



A Classification of Mengerian 4-uniform Hypergraphs Derived from Graphs

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ABSTRACT

In this paper, we give a classification of all Mengerian 4-uniform hypergraphs derived from graphs.

Keywords: Mengerian hypergraphs, Packing property, Normally torsion-free monomial ideals, Associated prime ideals

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1. Introduction

Assume that R is a commutative Noetherian ring and I is an ideal of R . Then a prime ideal $\mathfrak{p} \subset R$ is an *associated prime* of I if there exists an element h in R such that $\mathfrak{p} = (I :_R h)$, where $(I :_R h) = \{r \in R \mid rh \in I\}$. The *set of associated primes* of I , denoted by $\text{Ass}_R(I)$, is the set of all prime ideals associated to I . Brodmann [3] showed that the sequence $\{\text{Ass}_R(I^k)\}_{k \geq 1}$ of associated prime ideals is stationary for large k . This means that there exists a positive integer k_0 such that $\text{Ass}_R(I^k) = \text{Ass}_R(I^{k_0})$ for all $k \geq k_0$. The minimal such k_0 is called the *index of stability* of I and $\text{Ass}_R(I^{k_0})$ is called the *stable set* of associated prime ideals of I , which is denoted by $\text{Ass}^\infty(I)$.

Several questions arise in the context of Brodmann's result. An ideal $I \subset R$ is said to

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be *normally torsion-free* if, for all $k \geq 1$, we have $\text{Ass}_R(I^k) \subseteq \text{Ass}_R(I)$, see [24, Definition 4.3.28]. In particular, if I is a square-free monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K , then I is normally torsion-free if, for all $k \geq 1$, we have $\text{Ass}_R(I^k) = \text{Ass}_R(I)$.

Generally, finding classes of normally torsion-free ideals have been a theme of many papers, nevertheless, some classes of such ideals emanate from graph theory. Particularly, it is well-known that a finite simple undirected graph is bipartite if and only if its edge ideal is normally torsion-free, if and only if its cover ideal is normally torsion-free, see [9, 23] for more information. Furthermore, in [17, Theorem 3.3], the authors stated that the Alexander dual of the monomial ideal generated by the paths of maximal lengths in a rooted starlike tree is normally torsion-free. In [13, Theorem 3.2], it has been shown that the Alexander dual of path ideals generated by all paths of length 2 in rooted trees are normally torsion-free.

Let G be an undirected and simple graph. More recently, it has been verified in [18, Theorem 2.8] that if G is a strongly chordal graph, then both $NI(G)$ and $DI(G)$ are normally torsion-free, where $NI(G)$ (respectively, $DI(G)$) stands for the closed neighborhood ideal of G (respectively, dominating ideal of G). Moreover, when C_n denotes the simple cycle graph on n vertices, then based on [18, Theorem 4.3], $DI(C_n)$ is normally torsion-free if and only if $n \in \{3, 6, 9\}$, the definitions of closed neighborhood ideals and dominating ideals and their properties can be found in [16, 19].

In addition, Lemma 5.15 in [20] gives a characterization of all normally torsion-free t -spread principal Borel ideals. In fact, a monomial $x_{i_1}x_{i_2} \cdots x_{i_d} \in R = K[x_1, \dots, x_n]$ with $i_1 \leq i_2 \leq \cdots \leq i_d$ is called t -spread if $i_j - i_{j-1} \geq t$ for all $j = 2, \dots, d$. A monomial ideal I in R is called a t -spread monomial ideal if it is generated by t -spread monomials. Also, I is called a t -spread strongly stable ideal if for all t -spread monomials $u \in \mathcal{G}(I)$, all $j \in \text{supp}(u)$ and all $1 \leq i < j$ such that $x_i(u/x_j)$ is a t -spread monomial, it follows that $x_i(u/x_j) \in I$. Moreover, I is called a t -spread principal Borel if there exists a t -spread monomial $u \in \mathcal{G}(I)$ such that I is the smallest t -spread strongly stable ideal which contains u ; we denote it as $I = B_t(u)$. By [20, Lemma 5.15(i)], if $u = x_a x_b x_n$ is a t -spread monomial, $I = B_t(u)$, and $b < 2t + 1$, then I is normally torsion-free. In this direction, in [15, Theorem 4.6], the authors showed that the edge ideal of any t -spread d -partite hypergraph K_V^t is normally torsion-free. More information about t -spread monomial ideals and vector-spread monomial ideals can be found in [7, 8].

Furthermore, Herzog et. al., in [12, Corollary 4.6], proved that every transversal polymatroidal ideal is normally torsion-free. In particular, Olteanu [21] characterized all the lexsegment ideals which are normally torsion-free non-square-free monomial ideals.

In the following text, we focus on normally torsion-free square-free monomial ideals. Indeed, one of the motivations of this work originates from this fact that normally torsion-free square-free monomial ideals are closely related to Mengerian hypergraphs. A hypergraph is called *Mengerian* if it satisfies a certain min-max equation, which is known as the Mengerian property in hypergraph theory or has the max-flow min-cut property in integer programming. In fact, it is well-known that if \mathcal{C} is a clutter, i.e. an antichain w.r.t. the partial order induced by inclusion, and $I = I(\mathcal{C})$ its edge ideal, then I is normally

torsion-free if and only if $I^k = I^{(k)}$ for all $k \geq 1$, where $I^{(k)} = \bigcap_{\mathfrak{p} \in \text{Min}(I)} (I^k R_{\mathfrak{p}} \cap R)$ denotes the k -th symbolic power of I , if and only if \mathcal{C} is Mengerian, if and only if \mathcal{C} has the max-flow min-cut (MFMC for short) property, see [24, Theorem 14.3.6] for more details.

So far, there exist two well-known classifications of Mengerian 2-uniform hypergraphs and 3-uniform hypergraphs derived from graphs as follows:

- (i) ([23, Theorem 5.9]) Let G be a simple undirected graph and I its edge ideal. Then G is bipartite if and only if I is normally torsion-free.
- (ii) ([1, Theorem 3.9]) Let G be a connected simple undirected graph and $t \geq 2$ be an integer, and let $J = I_3(G)$ be the cubic path ideal of G . Then, $J^{(n)} = J^n$ for all $n \geq 1$ if and only if G is a path graph P_t or G is the cycle C_{3k} when $k = 1, 2, 3$.

In addition, according to [18, Proposition 3.2], the t -path ideals of path graphs are normally torsion-free for all $t \geq 0$.

The main aim of this work is to provide a classification of all Mengerian 4-uniform hypergraphs derived from graphs as we will give in the following theorem:

Theorem 1.1. *If $G = (V, E)$ is a connected graph, then $\mathcal{H}_3(G)$ is Mengerian if and only if $|V| = 4$, or $G = C_8$, or G is a path with double stars, or a star with an additional edge between two of its leaves.*

It is possible to ask whether the techniques used in this paper can be generalized to the general case? Our computations and observations show that investigating the cases greater than 4 is too complicated, and we may need to change our technique as in [23] the authors used the Rees Algebra and classic methods in commutative algebra for proving the case 2, and in [1] the authors utilized the symbolic powers technique for showing the case 3, and the present authors used the TDI (totally dual integral) method for proving the case 4; however, we left an open question in this direction at the end of this paper, see Question 4.1.

Throughout this paper, all graphs are simple, finite, connected, and undirected. Also, any necessary definitions related to graph theory (respectively, monomial ideals) can be found in [25] (respectively, [11]).

2. Preliminaries

In what follows, we recollect some necessary definitions related to hypergraph theory, which can be found in [2, 24].

A finite *hypergraph* \mathcal{H} on a vertex set $V(\mathcal{H}) = \{x_1, x_2, \dots, x_n\}$ is a collection of hyper-edges $E(\mathcal{H}) = \{E_1, \dots, E_m\}$ with $E_i \subseteq V(\mathcal{H})$ for all $i = 1, \dots, m$. The *incidence matrix* of \mathcal{H} is an $m \times n$ matrix $A = (a_{ij})$ such that $a_{ij} = 1$ if $x_j \in E_i$, and $a_{ij} = 0$ otherwise. A hypergraph \mathcal{H} is called *simple* if $E_i \subseteq E_j$ implies $i = j$. Simple hypergraphs are also known as *clutters*. Moreover, if $|E_i| = d$ for all $i = 1, \dots, m$, then \mathcal{H} is called a *d-uniform* hypergraph. A 2-uniform hypergraph \mathcal{H} is just a finite simple graph.

Let $V(\mathcal{H}) = \{x_1, x_2, \dots, x_n\}$ and $R = K[x_1, \dots, x_n]$ be the polynomial ring in n vari-

ables over a field K . The *edge ideal* of \mathcal{H} is given by

$$I(\mathcal{H}) = \left(\prod_{x_j \in E_i} x_j : E_i \in E(\mathcal{H}) \right).$$

Let G be a simple finite connected undirected graph. The *t -path hypergraph* of G , denoted by $\mathcal{H}_t(G)$, is a $(t+1)$ -uniform hypergraph on the vertex set $V(G)$ such that $e = \{v_1, \dots, v_{t+1}\}$ is an edge if contains a path of length t in G . The edge ideal of $\mathcal{H}_t(G)$ is called the *t -path ideal* of G and is denoted by $I_t(G)$. When $t = 1$, then $I_1(G)$ is simply the *edge ideal* $I(G)$ of G . If there is no t -path in G , then we set $I_t(G) = 0$.

To understand Theorem 2.4, we require to review the following definitions from combinatorial optimization.

Definition 2.1. ([24, Definition 14.3.4]) *A clutter \mathcal{C} satisfies the max-flow min-cut (MFMC) property if both sides of the LP-duality equation*

$$\min\{\langle c, x \rangle \mid x \geq 0; Ax \geq \mathbf{1}\} = \max\{\langle y, \mathbf{1} \rangle \mid y \geq 0; y^\top A \leq c^\top\}, \quad (1)$$

have integral optimum solutions x and y for each nonnegative integral vector c . The system $x \geq 0; Ax \geq \mathbf{1}$ is called *totally dual integral (TDI)* if the maximum in Eq. (1) has an integral optimum solution y for each integral vector c with finite maximum.

For a hypergraph \mathcal{H} , the *deletion* $\mathcal{H} \setminus x$ at a vertex $x \in V(\mathcal{H})$ has the effect of setting $x = 0$ in the edge ideal $I := I(\mathcal{H})$, and the *contraction* \mathcal{H}/x has the effect of setting $x = 1$ in the edge ideal $I(\mathcal{H})$, see [24, Definition 6.5.2]. We call an ideal I' a *minor* of a square-free monomial ideal I if I' can be obtained from I by a sequence of taking quotients (=deletion) and localizations (=contraction) at the variables, refer to [24, Definition 6.5.3].

A vertex $x \in V(\mathcal{H})$ is called an *isolated vertex* if $\{x\} \in E(\mathcal{H})$. It follows from the definition that if x is an isolated vertex of \mathcal{H} , then $\{x\}$ is the only edge in \mathcal{H} that contains x . A *vertex cover* of \mathcal{H} is a set of vertices that has a non-empty intersection with all of the edges of \mathcal{H} . The minimum cardinality of a vertex cover of \mathcal{H} is denoted by $\tau(\mathcal{H})$, consult [2, Page 53]. It should be noted that by the correspondence between minimal primes and minimal vertex covers, $\tau(I)$ is also the height of I (cf. [24, Proposition 13.1.6]).

We say the generators of a square-free monomial ideal I as being *independent* or a *matching* if the corresponding edges of the associated hypergraph are pairwise disjoint; that is, the generators have disjoint supports. We denote the maximum cardinality of an independent set in I by $\nu(I)$, see [2, Page 64]. Furthermore, note that a subset of the monomial generators of I is independent if and only if it forms a regular sequence. Thus, $\nu(I)$ is equal to the *monomial grade* of I , denoted by $\mathfrak{m}\text{-grade}(I)$, where the monomial grade of an ideal I is the maximum length of a regular sequence of monomials in I , see [24, Definition 13.1.5 and Proposition 13.1.6]. By weak linear programming duality we always have $\tau(I) \geq \nu(I)$.

A square-free monomial ideal I is said to satisfy the *König property* if $\tau(I) = \nu(I)$, refer to [24, Definition 13.1.4]. Consequently, I satisfies the König property if and only if

$\text{grade}(I) = \text{ht}(I) = \mathbf{m}\text{-grade}(I)$. Moreover, a square-free monomial ideal I has the *packing property* if I and all of its proper minors satisfy the König property (cf. [24, Definition 14.3.16]).

Definition 2.2. ([24, Definition 14.3.5]) *A clutter \mathcal{C} is called Mengerian if $\nu(\mathcal{C}^a) = \tau(\mathcal{C}^a)$ for all $a \in \mathbb{N}^n$, where \mathbb{N} denotes the set of nonnegative integers. In other words, \mathcal{C} is Mengerian if all its multiplications of vertices have the König property.*

In 1990 [5], Michele Conforti and Gérard Cornuéjols made up the following conjecture, and it is still open.

Conjecture 2.3. (*Conforti-Cornuéjols conjecture*) *A hypergraph has the packing property if and only if it is Mengerian.*

We finally summarize all of the above notions in the following theorem.

Theorem 2.4. ([24, Theorem 14.3.6]) *Let \mathcal{C} be a clutter and let A be its incidence matrix. The following are equivalent:*

- (i) $\text{gr}_I(R)$ is reduced, where $I = I(\mathcal{C})$ is the edge ideal of \mathcal{C} .
- (ii) $R[It]$ is normal and $\mathcal{Q}(A)$ is an integral polyhedron.
- (iii) $x \geq 0$; $Ax \geq \mathbf{1}$ is a TDI system.
- (iv) \mathcal{C} has the max-flow min-cut (MFMC) property.
- (v) $I^i = I^{(i)}$ for $i \geq 1$.
- (vi) I is normally torsion-free, i.e., $\text{Ass}(I^i) \subseteq \text{Ass}(I)$ for $i \geq 1$.
- (vii) \mathcal{C} is Mengerian, i.e., $\nu(\mathcal{C}^a) = \tau(\mathcal{C}^a)$ for all $a \in \mathbb{N}^n$.

3. Main Result

Given a connected graph $G = (V, E)$ with $|V| \geq 4$, we consider the 4-uniform hypergraph $\mathcal{H}_3(G) = (\mathcal{V}, \mathcal{F})$ such that $e = \{v_1, v_2, v_3, v_4\} \in \mathcal{F}$ if and only if contains a path of length 3 in G . Let A denote the incidence matrix of $\mathcal{H}_3(G)$. Then $\mathcal{H}_3(G)$ is Mengerian if and only if the system $Ax \geq \mathbf{1}$ with $x \geq 0$ is TDI. This in particular implies that the system is integral. In this case, we call the hypergraph *ideal*. It should be noted that if a hypergraph is not ideal, then it is not Mengerian. Furthermore, every empty clutter is considered Mengerian.

To show Proposition 3.4 here below, we require to employ the following definition, exercise, and theorem. Given a monomial ideal I , we denote by $\mu(I)$ the minimal number of monomial generators of I .

Definition 3.1. ([24, Definition 4.3.22]) *Let I be an ideal of a ring R and $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ the minimal primes of I . Given an integer $n \geq 1$, the n th symbolic power of I is defined to be the ideal $I^{(n)} = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_r$, where \mathfrak{q}_i is the primary component of I^n corresponding to*

\mathfrak{p}_i .

Exercise 3.2. ([24, Exercise 6.1.25]) If $\mathfrak{q}_1, \dots, \mathfrak{q}_r$ are primary monomial ideals of R with non-comparable radicals and I is an ideal such that $I = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_r$, then $I^{(n)} = \mathfrak{q}_1^n \cap \dots \cap \mathfrak{q}_r^n$.

Theorem 3.3. ([14, Theorem 4.8]) Let I be a square-free monomial ideal. If $I^n = I^{(n)}$ for every $n \leq \lceil \frac{\mu(I)}{2} \rceil$, then $I^n = I^{(n)}$ for every $n \in \mathbb{Z}_{>0}$.

Proposition 3.4. Let $C_8 = (V(C_8), E(C_8))$ denote the cycle graph with vertex set $V(C_8) = \{x_1, \dots, x_8\}$ and the following edge set

$$E(C_8) = \{\{x_i, x_{i+1}\} : i = 1, \dots, 7\} \cup \{\{x_1, x_8\}\}.$$

Let $\mathcal{H}_3(C_8)$ be the 4-uniform hypergraph on the vertex set $V(C_8)$. Then $\mathcal{H}_3(C_8)$ is Mengerian.

Proof. Let $I_3(C_8)$ be the edge ideal of $\mathcal{H}_3(C_8)$, and $R = K[x_1, \dots, x_8]$. For convenience of notation, we put $J := I_3(C_8)$. It is routine to check that

$$J = (x_1x_2x_3x_4, x_2x_3x_4x_5, x_3x_4x_5x_6, x_4x_5x_6x_7, x_5x_6x_7x_8, x_6x_7x_8x_1, x_7x_8x_1x_2, x_8x_1x_2x_3).$$

Hence, we deduce that $\mu(J) = 8$. Using *Macaulay2* [10], we obtain

$$\begin{aligned} \text{Ass}(J) = \text{Ass}(J^2) = \text{Ass}(J^3) = \text{Ass}(J^4) = & \{(x_5, x_1), (x_6, x_2), (x_7, x_3), (x_8, x_4), (x_6, x_3, x_1), \\ & (x_6, x_4, x_1), (x_7, x_4, x_1), (x_7, x_4, x_2), (x_7, x_5, x_2), \\ & (x_8, x_5, x_2), (x_8, x_5, x_3), (x_8, x_6, x_3)\}. \end{aligned}$$

Since J is a square-free monomial ideal, we deduce from [11, Corollary 1.3.6] that $\text{Ass}(J) = \text{Min}(J)$, in particular, $J = \bigcap_{\mathfrak{p} \in \text{Min}(J)} \mathfrak{p}$. Hence, we have $\text{Ass}(J^n) = \text{Min}(J)$ for all $1 \leq n \leq 4$. This means that J^n (for all $1 \leq n \leq 4$) has no embedded associated prime ideals. We can derive from Definition 3.1 and Exercise 3.2 that $J^n = J^{(n)} = \bigcap_{\mathfrak{p} \in \text{Min}(J)} \mathfrak{p}^n$ for all $1 \leq n \leq 4$. It follows now from Theorem 3.3 that $J^n = J^{(n)}$ for any $n \geq 1$. One can immediately conclude from Theorem 2.4 that J is normally torsion-free, and so $\mathcal{H}_3(C_8)$ is Mengerian, as required. \square

To formulate Lemma 3.8, we need the following definitions and corollary.

Definition 3.5. A tree T is called a path with double stars if it has at most two vertices which are adjacent to leaves. Note that this includes stars and paths.

Definition 3.6. ([24, Definition 1.5.5]) A matrix A is called totally unimodular if each $i \times i$ subdeterminant of A is 0 or ± 1 for all $i \geq 1$.

Corollary 3.7. ([24, Corollary 14.3.14]) If A is totally unimodular, then $\text{gr}_I(R)$ is reduced, and so the clutter is Mengerian.

Lemma 3.8. *If $G = (V(G), E(G))$ is a connected graph and A is the incidence matrix of $\mathcal{H}_3(G)$ such that G is a path with double stars, where the path has at least two edges, or G is a star plus an edge between two of its leaves, then A is totally unimodular, and thus $\mathcal{H}_3(G)$ is Mengerian.*

Proof. We proceed by induction on the number of vertices. If $|V(G)| \leq 4$, then the hypergraph is either empty or consists of one hyperedge and the matrix is one all 1-row. In both cases, the matrix A is totally unimodular. Let $|V(G)| > 4$ and v be a vertex of degree 1 and A' be a square submatrix of A . If A' does not intersect the column corresponding to v , then $\det(A') \in \{0, \pm 1\}$ by inductive assumption applied to $G \setminus v$. If A' intersects the column of A , then this column can have at most one 1. If it is all zero, then we have $\det(A') = 0$. Otherwise, using Laplacian expansion and again inductive assumption applied to $G \setminus v$, we again find $\det(A') \in \{0, \pm 1\}$. This completes the inductive step, and so the claim has been shown by induction. The last assertion can be deduced according to Corollary 3.7. \square

As was pointed out by an anonymous referee, the above proof does not apply to the case where we have a path consisting of just one edge with double stars. Thus, we need an extra lemma.

Lemma 3.9. *If $G = (V(G), E(G))$ is a path with double stars, where the path has exactly one edge, then A is totally unimodular, and thus $\mathcal{H}_3(G)$ is Mengerian.*

Proof. We construct a network from G as follows. Subdivide the inner edge, and direct all edges of the resulting tree such that all paths using four edges are directed. Call that directed tree T . Now add all arcs from all leaves of the first star to the leaves of the other star, i.e. the corresponding complete bipartite digraph, resulting in a directed graph D . Consider the totally unimodular *network matrix* (see [22] Chapter 19, Example 4) \tilde{A} we get from the digraph D and its tree T .

Then A is the submatrix of \tilde{A} we get by deleting all columns corresponding to arcs in T . As a submatrix of a totally unimodular matrix, A is totally unimodular as well. \square

Proposition 3.10. *If $G = (V(G), E(G))$ is a connected graph such that $|V(G)| = 4$, or $G = C_8$, or G is a path with double stars, or G is a star plus an edge between two of its leaves, then $\mathcal{H}_3(G)$ is Mengerian.*

Proof. The claim clearly is true if $|V(G)| = 4$. If $G = C_8$, by virtue of Proposition 3.4, we can conclude that $\mathcal{H}_3(C_8)$ is Mengerian. In any other case, $\mathcal{H}_3(G)$ is Mengerian on account of Lemmata 3.8 and 3.9. \square

In the remaining cases, we will show that the hypergraphs are not Mengerian because they are not ideal. For this purpose, we present in each case a fractional vertex of the covering polyhedron. We use the following well-known fact from linear programming.

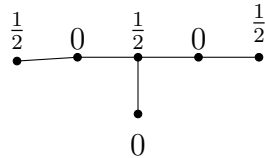
Lemma 3.11. (see [22, Theorem 8.4 and (23) in Section 8.5]) *Let $P = \{Ax \leq b\}$ denote a polyhedron. Then $v \in P$ is a vertex of P if and only if there is a subsystem $A'x \leq b'$ of $Ax \leq b$ such that A' has full column rank and $A'v = b'$.*

Lemma 3.12. *If G is a connected graph on 5 vertices, then $\mathcal{H}_3(G)$ is Mengerian if and only if G is a path with double stars or a star plus an edge between two of its leaves.*

Proof. Sufficiency follows from Proposition 3.10. Clearly, in a connected graph on 5 vertices $\{v_1, v_2, v_3, v_4, v_5\}$, every vertex v_i is in some connected subgraph on 4 vertices and at least two vertices are not in every such component. If exactly two vertices are not in every such component, G must be a path with four edges, or a path with double stars, or a star plus an edge between two of its leaves. If three vertices, say $\{v_1, v_2, v_3\}$, are not in every path with four vertices, then $x_4 = 0, x_5 = 0, x_1 + x_2 + x_4 + x_5 = 1 = x_1 + x_3 = x_2 + x_3 = 1$ yields the solution $\frac{1}{2}(1, 1, 1, 0, 0)$, which is a vertex by Lemma 3.11. This implies that the hypergraph is not ideal, and thus also not Mengerian. Similarly, in the case of 4 vertices not in every connected component, we find the vertex $\frac{1}{3}(1, 1, 1, 1, 0)$ and finally, if G is 2-connected, then the required vertex is $\frac{1}{4}(1, 1, 1, 1, 1)$. Thus, in any of these cases, the hypergraph is neither ideal nor Mengerian, and the proof is over. \square

Proposition 3.13. *If G is the following graph, then $\mathcal{H}_3(G)$ is not Mengerian.*

Proof. The numbers indicated at the vertices clearly define a non-integral vertex. Thus, $\mathcal{H}_3(G)$ is neither ideal nor Mengerian.



\square

In the following result we are using a result from the paper [4] which has been published hundred years ago (in 1923) and here is the first citation of this result.

Lemma 3.14. *If $G = C_k$ is a cycle with $k \geq 5$ and $k \neq 8$, then $\mathcal{H}_3(G)$ is not ideal, and thus not Mengerian.*

Proof. By [6, Theorem 4.1], these hypergraphs are not ideal, and thus not Mengerian either. We will present fractional vertices for completeness. If k is odd, then [4, Theorem V] implies that $\det(A) = 4$, where A denotes the incidence matrix of $\mathcal{H}_3(G)$. So, $\frac{1}{4}(1, \dots, 1)$ is a vertex. If $k \equiv 2 \pmod{4}$, then $\frac{1}{2}(1, 0, 1, 0, 1, \dots, 1, 0)$ is a vertex since $\frac{k}{2}$ non-negativity constraints and all hyperedge constraints are satisfied with equality and the matrix is of full rank. Finally, if $k \equiv 0 \pmod{4}$ and $k \geq 12$ using the same approach as for $k \equiv 0 \pmod{2}$ would yield a system which is not of full rank. But $\frac{1}{2}(1, 1, 0, 1, 0, 1, 1, 0, 1, 0, 1, \dots, 0, 1, 0)$

is a vertex satisfying $\frac{k}{2} - 1$ non-negativity constraints and $k - 4$ hyperedge constraints with equality. We thus have in total $\frac{3}{2}k - 5 > k$ equalities, and the system is of full rank. This completes the proof. \square

A remark here what makes the case $k = 8$ special is certainly in order.

Lemma 3.15. *If $G = (V, E)$ is a connected graph with $|V(G)| \geq 5$ and G is neither C_8 nor a path with double stars nor a star plus an edge between two of its leaves, then $\mathcal{H}_3(G)$ is not Mengerian.*

Proof. If G is a tree which is not a path with double stars, then it has three vertices which are adjacent to leaves. Hence, the following graph H (see Figure 1) is a minor, and thus $\mathcal{H}_3(G)$ is not Mengerian by Lemma 3.12. If G contains a triangle and is not itself a star plus an edge between two of its leaves, then we find an induced subgraph on 5 vertices which is not Mengerian by Lemma 3.12. If G contains a cycle of length 4, again we find an induced subgraph on 5 vertices which is not Mengerian by Lemma 3.12.

If G contains a cycle of length $k \geq 6$ and $k \neq 8$, then $\mathcal{H}_3(G)$ is not Mengerian by Proposition 3.13. If the only cycle in G is the cycle C_8 , since $G \neq C_8$, we find a vertex adjacent to only one vertex of C_8 , and so again we have the following graph H which is not Mengerian by Lemma 3.12 as a minor. Any other cycle is a cycle of length ≥ 6 and the hypergraph is not Mengerian by Lemma 3.14. This finishes our argument.

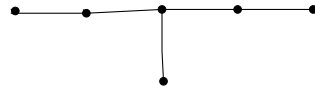


Figure 1. The graph H

\square

The main result of this paper follows combining Proposition 3.10 with Lemma 3.15.

Theorem 3.16. *If $G = (V, E)$ is a connected graph, then $\mathcal{H}_3(G)$ is Mengerian if and only if $|V| = 4$, or $G = C_8$, or G is a path with double stars, or a star with an additional edge between two of its leaves.*

Corollary 3.17. *If $G = (V, E)$ is a connected graph with $|V| = 4$, or $G = C_8$, or G is a path with double stars, or a star with an additional edge between two of its leaves, then $\mathcal{H}_3(G)$ satisfies the Conforti-Cornuéjols conjecture.*

4. Conclusion and Outlook

In this paper we considered the 4-uniform hypergraphs $\mathcal{H}_3(G)$, where the hyperedges are defined by the vertex sets $h \in \binom{V}{4}$ of a graph $G = (V, E)$, that contain a path on 3 edges. We characterized the graphs where the corresponding hypergraphs are Mengerian. Except

for $\mathcal{H}_3(C_8)$, the corresponding matrices where all even totally unimodular. Moreover, we found a dichotomie that in this class every hypergraph is either Mengerian or the corresponding matrix is non-ideal. In view of Theorem 4.1(i) in [6], we wonder whether

Question 4.1. *Is it true that for $t \geq 4$ every matrix of the $(t + 1)$ -uniform hypergraph $\mathcal{H}_t(G)$ either is totally unimodular or non-ideal?*

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