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A New Series of Affine Resolvable PBIB(4) Designs in Two Replicates

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ABSTRACT

A new series of four-associate class partially balanced incomplete block designs in two replications has been proposed. The blocks of these designs are of two different sizes. The blocks can be divided into two groups such that every treatment appears in each group exactly once, and any two blocks belonging to two different groups have a constant number of treatments in common, *i.e.*, these designs are affine resolvable.

Keywords: Association scheme, Affine resolvable, Partially balanced incomplete block designs, Unequal block sizes

2020 Mathematics Subject Classification: 11A99, 11B75, 68R05

1. Introduction

Partially balanced incomplete block (PBIB) designs form an important class of block designs. If the experimenter is constrained of resources, PBIB designs with higher associate classes in minimum replications are an alternative to the more popular class of block designs, *i.e.*, balanced incomplete block (BIB) designs or two-associate class PBIB designs. Furthermore, it is advisable to opt for a resolvable block design for situations where experimentation is to be done, one complete replication at one location/season,

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considering spatial/temporal variations. Some distinguished classes of resolvable PBIB designs are available in the literature (Rao [3]; Williams et al. [5]; Kageyama [2]; Varghese and Sharma [4]; Agrawal et al. [1]). Again, equally sized blocks may not always be feasible in every experimental situation. Hence, it might be of interest to the experimenters to have resolvable PBIB designs in fewer replications with unequal block sizes.

We define a new association scheme in Section 2. A general method of constructing resolvable PBIB designs in unequal block sizes based on this association scheme is given in Section 3. In Section 4, an outline of the analysis of these designs is given.

2. Association Scheme

Let the number of treatments be $v = 2mt(t\text{-}1)$ where, $m \geq 1$, $t \geq 3$. Arrange these treatments in a rectangular array consisting of t-1 groups, each group containing t rows and two columns and each row-column intersection having m treatments is shown in Table 1.

For any given treatment θ appearing in the i^{th} row of any group, remaining treatments appearing in the same row of the same group are first associates, treatments appearing in all other rows of the same group are second associates, treatments belonging to the i^{th} row of the other groups are third associates and the rest are fourth associates. The parameters of the association scheme are derived as follows:

$v = 2mt(t-1), n\_1 = 2m-1, n\_2 = 2m(t-1), n\_3 = 2m(t-2), n\_4 = 2m(t-1)(t-2),$

$$P_1 = \begin{bmatrix} 2(m-1) & 0 & 0 & 0 \\ 0 & 2(t-1)m & 0 & 0 \\ 0 & 0 & 2(t-2)m & 0 \\ 0 & 0 & 0 & 2(t-1)(t-2)m \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 0 & 2m-1 & 0 & 0 \\ 2m-1 & 2(t-2)m & 0 & 0 \\ 0 & 0 & 0 & 2(t-2)m \\ 0 & 0 & 2(t-2)m & 2m(t-2)^2 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 0 & 0 & 2m-1 & 0 \\ 0 & 0 & 0 & 2(t-1)m \\ 2m-1 & 0 & 2(t-3)m & 0 \\ 0 & 2(t-1)m & 0 & 2(t-1)(t-3)m \end{bmatrix},$$

and

$$P_4 = \begin{bmatrix} 0 & 0 & 0 & 2m-1 \\ 0 & 0 & 2m & 2(t-2)m \\ 0 & 2m & 0 & 2(t-3)m \\ 2m-1 & 2(t-2)m & 2(t-3)m & 2(t-2)(t-3)m \end{bmatrix}.$$

Example 2.1. Let $m = 2$ and $t = 4$, resulting in $v = 48$ treatments. Arrange these treatments in a rectangular array having three groups, each group containing four rows and two columns, and each row-column intersection has two treatments (see Table 2):

Groups	**Rows**	**Columns: Set 1**				**Columns: Set 2**			
		1	2	$\cdots$	m	$1+tm$	$2+tm$	$\cdots$	$m+tm$
1	1	1	2	$\cdots$	m	$1+tm$	$2+tm$	$\cdots$	$m+tm$
	2	$1+m$	$2+m$	$\cdots$	$2m$	$1+(t+1)m$	$2+(t+1)m$	$\cdots$	$(t+2)m$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	t	$1+(t-1)m$	$2+(t-1)m$	$\cdots$	tm	$1+(2t-1)m$	$2+(2t-1)m$	$\cdots$	$2tm$
2	1	$1+2tm$	$2+2tm$	$\cdots$	$m+2tm$	$1+3tm$	$2+3tm$	$\cdots$	$m+3tm$
	2	$1+(2t+1)m$	$2+(2t+1)m$	$\cdots$	$(2t+2)m$	$1+(3t+1)m$	$2+(3t+1)m$	$\cdots$	$(3t+2)m$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	t	$1+(3t-1)m$	$2+(3t-1)m$	$\cdots$	$3tm$	$1+(4t-1)m$	$2+(4t-1)m$	$\cdots$	$4tm$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$t-1$	1	$1+2t(t-2)m$	$2+2t(t-2)m$	$\cdots$	$m+2t(t-2)m$	$1+[2(t-1)-1]tm$	$2+[2(t-1)-1]tm$	$\cdots$	$m+[2(t-1)-1]tm$
	2	$1+m+2t(t-2)m$	$2+m+2t(t-2)m$	$\cdots$	$2m+2t(t-2)m$	$1+[(2(t-1)-1)t+1]m$	$2+[(2(t-1)-1)t+1]m$	$\cdots$	$m+[(2(t-1)-1)t+1]m$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	t	$1+(t-1)m+2t(t-2)m$	$2+(t-1)m+2t(t-2)m$	$\cdots$	$tm+2t(t-1)m$	$1+[(2(t-1)-1)t+t-1]m$	$2+[(2(t-1)-1)t+t-1]m$	$\cdots$	$2t(t-1)m$

Table 1.

Groups	Rows	Columns 1		Columns 2	
		A	**B**	**C**	**D**
1	1	1	2	9	10
	2	3	4	11	12
	3	5	6	13	14
	4	7	8	15	16
2	1	17	18	25	26
	2	19	20	27	28
	3	21	22	29	30
	4	23	24	31	32
3	1	33	34	41	42
	2	35	36	43	44
	3	37	38	45	46
	4	39	40	47	48

Table 2. Improved Table with Symmetric Entries

Different associates of selected treatments, say, treatment 1, 20 and 40, are listed in Table 3.

Treatments	First Associates	Second Associates	Third Associates	Fourth Associates
1	2,9,10	3,4,5,6,7,8,11, 12,13,14,15,16	17,18,25,26,33,34,41,42	19,20,21,22,23,24,27,28,29,30,31,32,35,36, 37,38,39,40,41,42,43,44,45,46,47,48
20	19,27,28	17,18,21,22,23,24,25,26,29,30,31,32	3,4,11,12,35,36,43,44	1,2,5,6,7,8,9,10,13,14,15,16,33,34,37,38,39,40,41,42,45, 46,47,48
40	39,47,48	33,34,35,36,37,38,41,42,43,44,45,46	7,8,15,16,23,24,31,32	1,2,3,4,5,6,9,10,11,12,13,14,17,18,19,20,21,22,25,26,27, 28,29,30

Table 3. Different associates of treatments

The method of construction is summarized below:

3. Method of Construction

The first $b_1 = t$-1 blocks of size $k_1 = 2mt$ can be obtained by treating the t-1 groups of the above array as blocks. The i^{th} block of remaining $b_2 = t$ blocks of size $k_2 = 2m(t$-1) can be obtained by appending the i^{th} $(1 < i \leq t)$ row of each group of the above array one after another. This will yield a class of affine resolvable block designs having unequal block sizes with parameters:

$$\begin{cases} v = 2mt(t-1), \quad b_1 = t-1, \quad b_2 = t, \quad r = 2, \\ k_1 = 2mt, \quad k_2 = 2m(t-1), \\ \lambda_1 = 2, \quad \lambda_2 = 1, \quad \lambda_3 = 1, \quad \lambda_4 = 0, \\ n_1 = 2m-1, \quad n_2 = 2m(t-1), \quad n_3 = 2m(t-2), \quad n_4 = 2m(t-1)(t-2). \end{cases} \tag{1}$$

Here, the blocks of the design can be divided into two groups such that in each group, every treatment appears precisely once, and any two blocks belonging to two different groups have $\frac{k_1 k_2}{v} = 2m$ treatments in common. These designs are partially variance balanced with an underlying association scheme defined above.

It has been verified that all the parametric relationships of the association scheme and designs are satisfied for the above class of designs.

$$\begin{cases} 1. \quad \sum_{i=1}^{4} n_i = v-1, \\ 2. \quad \sum_{k=1}^{4} p^i_{jk} = n_j - \delta_{ij}, \quad \delta_{ij} = \begin{cases} 0, & i \neq j = 1,2,3,4, \\ 1, & i = j = 1,2,3,4, \end{cases} \\ 3. \quad n_i p^i_{jk} = n_j p^j_{ik} = n_k p^k_{ij}, \quad i,j,k = 1,2,3,4, \\ 4. \quad vr = \sum_{j=1}^{2} b_j k_j, \quad j = 1,2,3,4, \\ 5. \quad \sum_{i=1}^{4} n_i \lambda_i = \sum_{l=1}^{2} (k_l - 1). \end{cases} \tag{2}$$

Example 3.1. To form the design corresponding to Illustration 1, treatments belonging to the first group are taken together to form the first block and those of the second group as the second block. The third block of size 12 can be obtained by combining the treatments of the first row of each of the three groups. Similarly, the remaining blocks of size 12 can be obtained by taking together the treatments appearing in the corresponding rows of each group. Hence, the resultant resolvable design for 48 treatments, each replicated twice in three blocks of size 16 and four blocks of size 12, is obtained (see Table 4):

Replications	Blocks	Treatments															
I	i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	ii	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
	iii	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
II	iv	1	2	9	10	17	18	25	26	33	34	41	42				
	v	3	4	11	12	19	20	27	28	35	36	43	44				
	vi	5	6	13	14	21	22	29	30	37	38	45	46				
	vii	7	8	15	16	23	24	31	32	39	40	47	48				

Table 4.

4. Outline of Analysis

The analysis of these designs may be carried out using the same approach as PBIB designs. However, as the proposed designs are resolvable, the following nested model is appropriate for a set-up of v treatments in b (=2t-1) blocks where t-1 blocks are of size $2mt$ and t blocks are of size $2m(t\text{-}1)$:

$$y_{\text{ijm}} = \mu + \tau_i + R_m + \beta_{j(m)} + e_{\text{ijm}}. \tag{3}$$

Here, y_{ijm} is the response on plot i ($i = 1, 2, \ldots, v$) in block j ($j = 1, 2, \ldots, b$), nested within the mth ($m = 1, 2, \ldots, r$) replication. μ is the general mean, τ_i is the effect of the ith treatment, R_m is the effect of the mth replicate, $\beta_{j(m)}$ is the effect of the jth block in the mth replicate, and e_{ijm} are independent random errors normally distributed with mean zero and variance σ^2.

Solving the normal equations, obtained by minimizing the residual sum of squares, will lead to the following general expressions for variances of elementary contrasts between two estimated treatment effects:

$$\text{Var}(\hat{\tau}_i - \hat{\tau}_{i'}) = \begin{cases} v_1 = \sigma^2, & \text{if } i \text{ and } i' \text{ are first associates;} \\ v_2 = \frac{2m(t-1)+1}{2m(t-1)}\sigma^2, & \text{if } i \text{ and } i' \text{ are second associates;} \\ v_3 = \frac{2mt+1}{2mt}\sigma^2, & \text{if } i \text{ and } i' \text{ are third associates;} \\ v_4 = \frac{2mt(t-1)+2t-1}{2mt(t-1)}\sigma^2, & \text{otherwise.} \end{cases}$$

The average variance of elementary contrasts between two estimated treatment effects is then calculated as:

$$\bar{V} = \frac{n_1 v_1 + n_2 v_2 + n_3 v_3 + n_4 v_4}{n_1 + n_2 + n_3 + n_4} = \frac{2(mt^2 - mt + t - 2)}{2mt^2 - mt - 1}\sigma^2.$$

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