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3-Trees with Diameter 2

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ABSTRACT

A k-tree is a graph that can be formed by starting with K_{k+1} and iterating the operation of making a new vertex adjacent to all the vertices of a k-clique of the existing graph. A structural characterization of 3-trees with diameter at most 2 is proven. This implies a corollary for planar 3-trees which leads to a description of their degree sequences.

Keywords: k-tree, Diameter, Planar graph, Degree sequence

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1. Introduction

In this paper, we seek a structural (constructive) characterization of 3-trees with diameter at most 2.

Definition 1.1. A *k-tree* is a graph that can be formed by starting with K_{k+1} and iterating the operation of making a new vertex adjacent to all the vertices of a k-clique (the *root*) of the existing graph. A *deletion sequence* of a graph G is an ordering $v_1, \ldots, v_n$ of $V(G)$ such that each v_i has minimum degree in the induced subgraph $G[\{v_i, v_{i+1}, \ldots, v_n\}]$.
A *k-leaf* is a degree k vertex of a k-tree.

See [5] for a survey of results on k-trees. There are many results describing and char-

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acterizing the structure of k-trees. Graphs with diameter 2 have been studied in relation to many other graph classes, such as cages and planar graphs [11].

Definition 1.2. The *distance* between vertices u and v, $d(u,v)$, is the length of a shortest $u-v$ path. The *diameter* of a graph G is the maximum distance between any pair of vertices in G.

A k-tree has diameter 1 if and only if it is K_{k+1}. For 2-trees, the following theorem was proved in [6].

Proposition 1.3. [6] *The following are equivalent for a 2-tree G:*

1. G is has diameter at most 2.

2. G does not contain P_6^2.

3. G is $T+K_1$ for any tree T, or any graph formed by adding any number of vertices adjacent to pairs of vertices of K_3.

Note that $1 \Leftrightarrow 2$ is a forbidden subgraph characterization, while $1 \Leftrightarrow 3$ is a structural (constructive) characterization. In [4], Proposition 1.3 was generalized to a structural characterization of maximal 2-degenerate graphs with diameter 2. In [2], a forbidden subgraph characterization was found for k-trees with diameter $d \geq 2$. Thus the next natural questions are to find a structural characterization of 2-trees with diameter 3 and 3-trees with diameter 2.

Definitions of terms and notation not defined here appear in [3]. In particular, $n(G)$ is the number of vertices of a graph G. The neighborhood of a vertex v is denoted $N(v)$, and the closed neighborhood is denoted $N[v]$. The square G^2 is formed by adding all edges between pairs of vertices with distance 2 in G. The join of graphs G and H is denoted $G+H$.

2. Preliminaries

One way for a k-tree to have diameter at most 2 is for there to be a vertex adjacent to all other vertices.

Definition 2.1. A *dominating vertex* of a graph is a vertex adjacent to all other vertices. When constructing a k-tree, we *duplicate* a k-leaf by adding another k-leaf with the same neighborhood.

The following observations should be immediate.

Lemma 2.2. *Let T be a k-tree with diameter at least 2.*

a. Adding a k-leaf to T cannot reduce the diameter.

b. Duplicating a k-leaf arbitrarily many times will not change the diameter.

Proposition 2.3. *A k-tree has diameter at most 2 if and only if any two k-leaves of G*

have a common neighbor.

Proof. A k-tree G has diameter at most 2 if and only if the distance between any two vertices of G is at most 2. By Lemma 2.2, this will be the case if and only if any two k-leaves are at distance at most 2. This will hold if and only if any two k-leaves of G have a common neighbor. □

Definition 2.4. A *k-path graph* G is an alternating sequence of distinct k- and $k+1$-cliques $e_0, t_1, e_1, t_2, ..., t_p, e_p$, starting and ending with a k-clique and such that t_i contains exactly two k-cliques e_{i-1} and e_i.

For order $n > k+1$, k-paths are just the k-trees with exactly two k-leaves [10]. See Figures 1 and 2 for examples of k-paths.

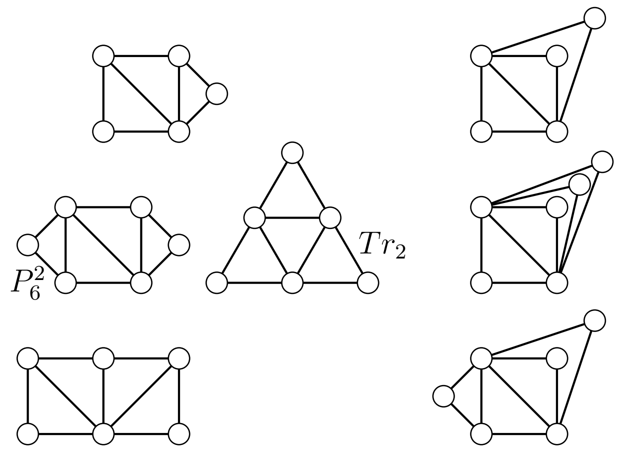


Fig. 1. The 2-trees of order 5 and 6 are shown above. Those in the first column are 2-paths. The one in the second column is outerplanar but not a 2-path. The rest are not outerplanar

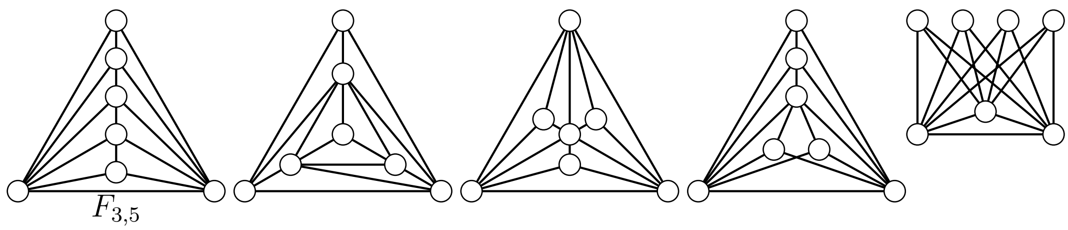


Fig. 2. The 3-trees with order 7. The leftmost two are 3-paths, and the leftmost three are maximal planar.

Lemma 2.5. *A graph T of order $n > k+1$ is a k-tree if and only if $T+K_1$ is a $(k+1)$-tree. Moreover, T is a k-path if and only if $T+K_1$ is a $(k+1)$-path.*

Proof. (⇒) Any k-tree T has a deletion sequence $v_1 \cdots v_n$ so that $d(v_i) = \max\{k, n-i\}$ when v_i is deleted. Joining a vertex x to T results in a graph T' with a deletion sequence $v_1 \cdots v_n x$ so that $d(v_i) = \max\{k+1, n+1-i\}$ when v_i is deleted. Thus T' is a $(k+1)$-tree.

(⇐) Let $T + K_1$ have the K_1 denoted x. Then $T + K_1$ has a deletion sequence $v_1 \cdots v_n x$ so that $d(v_i) = \max\{k+1, n+1-i\}$ when v_i is deleted. Thus T has a deletion sequence $v_1 \cdots v_n$ so that $d(v_i) = \max\{k, n-i\}$ when v_i is deleted, so T is a k-tree.

The proof for k-paths is essentially the same. □

3. 2-trees with Diameter 3

In this section, we characterize 2-trees with diameter at most 3.

Definition 3.1. A *dominating triple* is three vertices $\{x, y, z\}$ that form a triangle of a 2-tree T so that any 2-leaf of T is adjacent to at least one of them. A *private neighbor* of x (in a dominating triple) is adjacent to x, but not y or z.

A *common triple* is three vertices $\{x, y, z\}$ that form a triangle of a 2-tree T so that any 2-leaf of T is adjacent to at least two of them.

Theorem 3.2. *A 2-tree T has diameter at most 3 if and only if T has a dominating triple.*

Proof. (⇐) If T has a dominating triple, then there is a path of length at most 3 between any two vertices of T.

(⇒) Suppose that T has diameter at most 3. The result is obvious for diameter 1 or 2, so suppose T has diameter 3.

We use induction on n. Assume the result holds for 2-trees with order n, and let T have order $n+1$, and 2-leaf v. Now $T - v$ has diameter at most 3, so it has a dominating triple $t = \{x, y, z\}$. If v is adjacent to any of its vertices, T also has a dominating triple and we are done. Thus we assume that T has no dominating triple with a vertex adjacent to v.

Deleting two vertices of t (say x and y) will disconnect v from the third (z). Thus there is a vertex w adjacent to x and y in the same component of $T - x - y$ as v. We may assume that z has no private neighbor a, since else $d(v, a) = 4$. But then $\{x, y, w\}$ is also a dominating triple. Thus by our assumption, v is not adjacent to w. Say $d(v, x) = 2$. Then y has no private neighbor b, since else $d(v, b) = 4$. But then x is a dominating vertex of $T - v$. Let $N(v) = \{u_1, u_2\}$. Then T has a dominating triple $\{x, u_1, u_2\}$. □ A

fan is $P_r + K_1$, where P_r is a path. Call K_1 the *center* of the fan.

Proposition 3.3. *A 2-tree T has a dominating triple $t = \{x, y, z\}$ if and only if it has a covering by fans centered at the three vertices of t.*

Proof. (⇐) If this holds, any vertex of T is adjacent to a vertex of t.

(⇒) Let T have a dominating triple $t = \{x, y, z\}$. Let v be a vertex not in t, so v is adjacent to a vertex x of t. If v is adjacent to two vertices of t, it is contained in a fan centered at x. Else v is adjacent to x and a vertex u not in t. Now u is adjacent to x, and the argument can be repeated, producing a fan centered at x. □

4. 3-trees with Diameter 2

To characterize 3-trees with diameter 2, we use the strategy of starting with a 3-tree with a dominating vertex, and then considering what can be added while maintaining diameter 2.

Definition 4.1. A *k-fan* $F_{k,r}$ is $K_{k-1} + P_r$. Call the K_2 in a 3-fan its *base*.

Thus a 2-fan is just a fan. Any k-fan is a k-path, and hence also a k-tree. Any 3-fan is maximal planar (see Figure 2).

We may be able to identify a triangle of a 3-fan with a triangle of a 3-tree (with the base as one of the identified edges) while maintaining diameter 2. Call this operation *fan overlapping*. Fan overlapping produces only 3-trees since identifying k-cliques of two k-trees produces another k-tree.

Theorem 4.2. *Let T be 3-tree. Then T has diameter at most 2 if and only if it is formed in one of the following ways.*

1. $T = H + K_1$, where H is a 2-tree.

2. Let K_4 have vertices $\{u, x, y, z\}$. Then T is formed by fan overlapping, where the base of the fan must be ux, uy, or xy, and adding 3-leaves with root $\{x, y, z\}$.

3. Let uxy be the K_3 in $K_3 + \overline{K}_r$, $r \geq 1$. Then T is formed by fan overlapping, where the base of the fan must be ux, uy, or xy.

Proof. (⇐) Clearly each construction produces a 3-tree. In Case 1, there is a dominating vertex. In Case 2, every pair of vertices not in $\{u, x, y, z\}$ has a neighbor in $\{u, x, y, z\}$. In Case 3, every pair of vertices not in $\{u, x, y\}$ has a neighbor in $\{u, x, y\}$. Thus each 3-tree has diameter at most 2.

(⇒) Assume the hypotheses. Let u have maximum degree in T, $S = V(T) - N[u]$, and $H = N(u)$. Now H is a 2-tree [7], so $T - S$ is a 3-tree. Thus if u is a dominating vertex, $T - u$ is a 2-tree by Lemma 2.5. Thus we assume T has no dominating vertex, so S is nonempty.

Clearly, every vertex in S neighbors a vertex in H. Let R be all vertices in H with neighbors in S. Every vertex in R is contained in a triangle of H, and each pair of these triangles must have a have a nonempty intersection, since else two vertices of S have no common neighbor. Then R is a union of triangles, and the graph induced by R has diameter 2. It is contained in a minimal 2-tree T' which has diameter 2, so by Proposition 1.3, T' has a dominating vertex or a common triple.

Suppose T' has a dominating vertex x. Now H must have diameter 2, since else some vertex in S would be more than 2 away from a vertex in H. If x is dominating in H, x is also dominating in T. By assumption, we can exclude this case. Thus H has a common triple, so T' does also.

Next we assume that $T' = K_3$, whose vertices are $\{x, y, z\}$, none of which is dominating in H. There is at least one vertex v in S whose neighbors are T'. Then any other vertex in H is adjacent to a vertex in T' and u. Then T is formed by fan overlapping with bases ux, uy, or xy, and adding at least one 3-leaf with root $\{x, y, z\}$.

Next we assume that $T' = K_2 + \overline{K}_s$, the vertices of K_2 are $\{x, y\}$, neither of which is dominating in H. Then for each vertex w in the $\overline{K}_s$, there is at least one vertex in S not adjacent to it (else we return to the previous case). Then every vertex of T not in $D = \{u, x, y\}$ is adjacent to at least two vertices in D. Thus every vertex not in D is part of a 3-fan with base ux, uy, or xy, and there is at least one vertex of T adjacent to all vertices of D.

Finally, we assume T' contains a triangle xyz, any other vertex of T' is adjacent to exactly two of $\{x, y, z\}$, and each pair (xy, xz, and yz) has at least one additional neighbor in T'. Now each vertex of T' is adjacent to at least one vertex in S, so $H = T'$. Thus each vertex in S is adjacent to at least two vertices in $\{x, y, z\}$. Thus each vertex in T is adjacent to at least two vertices in $\{x, y, z\}$. But then we can return to the previous case by giving $\{x, y, z\}$ the roles of $\{u, x, y\}$. □ This characterization allows us to evaluate

or bound parameters of 3-trees with diameter 2. In the following results, we refer to the three graph classes in the statement of Theorem 4.2 as Cases 1, 2, and 3.

Corollary 4.3. A 3-tree with diameter 2 and order $n \geq 5$ and maximum degree Δ has $n \leq \frac{5\Delta - 5}{3}$.

Proof. In Case 1, a 3-tree with a dominating vertex has $\Delta = n - 1$, so $n = \Delta + 1$.

In Case 2, for each vertex $v \in \{u, x, y, z\}$, let S_v be the set of vertices not adjacent to v. To have $\Delta(T) = n - 1 - r$, each S_v must contain at least r vertices. Now vertices in S_u are adjacent to triangle xyz, so they are only in S_u.

The vertices in S_x may be in S_y or S_z (not both), and similarly for the vertices in S_y or S_z. However, there must be at least one vertex only in one of the sets S_x, S_y, or S_z. If there is exactly one vertex adjacent to (say) $\{u, y, z\}$, then $\Delta(T) = n - 1 - r$ requires at least r vertices each in S_y and S_z, and none in both. Thus $n \geq 4 + 3r + 1 = 3r + 5$ and $\Delta(T) \geq n - 1 - \frac{n-5}{3} = \frac{2}{3}n + \frac{2}{3}$, so $n \leq \frac{3\Delta - 2}{2}$.

Suppose there are two vertices in only one of the sets S_x, S_y, or S_z, say one each in sets S_x and S_y. Any other vertex can be in any two of the three sets. Then s vertices in $S_x \cup S_y \cup S_z$ yield $|S_x| \cup |S_y| \cup |S_z| \leq 2s - 2$, so $r \leq \frac{2s-2}{3}$. Now $n = 4 + s + r \geq 4 + \frac{3r+2}{2} + r = \frac{5r+10}{2}$, so $r \leq \frac{2n-10}{5}$. Then $\Delta(T) \geq n - 1 - \frac{2n-10}{5} = \frac{3n}{5} + 1$. Thus $n \leq \frac{5\Delta-5}{3}$.

In Case 3, $K_3 + \overline{K}_r$ has $r \geq 1$, with uxy being the K_3. There are at most $n - 4$ vertices with exactly two neighbors in $K_3 + \overline{K}_r$. These vertices split into three sets based on which of $\{u, x, y\}$ they are not adjacent to. When Δ is minimum, one of these sets contains at least $\frac{n-4}{3}$ vertices. Then $\Delta \geq n - 1 - \frac{n-4}{3} = \frac{2n+1}{3}$, so $n \leq \frac{3\Delta - 1}{2}$. □ The smallest possible

Δ for a 3-tree with diameter 2 and order n is $n-1$ for $3 \le n \le 7$ and $\lceil \frac{3n}{5} + 1 \rceil$ for $n \ge 5$.

5. Planar 3-trees

Next we consider an important special class of k-trees.

Definition 5.1. A *simple k-tree* is defined recursively by starting with K_{k+1} and iteratively adding a vertex adjacent to all vertices of a k-clique Q not previously used as the neighborhood of a k-leaf.

A *plane drawing* of a graph is a drawing in the plane that has no crossings. A graph is *outerplanar* if it has a plane drawing with all vertices on the boundary of the exterior region. A graph is a *maximal outerplanar graph (MOP)* if no edge can be added so that the resulting graph is still outerplanar.

An *Apollonian network* is a planar 3-tree.

The MOPs are exactly the simple 2-trees, and the planar 3-trees are exactly the simple 3-trees [10]. See Figures 1 and 2 for examples of these graphs.

Corollary 5.2. Let T be planar 3-tree. Then T has diameter at most 2 if and only if it is formed in one of the following ways.

1. $T = H + K_1$, where H is a MOP.
2. Let K_4 have vertices $\{u, x, y, z\}$. Then T is formed by fan overlapping with bases ux, uy, uz, and only triangles of K_4 are used for overlapping, each at most once. A single 3-leaf may be added with root $\{x, y, z\}$.
3. Let uxy be the K_3 in $K_3 + \overline{K}_r$, $1 \le r \le 2$. Then T is formed by fan overlapping with bases ux, uy, or xy. Only triangles of $K_3 + \overline{K}_r$ are used for overlapping, and each at most once.

Proof. In Case 1, for T to be planar, H must be outerplanar.

In Cases 2 and 3, for T to be planar, it must be a simple 3-tree, so each root is used at most once. Thus each triangle of K_4 or $K_3+\overline{K}_r$ can be used at most once for overlapping, and no other triangle can be used for overlapping. In Case 3, $r \le 2$, since $K_3 + \overline{K}_3$ is not planar. □

Seyffarth [11] studied maximal planar graphs with diameter 2. Seyffarth showed that such graphs have $n \le \frac{3}{2}\Delta + 1$ and found two infinite classes of maximal planar graphs that show this bound is sharp. This is not claimed to be a complete characterization.

Of course, maximal planar graphs with diameter 2 need not be 3-trees. For example, the double wheel $\overline{K}_2+C_{n-2}$ has minimum degree 4 and diameter 2. Seyffarth's two classes both contain subgraphs with minimum degree 4. Thus it appears that the bound on n can be improved when we only consider planar 3-trees.

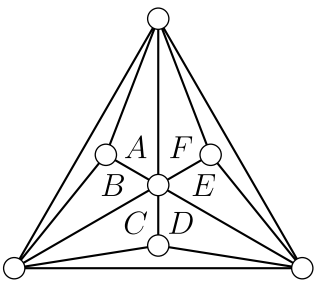
Corollary 5.3. A planar 3-tree with diameter 2 with order $n \ge 4$ and maximum degree Δ has $n \le \frac{3}{2}\Delta - \frac{1}{2}$.

Proof. In Case 1, a 3-tree with a dominating vertex has $\Delta = n-1$, so $n = \Delta + 1$.

In Case 2, there can only be one vertex not adjacent to u, so $\Delta \geq n-2$, and $n \leq \Delta + 2$.

In Case 3, $K_3 + \overline{K}_r$ has $1 \leq r \leq 2$, with uxy being the K_3. There are at most $n-4$ vertices with exactly two neighbors in $K_3 + \overline{K}_r$. These vertices split into three sets based on which of $\{u, x, y\}$ they are not adjacent to. When Δ is minimum, one of these sets contains at least $\frac{n-4}{3}$ vertices. Then $\Delta \geq n - 1 - \frac{n-4}{3} = \frac{2n+1}{3}$, so $n \leq \frac{3\Delta - 1}{2}$. □ Thus no planar 3-tree can be an extremal graph for Seyffarth's theorem.

We may be interested to characterize the degree sequences of planar 3-trees with diameter 2. Note that in Case 1, G has a dominating vertex u if and only if $G - u$ is a MOP. Consequently, we can determine whether a list of numbers is a degree sequence of a planar 3-tree with a dominating vertex if and only if we can determine whether a corresponding list is the degree sequence of a MOP. However, no characterization of degree sequences of MOPs is known. See [1, 8, 9] for partial results. Thus we instead consider graphs for Cases 2 and 3 that are not covered by Case 1.

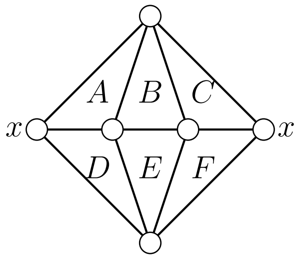


Case 2: To avoid a dominating vertex, we assume there is a vertex rooted on each triangle of the K_4 with vertex set $\{u, x, y, z\}$. We designate the six triangles A-F in order around u (see the graph above). We can break down the cases by how many vertices are in each of the 6 triangles. Note that if there are no vertices in D, we can move the vertices in C to B without changing the degree sequence.

We organize cases based on how many degree 5 vertices there are rooted on the K_4. We can reduce the cases to possibly adding vertices inside ACE, ABD, ABCD, or ABCDEF. Suppose a vertices are added inside A, and similarly for the other triangles. We require $a, b \geq 1$ when A and B are both listed in a case, and similarly for the pairs $\{C, D\}$ and $\{E, F\}$, but not otherwise. We obtain the following possible degree sequences (d^r indicates r vertices of degree d).

Triangles	Degree Sequence
ACE	$(n-2)^1 (6+a)^1 (6+c)^1 (6+e)^1 4^{a+c+e} 3^4$
ABD	$(n-2)^1 (6+a)^1 (6+b)^1 (6+d)^1 5^1 4^{a+b+d-2} 3^5$
ABCD	$(n-2)^1 (6+a)^1 (6+b+c)^1 (6+d)^1 5^2 4^{a+b+c+d-4} 3^6$
ABCDEF	$(n-2)^1 (6+a+f)^1 (6+b+c)^1 (6+d+e)^1 5^3 4^{a+b+c+d+e+f-6} 3^7$

Case 3: To avoid a dominating vertex, there must be fans attached to at least one of each of the three pairs $\{A, D\}$, $\{B, E\}$, and $\{C, F\}$ (see the graph above). Thus the triangles where fans are attached are (up to symmetry) ABC, ABF, ABCD, ABDF,



ABCDE, or ABCDEF. Suppose $a \geq 1$ vertices are added inside A, and similarly for the other triangles. We obtain the following possible degree sequences.

Triangles	Degree Sequence
ABC	$(4+a+b)^1 (4+a+c)^1 (4+b+c)^1 6^1 4^{a+b+c-3} 3^4$
ABF	$(4+a+b)^1 (4+a+f)^1 (4+b+f)^1 5^1 4^{a+b+f-2} 3^3$
ABCD	$(4+a+b+d)^1 (4+a+c+d)^1 (4+b+c)^1 6^1 4^{a+b+c+d-3} 3^4$
ABDF	$(4+a+b+d)^1 (4+a+d+f)^1 (4+b+f)^1 5^2 4^{a+b+d+f-4} 3^4$
ABCDE	$(4+a+b+d+e)^1 (4+a+c+d)^1 (4+b+c+e)^1 6^1 5^1 4^{a+b+c+d+e-5} 3^5$
ABCDEF	$(4+a+b+d+e)^1 (4+a+c+d+f)^1 (4+b+c+e+f)^1 6^2 4^{a+b+c+d+e+f-6} 3^6$

For example, suppose we have the degree sequence $S = 8^2 6^1 5^2 4^1 3^4$. We see $n = 10$ and $\sum d_i = 48 = 2(3n-6)$, so we have the right degree sum for a 3-tree [5]. The 3^4 and 5^2 shows it must fall under Case 3, subcase ABDF. Then $a+b+d+f = 5$, so $a = 2$ and $b = d = f = 1$. Thus S is the degree sequence of a 3-tree with diameter 2.

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References

[1] A. Bar-Noy, T. Böhnlein, D. Peleg, Y. Ran, and D. Rawitz. Approximate realizations for outerplanaric degree sequences. *Journal of Computer and System Sciences*, 148:103588, 2025. https://doi.org/10.1016/j.jcss.2024.103588.

[2] A. Bickle. k-paths of k-trees. In *Springer Proc. Math. Stat.* Volume 388, pages 287–291. 2020.

[3] A. Bickle. *Fundamentals of Graph Theory.* AMS, 2020.

[4] A. Bickle. Maximal k-degenerate graphs with diameter 2. *International Journal of Mathematical Combinatorics*, 2:67–78, 2021.

[5] A. Bickle. A survey of maximal k-degenerate graphs and k-trees. *Theory and Applications of Graphs*, 0(1):Article 5, 2024. 10.20429/tag.2024.000105.

[6] A. Bickle and Z. Che. Wiener indices of maximal k-degenerate graphs. *Graphs and Combinatorics*, 37(2):581–589, 2021. https://doi.org/10.1007/s00373-020-02264-8.

[7] R. Froberg. A characterization of k-trees. *Discrete Mathematics*, 104(3):307–309, 1992. https://doi.org/10.1016/0012-365X(92)90452-L.

[8] D. Y. Li and J. Z. Mao. Degree sequences of maximal outerplanar graphs. *J Central China Normal Univ Natur Sci*, 26(3):270–273, 1992.

[9] Z. Li and Y. Zuo. On the degree sequences of maximal outerplanar graphs. *Ars Combinatoria*, 140:237–250, 2018.

[10] L. Markenzon, C. M. Justel, and N. Paciornik. Subclasses of k-trees: characterization and recognition. *Discrete Applied Mathematics*, 154(5):818–825, 2006. https://doi.org/10.1016/j.dam.2005.05.021.

[11] K. Seyffarth. Maximal planar graphs of diameter two. *Journal of Graph Theory*, 13(5):619–648, 1989.