theorem 1, $diam_{mt}(G) \geq 2$. If $diam_{mt}(G) \geq 4$, then by Theorem 7, $mt(G) \leq p - 3$, it contradicts our assumption. Hence $diam_{mt}(G) = 2$ or 3. If G is a tree, then G is either a star or a double star. If G is a star, then by Corollary 1, $mt(G) = p - 1$, it contradicts our assumption. If G is a double star, then by Corollary 1, $mt(G) = p - 2$. Now, let G be not a tree. Let k denote the length of the longest cycle with no inner chords in G. Since $diam_{mt}(G) = 2$ or 3, we have $k \leq 5$. Now we have three cases.

Case 1. $k = 5$. Let 5-cycle in G be $C_5 = v_1, v_2, v_3, v_4, v_5, v_1$.

Then mt-set of G is $M = (V(G) - V(C_5)) \cup \{v_1, v_3\}$ and hence $mt(G) \leq p - 3$, it contradicts our assumption.

Case 2. $k = 4$.

Let 4-cycle in G be $C_4 = v_1, v_2, v_3, v_4, v_1$. Since $p \geq 5$ and G is connected, there exists a vertex, say u, not on C_4 such that u is adjacent to some vertices in C_4. If u is adjacent to exactly one vertex, say v_1, then an mt-set of G is $M = V(G) - \{v_1, v_2, v_4\}$ and hence $mt(G) \leq p - 3$, it contradicts our assumption. If u is adjacent to two consecutive vertices of C_4, say v_1 and v_2, then also M is an mt-set of G and so $mt(G) = p - 3$, it contradicts our assumption. If u is adjacent to two non-consecutive vertices of C_4, say v_1 and v_3, then $M_1 = V(G) - \{u, v_2, v_4\}$ is an mt-set of G and so $mt(G) = p - 3$, it contradicts our assumption.

Case 3. $k = 3$.

If G contains 2 or more edge distinct triangles, then $diam_{mt}(G) \geq 4$. Then by Theorem 7, $mt(G) \leq p-3$, it contradicts our assumption. If G contains two common edge triangles, then G_1 or G_2 in Figure 3 is a subgraph of G. As a result, mt-set of G is $M = V(G) - \{x_1, x_2, x_3\}$ and hence $mt(G) \leq p - 3$, it contradicts our assumption. If G contains three or more common edge triangles, then G_3 in Figure 3 is a subgraph of G. As a result, mt-set of G is $M = V(G) - \{x_1, x_2, x_3\}$ and hence $mt(G) \leq p - 3$, it contradicts our assumption. Hence G contains a unique triangle $C_3 = v_1, v_2, v_3, v_1$. If there are two or three vertices of C_3 having degree 3 or more, then $diam_{mt}(G) \geq 4$, it contradicts our assumption. Thus exactly one vertex in C_3 has degree 3 or more. Since $diam_{mt}(G) = 3$, it follows that $G = K_{1,p-1} + e$.

□

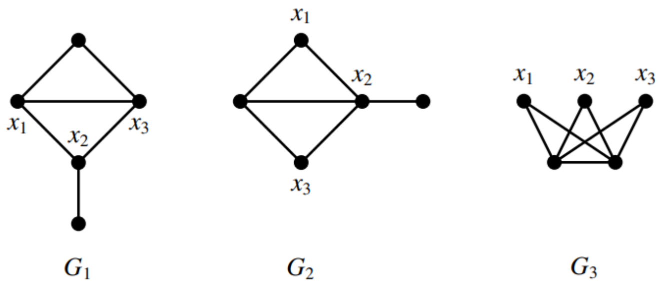

Figure 3. The Subgraphs of G in Case 3 of Theorem 10

Remark 3. *Let G be a graph with order $p \leq 4$. Then $mt(G) = p - 2$ if and only if G is any connected graph of order four other than a star.*

3. Realization Result

Theorem 11. *If G is any connected graph of order p, then $2 \leq mt(G) \leq m(G) \leq g(G) \leq p$.*

Proof. Obviously, every monophonic path is an mt-path and every geodesic is a monophonic path. Hence every monophonic set is an mt-set and every geodetic set is a monophonic set, and so $mt(G) \leq m(G) \leq g(G)$. Then the result follows from Proposition 1. □

Theorem 12. *For each triple (k, l, n) with $2 \leq k \leq l \leq n$, there exists a connected graph G with $mt(G) = k$, $m(G) = l$ and $g(G) = n$.*

Proof. If $2 \leq k = l = n$, then take G as a tree with number of end-vertices k. Then by Theorems 2, 4 and Corollary 1, we have $mt(G) = m(G) = g(G) = k$. In other cases, we construct a graph G as follows: Let P represent a path u_1, u_2, u_3, u_4 of length 3 and P_i represent $n - l$ copies of a path x_i, y_i $(1 \leq i \leq n - l)$ of length 1. Assume G is the graph formed by connecting each x_i $(1 \leq i \leq n - l)$ to u_2, connecting each y_i $(1 \leq i \leq n - l)$ to u_4, and adding $l - 1$ new vertices $v_1, v_2, \ldots, v_{k-1}, w_1, w_2, \ldots, w_{l-k}$ and connecting each v_i $(1 \leq i \leq k - 1)$ to u_4, and connecting each w_i $(i \leq i \leq l - k)$ to u_3 and u_4. The graph G is shown in Figure 4.

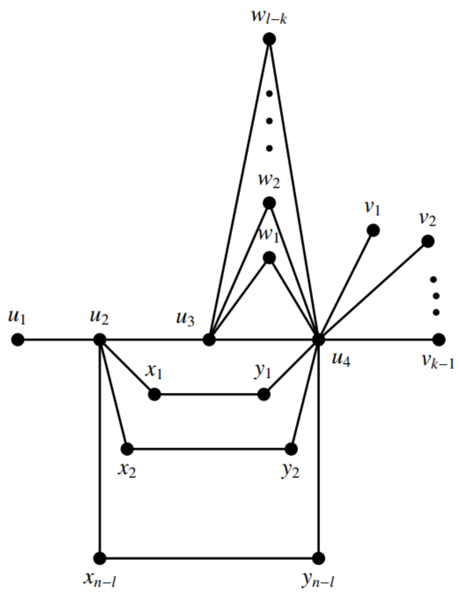

Figure 4. The Graph G in Theorem 12

The set of simplicial vertices is $M' = M \cup \{w_1, w_2, \ldots, w_{l-k}\}$, where $M = \{u_1, v_1, v_2, \ldots, v_{k-1}\}$ is the set of end-vertices. By Theorem 5(i), every member of M is included in each mt-set of G.

As M is itself an mt-set, $mt(G) = |M| = k$. Similarly, by Theorem 2, every member of M' is included in each monophonic set of G. Obviously, M' is a monophonic set of G and hence $m(G) = |M'| = l$. Now, by Theorem 4, every member of M' is included in each geodetic set of G. Obviously, the vertices x_i and y_i $(1 \leq i \leq n-l)$ do not lie on any geodesic connecting any two vertices in M'. As a result, M' is not a geodetic set of G and hence $g(G) > |M'|$. We can easily verify that either x_i or y_i $(i \leq i \leq n-l)$ must belong to every geodetic set of G. Let $M'' = M' \cup \{x_1, x_2, \ldots, x_{n-l}\}$. Now, every vertex of G exists on a geodesic joining two vertices in M''. Hence M'' is a geodetic set of G and so $g(G) = |M''| = n$. In the next theorem, we construct a graph of prescribed order, mt-diameter and mt-number under suitable conditions. □

Theorem 13. *For each triple (k, l, p) with $2 \leq l \leq p-k+1$ and $k \geq 3$, there exists a connected graph G with $diam_{mt}(G) = k$, $mt(G) = l$ and $|V(G)| = p$.*

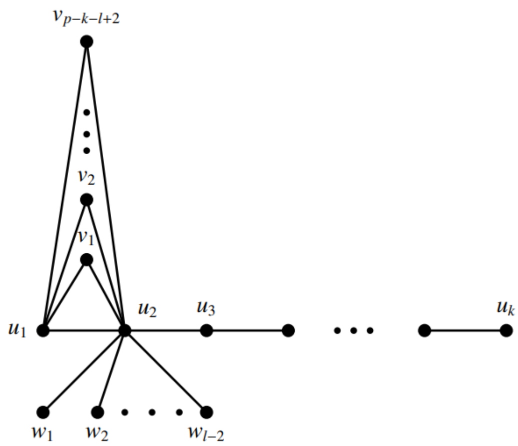

Figure 5. The Graph G in Theorem 13

Proof. Assume G is the graph formed from the path $P_k : u_1, u_2, \ldots, u_k$ of order k by adding $p-k$ new vertices $v_1, v_2, \ldots, v_{p-k-l+2}, w_1, w_2, \ldots, w_{l-2}$ and connecting each w_i $(1 \leq i \leq l-2)$ to u_2, and joining each v_i $(1 \leq i \leq p-k-l+2)$ to both u_1 and u_2. The graph G is shown in Figure 5 and the order of the graph G is p. Obviously, for any $x \in V(G)$, $2 \leq e_{mt}(x) \leq k$ and for any $y \in \{u_1, u_k, v_1, v_2, \ldots, v_{p-k-l+2}\}$, $e_{mt}(y) = k$. Hence $diam_{mt}(G) = k$. Now, we claim that $mt(G) = l$. Let the end-vertices set of G be $M = \{w_1, w_2, \ldots, w_{l-2}, u_k\}$. Then by Theorem 5(i), every member of M is included in each mt-set of G and hence $mt(G) \geq |M| = l-1$. Obviously, the vertices $u_1, v_1, v_2, \ldots, v_{p-k-l+2}$ do not exist on any $u-v$ monophonic-triangular path for every $u, v \in M$. Let $M' = M \cup \{u_1\}$. Since the vertices $u_1, v_1, v_2, \ldots, v_{p-k-l+2}$ lie on an $u_1 - y$ monophonic-triangular path for some $y \in M$, a minimum mt-set of G is M' and hence $mt(G) = |M'| = k$. Hence the result. □

Ostrand [22] has shown that every two positive integers k and l with $k \leq l \leq 2k$ are realisable as the radius and diameter of a graph, respectively, because $rad(G) \leq diam(G) \leq 2\, rad(G)$. Santhakumaran et al. [16] have shown that every two positive integers k and l with $k \leq l$ are realisable as the monophonic radius and monophonic diameter of a connected graph, respectively, because $rad_m(G) \leq diam_m(G)$. Similarly, since $rad_{mt}(G) \leq diam_{mt}(G)$, it was demonstrated by Titus et.al. [4] that all positive integers k and l with $k \leq l$, are realisable as the mt-radius and mt-diameter of a connected graph, respectively. This theorem can also be extended so that the mt-number can be prescribed when $rad_{mt}(G) < diam_{mt}(G)$.

Theorem 14. *For any triple (k, l, n) with $2 \leq k < l$ and $n \geq 3$, there is a connected graph G with $rad_{mt}(G) = k$, $diam_{mt}(G) = l$ and $mt(G) = n$.*

Proof. We consider three cases to prove this theorem.
Case 1. $k+2 \leq l \leq 2k$.

Let the cycles of order $k+2$ and $l-k+2$ be $C_1 : u_0, u_1, \ldots, u_{k+1}, u_0$ and $C_2 : v_0, v_1, \ldots, v_{l-k+1}, v_0$, respectively.

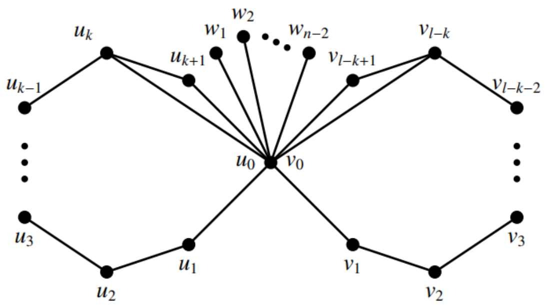

Figure 6. The Graph G in Case 1 of Theorem 14

Assume H is the graph formed from C_1 and C_2 by (i) merging the vertices u_0 of C_1 and v_0 of C_2, (ii) connecting the vertices u_0 and u_k of C_1, and (iii) connecting the vertices v_0 and v_{l-k} of C_2. Assume G is the graph formed from H by adding $n-2$ new vertices $w_1, w_2, \ldots, w_{n-2}$ and connecting each vertex $w_i (1 \le i \le n-2)$ to the vertex v_0 in H. In Figure 6, the graph G is shown.

Obviously, the mt-eccentric vertex of u_0 is u_2, the mt-eccentric vertex of u_2 is v_2, and so $e_{mt}(u_0) = k$ and $e_{mt}(u_2) = l$. Also, for any vertex x in G, $k \le e_{mt}(x) \le l$. Hence $rad_{mt}(G) = k$ and $diam_{mt}(G) = l$. Let $M = \{w_1, w_2, \ldots, w_{n-2}\}$ be the end-vertices set of G. By Theorem 5(i), every member of M is included in any mt-set of G and hence $mt(G) \ge |M| = n-2$. Also, it is evident that every mt-set of G contains at least one vertex from each cycle C_1 and C_2. Hence $mt(G) \ge n$. Let $M' = M \cup \{u_2, v_2\}$. Clearly, M' is an mt-set of G and hence $mt(G) = |M'| = n$.

Case 2. $l = k+1$.

Assume G is a graph formed from a cycle $C : u_0, u_1, \ldots, u_{k+1}, u_0$ by connecting the vertices u_0 and u_k, and adding $n-1$ new vertices $v_1, v_2, \ldots, v_{n-1}$ and connecting each vertex $v_i (1 \le i \le n-2)$ to the vertex u_0 of C. In Figure 7, the graph G is shown.

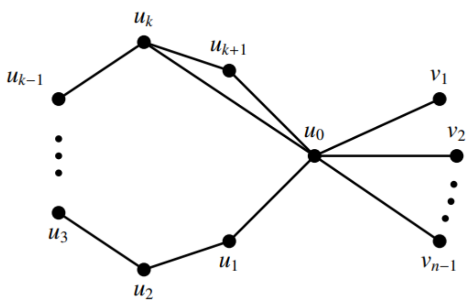

Figure 7. The Graph G in Case 2 of Theorem 14

Obviously, the mt-eccentric vertex of u_0 is u_2, the mt-eccentric vertex of v_1 is u_2, and $e_{mt}(x)$ is either k or $k+1$ for any vertex x in G. Hence $rad_{mt}(G) = k$ and $diam_{mt}(G) = k+1 = l$. Let the end-vertices set of G be $M = \{v_1, v_2, \ldots, v_{n-1}\}$. By Theorem 5(i), every member of M is included in any mt-set of G and hence $mt(G) \ge |M| = n-1$. It is evident that the vertices in $V(C) - \{u_0\}$ do not lie on any $x - y$ mt-path in G for any $x, y \in M$. Hence M is not an mt-set of G and so $mt(G) > |M| = n-1$. Let $M' = M \cup \{u_2\}$. Clearly, M' is an mt-set of G and hence $mt(G) = |M'| = n$.

Case 3. $l > 2k$.

Let $W = K_1 + C_{l+2}$ be a wheel with $V(K_1) = \{x\}$ and $V(C_{l+2}) = \{v_1, v_2, \ldots, v_{l+2}\}$, and let $C : u_0, u_1, \ldots, u_{k+1}, u_0$ be a cycle of order $k+1$. Assume G is the graph formed from the wheel W and the cycle C by merging the vertex x of W and u_0 of C, and by adding $n-3$ new vertices $w_1, w_2, \ldots, w_{n-3}$ and connecting each w_i $(1 \le i \le n-3)$ to the vertex x of W. In Figure 8, the graph G is shown.

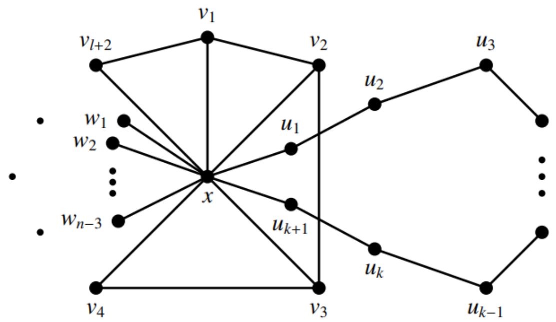

Figure 8. The Graph G in Case 3 of Theorem 14

Obviously, u_k is an mt-eccentric vertex of x, v_{l+2} is an mt-eccentric vertex of v_1, and so $e_{mt}(x) = k$ and $e_{mt}(v_1) = l$. Also, for any vertex u in G, $k \leq e_{mt}(u) \leq l$. Hence $rad_{mt}(G) = k$ and $diam_{mt}(G) = l$. Let $M = \{w_1, w_2, \ldots, w_{n-3}\}$ be the end-vertices set of G. By Theorem 5(i), every member of M is included in any mt-set of G and hence $mt(G) \geq |M| = n - 3$. It is clear that every mt-set of G contains at least one vertex from the cycle C and at least two vertices from the cycle C_{l+2} of the wheel W. Hence $mt(G) \geq n$. Let $M' = M \cup \{u_k, v_1, v_{l+1}\}$. Clearly, M' is an mt-set of G and hence $mt(G) = |M'| = n$. □

Problem 1. *For any pair (k, n) with $k \geq 2$ and $n \geq 2$, is there any connected graph G with $rad_{mt}(G) = diam_{mt}(G) = k$ and $mt(G) = n$?*

4. Conclusion

In this paper, we presented bounds and realization results for the mt-number of a connected graph. Additionally, we provided some characterization results for the parameter $mt(G)$. These findings contribute to a deeper understanding of the parameter monophonic-triangular number.

Furthermore, to enhance the security and efficiency of networks, we aim to implement a routing protocol based on the monophonic-triangular number of a graph. This protocol has the potential to optimize routing paths, reduce latency, and increase the overall robustness of network communications. By leveraging the unique properties of mt-numbers, we can develop innovative solutions for secure and efficient data transmission, which could be particularly beneficial for applications in cybersecurity, telecommunications, and large-scale distributed systems.

In summary, our work not only advances the theoretical understanding of the mt-number in graphs but also paves the way for practical applications that could significantly impact various fields that rely on complex network structures.

Declaration of Competing Interest

There is no conflict of interest related to this work.

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