

Generating hypergraphs of finite groups

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ABSTRACT

In a recent paper Cameron, Lakshmanan and Ajith [6] began an exploration of hypergraphs defined on algebraic structures, especially groups, to investigate whether this can add a new perspective. Following their suggestions, we consider suitable hypergraphs encoding the generating properties of a finite group. In particular, answering a question asked in their paper, we classified the finite solvable groups whose generating hypergraph is the basis hypergraph of a matroid.

Keywords: Hypergraphs, Generating hypergraph, Matroid

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1. Efficient domination and total coloring of graphs

The generating graph of a finite group G is the graph defined on the elements of G in such a way that two distinct vertices are connected by an edge if and only if they generate G . It was defined by Liebeck and Shalev in [17], and has been further investigated by many authors: see for example [2, 3, 4, 5, 8, 12, 19, 20, 21] for some of the range of questions that have been considered. Clearly the generating graph of G is an edgeless graph if G is not 2-generated. It is natural to think that for non-2-generated finite groups, the concept of generating graph can be generalized by making use of appropriate hypergraphs.

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Following the suggestions and considerations of Cameron, Lakshmanan and Ajith in their recent paper [6], we concentrate our attention on hypergraphs that encode the generating properties of finite groups. We propose two possible definitions in this direction. For a finite group G denote by $d(G)$ the smallest size of a generating set of G . If G is non-cyclic, then we define the *generating hypergraph* $\Gamma(G)$ associated to a finite group G as the undirected hypergraph whose vertices are the elements of G and where the hyperedges are the generating sets of G of size $d(G)$ (this definition could be given also when G is cyclic, but we prefer to exclude this case from our discussion, since it would have a different peculiar behavior; for example $\Gamma(G)$ would have no hyperedges if we did not admit loops). A minimal generating set of G can have size larger than $d(G)$, so we may also consider another hypergraph $\Delta(G)$, whose vertices are again the elements of G but where the set of hyperedges consists of all the minimal generating sets. We will use for $\Delta(G)$ the name *global generating hypergraph*. The finite groups G with the property that $\Gamma(G) = \Delta(G)$ have been classified by P. Apise and B. Klopsch [1]: they are solvable, with a quite restrictive structure.

In particular we concentrate our attention on the following aspects. In Section 2 we will investigate whether the hypergraphs obtained from $\Gamma(G)$ and $\Delta(G)$ by removing their isolated vertices are connected. We will prove that the answer is affirmative for the global generating hypergraph and for the generating hypergraph with the possible exclusion, in the second case, of the 2-generated finite groups. In Section 3 we will indicate how relevant structural properties of a finite group G (for example the p -solvability property) can be detected from the knowledge of the global generating hypergraph $\Delta(G)$. Finally in Section 4, answering a question asked in [6], we classify the finite solvable groups whose generating hypergraph is the basis hypergraph of a matroid.

This paper uses standard notations. To learn the fundamental concepts and symbols of group theory that are not covered here, the reader can consult [27].

2. Connectedness of the generating hypergraphs

An undirected hypergraph H is a pair $H = (V, E)$ where V is a set of elements called nodes or vertices and E is a non-empty subset of $\mathcal{P}(V)$ (power set of V) called hyperedges.

A path in a hypergraph $H = (V, E)$ between two distinct vertices x_1 and x_k is a sequence $x_1, e_1, \dots, x_{k-1}, e_{k-1}, x_k$ with the following properties:

- (a) x_1, \dots, x_k are distinct vertices,
- (b) e_1, \dots, e_{k-1} are hyperedges (not necessarily distinct),
- (c) $x_j, x_{j+1} \in e_j$ for $j = 1, 2, \dots, k-1$.

Two vertices $v, w \in V$ are said to be connected in H if there exists a path between v and w in H . A hypergraph H is connected if every pair of vertices $v, w \in V$ is connected in H .

Let G be a finite non-cyclic group. Recall that the generating hypergraph $\Gamma(G)$ has as vertices the elements of G and as hyperedges the generating sets of G of size $d(G)$, while the hyperedges of the global generating graph $\Delta(G)$ are the minimal generating sets of G .

An obstacle to the connectedness of the generating and global generating hypergraph of a finite group G is the possible presence of isolated vertices (certainly the identity element of G is an isolated vertex in both hypergraphs). It is so interesting to describe the set of the isolated vertices of $\Gamma(G)$ and $\Delta(G)$ and to investigate whether the hypergraphs $\tilde{\Gamma}(G)$ and $\tilde{\Delta}(G)$ obtained from $\Gamma(G)$ and $\Delta(G)$ by removing the isolated vertices are connected.

In the case of the global generating hypergraph, the answer to the question follows immediately from the results obtained in [18]. In that paper it is defined and investigated the *independence graph* of G : its vertices are the elements of G and two vertices x_1 and x_2 are joined by an edge if and only if $x_1 \neq x_2$ and there exists a minimal generating set of G containing x_1 and x_2 . In particular Lemma 4 and Theorem in [18] imply:

Proposition 2.1. *Let G be a non-cyclic finite group. Then $g \in G$ is an isolated vertex of the independence graph of G if and only if $g \in \text{Frat}(G)$. Moreover the subsubgraph of the independence graph of G induced by its non-isolated vertices is connected.*

Clearly the independence graph and the global generating hypergraph of G have the same isolated vertices and $\tilde{\Delta}(G)$ is connected if and only if it is connected the subgraph of the independence graph of G induced by its non-isolated vertices. Hence the previous proposition leads to the following conclusion:

Theorem 2.2. *Let G be a finite group. The set of isolated vertices of $\Delta(G)$ coincides with the Frattini subgroup $\text{Frat}(G)$ of G and the hypergraph $\tilde{\Delta}(G)$ is connected.*

Moving from $\Delta(G)$ to $\Gamma(G)$ the role of the independence graph is played by the *rank graph* of G , in which two distinct vertices x_1 and x_2 are joined by an edge if and only if $x_1 \neq x_2$ and there exists a generating set of G of size $d(G)$ containing x_1 and x_2 . Again the isolated vertices of the rank graph and of the generating hypergraph are the same, but it is difficult to give a good description of the set of these isolated vertices. It is in general larger than the Frattini subgroup (for example it contains all the elements of any normal subgroup N of G with the property that $d(G) = d(G/N)$) and it is not in general a subgroup. When $d(G) = 2$, ‘generating graph’, ‘rank graph’ and ‘generating hypergraph’ are different names for the same object. It seems reasonable to conjecture that the subgraph of the generating graph of a 2-generated finite group G induced by its non-isolated vertices is connected. This has been proved under additional assumptions, for example if G is solvable [8] or if G is a direct product of simple groups [7], but the question whether this is true in general is quite difficult and still widely open. However the situation is better when $d(G) \geq 3$. In a recent joint paper with D. Nemmi [25] we proved that, if $d(G) \geq 3$, then the subgraph of the rank graph of G induced by its non-isolated vertices is connected. This implies the following theorem:

Theorem 2.3. *Let G be a finite group. If $d(G) \geq 3$, then the hypergraph $\tilde{\Gamma}(G)$ is connected.*

3. Recognizing a finite group by its generating hypergraphs

For a given $t \in \mathbb{N}$, denote by $P_G(t)$ the probability that an element (g_1, \dots, g_t) of G^t is a generating t -uple for G , i.e. $G = \langle g_1, \dots, g_t \rangle$. If $t \in \mathbb{N}$, then (g_1, \dots, g_t) is a generating t -uple of G if and only if $\{g_1, \dots, g_t\}$ contains an hyperedge of $\Delta(G)$. So from the knowledge of the global generating hypergraph $\Delta(G)$ we can determine $P_G(t)$ for every $t \in \mathbb{N}$. When we know the value of $P_G(t)$ for every $t \in \mathbb{N}$ (or at least for $|G|$ different values of t), we can deduce a great amount of information on the structure of the finite group G . Let us explain why. For any finite group G we may define a sequence of integers $\{a_n(G)\}_{n \in \mathbb{N}}$ as follows:

$$a_n(G) = \sum_{|G:H|=n} \mu_G(H).$$

Here μ_G is the Möbius function defined on the subgroup lattice of G as $\mu_G(G) = 1$ and $\mu_G(H) = -\sum_{H < K} \mu_G(K)$ for any $H < G$. We recall the following nice result, proved by P. Hall [15].

Theorem 3.1. *Let G be a finite group. Then, for every positive integer t ,*

$$P_G(t) = \sum_{n \in \mathbb{N}} a_n(G)/n^t.$$

Lemma 3.2. *Let G be a finite group. Then the knowledge of the hypergraph $\Delta(G)$ allows to determine $a_n(G)$ for every $n \in \mathbb{N}$.*

Proof. Let $m = |G|$. It follows from Theorem 3.1 that $(a_1(G), \dots, a_m(G))$ is a solution of the system

$$\begin{cases} \frac{x_1}{1} + \frac{x_2}{2} + \dots + \frac{x_m}{m} = P_G(1), \\ \frac{x_1}{1} + \frac{x_2}{2^2} + \dots + \frac{x_m}{m^2} = P_G(2), \\ \vdots \quad \vdots \quad \dots \quad \vdots \quad \quad \quad \vdots \\ \frac{x_1}{1} + \frac{x_2}{2^m} + \dots + \frac{x_m}{m^m} = P_G(m). \end{cases}$$

Since

$$\det \begin{pmatrix} 1 & 2 & \dots & m \\ 1 & 2^2 & \dots & m^2 \\ \vdots & \vdots & \dots & \vdots \\ 1 & 2^m & \dots & m^m \end{pmatrix} \neq 0,$$

the system has a unique solution, hence the Dirichlet polynomial $P_G(s)$ is uniquely determined from the knowledge of $P_G(1), P_G(2), \dots, P_G(m)$ and, consequently, from the knowledge of the hypergraph $\Delta(G)$. \square

Thanks to Lemma 3.2, we can immediately say that every property of a group G that can be deduced from the knowledge of the integers $\{a_n(G)\}_{n \in \mathbb{N}}$ can be also deduced from the knowledge of the hypergraph $\Delta(G)$. So it is useful to recall some results relating properties of $\{a_n(G)\}_{n \in \mathbb{N}}$ to structural properties of G .

Lemma 3.3. *A finite group G is p -solvable if and only if, for every $c, d \in \mathbb{N}$, $a_{p^c d}(G) = a_{p^c}(G)a_d(G)$, whenever p and d are relatively prime. In particular G is solvable if and only if $a_{rs}(G) = a_r(G)a_s(G)$, whenever r and s are relatively prime.*

Proof. See [11, Theorem 1] and [9, Theorem 1.2]. □

Lemma 3.4. *Let G be a finite non-abelian simple group. If $a_n(X) = a_n(G)$ for every $n \in \mathbb{N}$, then $X/\text{Frat}(X)$ is a finite non-abelian simple group.*

Proof. See [10, Theorem 7]. □

Theorem 3.5. *Let G be a finite group and let p be a prime divisor of $|G|$. From the knowledge of the hypergraph $\Delta(G)$ it is possible to detect whether G is p -solvable. In particular it can be recognized whether G is solvable.*

Proof. The proof follows from Lemma 3.2 combined with Lemma 3.3. □

Theorem 3.6. *Let G be a finite non-abelian simple group and let X be a finite group. If $\Delta(X) \cong \Delta(G)$, then $X \cong G$.*

Proof. By Lemma 3.2, $a_n(X) = a_n(G)$ for every $n \in \mathbb{N}$. So, by Lemma 3.4, $X/\text{Frat}(X)$ is a finite non-abelian simple group. Moreover $\text{Frat}(G) = 1$, hence, by Theorem 2.2, the identity element is the unique isolated vertex of $\Delta(G)$. Thus $\Delta(X)$ has the same property, and consequently $\text{Frat}(X) = 1$. Hence X and G are simple groups with the same order. By [16, Theorem 5.1], if G and X are not isomorphic, then G and X either are $A_2(4)$ and $A_3(2)$ or are $B_n(q)$ and $C_n(q)$ for some $n \geq 3$ and some odd q . However, a direct computation shows that $P_2(A_2(4)) \neq P_2(A_3(2))$, and in [21, Section 7] it is proved that $P_2(B_n(q)) \neq P_2(C_n(q))$. Hence $X \cong G$. □

The procedure described in the proof of Theorem 3.5 in order to recognize from the knowledge of the global generating hypergraph $\Delta(G)$ whether the finite group G is solvable is quite complicate. One has to determine the rational number $P_G(t)$ for several values of t (this is in principle an elementary task but requires a boring enumeration task) and to solve a system with many equations and indeterminates. So it is natural to ask whether there is some other more directed criterion, which relies for example on the topological properties of the hypergraph $\Delta(G)$.

4. Basis hypergraph of a matroid

Recall that a matroid is a pair (X, \mathcal{B}) , where X is a set and \mathcal{B} is a non-empty family of subsets of X (the bases of the matroid) having the following properties:

- (a) no member of \mathcal{B} contains another;

- (b) (the Exchange Axiom): if A, B are bases and $a \in A \setminus B$, then there exists $b \in B \setminus A$ such that $(A \setminus \{a\}) \cup \{b\}$ is a basis.

This definition represents an axiomatisation of the notion of basis in a vector space. There are other many possible definitions of a matroid, in terms of the independent sets, the rank function, the circuits (minimal dependent sets), the hyperplanes, the subspaces, etc. (see [28] and [26] for this). An important class of hypergraphs consists of the basis hypergraphs of matroids. Indeed given a matroid (X, \mathcal{B}) , we may define an hypergraph whose vertices are the elements of X and whose hyperedges are the subset in \mathcal{B} . The authors of [6] asked the following question:

Question 4.1. *Is it possible to describe groups whose generating hypergraph is the basis hypergraph of a matroid?*

Notice that it follows from the Exchange Axiom that any two bases have the same number of elements; that is, the basis hypergraph is uniform, i.e. all the hyperedges have the same size. So if the global generating graph of a finite group G is the basis hypergraph of a matroid, then all the minimal generating sets of G have the same size (in which case $\Gamma(G)$ and $\Delta(G)$ coincide).

Clearly no minimal generating set of a finite group G contain another, so Question 4.1 is equivalence to ask whether the generating sets of G of minimal size satisfy the Exchange Axiom. With this motivation, we want to study the finite groups G which satisfy the following property: if $x_1, \dots, x_{d(G)}, y_1, \dots, y_{d(G)}$ are elements of G and $G = \langle x_1, \dots, x_{d(G)} \rangle = \langle y_1, \dots, y_{d(G)} \rangle$, then for every $1 \leq i \leq d(G)$, there exists j such that

$$\langle x_1, \dots, x_{i-1}, y_j, x_{i+1}, \dots, x_{d(G)} \rangle = G.$$

We call this property the ‘minimal generating set exchange property’ (MGSE property). If the previous property fails for the two generating sets $X = \{x_1, \dots, x_{d(G)}\}$ and $Y = \{y_1, \dots, y_{d(G)}\}$ we will say that X and Y fail the MGSE property.

We start our investigation with the following elementary observation.

Lemma 4.2. *Let G be a finite group. Then G satisfies the MGSE property if and only if $G/\text{Frat}(G)$ satisfies the MGSE property.*

We will use the following result by Gaschütz [13] to prove that the MGSE property is inherited by the quotients.

Lemma 4.3. *Let N be a normal subgroup of G and $\langle g_1N, \dots, g_rN \rangle = G/N$. If $r \geq d(G)$, then we can find u_1, \dots, u_r in N so that $\langle g_1u_1, \dots, g_ru_r \rangle = G$.*

Lemma 4.4. *Let G be a finite group. If G satisfies the MGSE property, then G/N satisfies the MGSE property for every normal subgroup N of G .*

Proof. Let $N \trianglelefteq G$, $u = d(G/N)$, $v = d(G)$. Suppose that the two minimal generating

sets $\tilde{X} = \{x_1N, \dots, x_uN\}$ and $\tilde{Y} = \{y_1N, \dots, y_uN\}$ fail the MGSE property. We may assume that $\{y_jN, x_2N, \dots, x_uN\} \neq G/N$ for every $1 \leq j \leq u$. By Lemma 4.3, there exist $m_1, \dots, m_v, n_1, \dots, n_v \in N$, such that

$$G = \langle x_1m_1, \dots, x_um_u, m_{u+1}, \dots, m_v \rangle = \langle y_1n_1, \dots, y_un_u, n_{u+1}, \dots, n_v \rangle.$$

We claim that the minimal generating sets

$$X = \{x_1m_1, \dots, x_um_u, m_{u+1}, \dots, m_v\} \text{ and } Y = \{y_1n_1, \dots, y_un_u, n_{u+1}, \dots, n_v\},$$

of G fail the MGSE property. Indeed, for every $y \in Y$,

$$\langle yN, x_2m_2N, \dots, x_um_uN, m_{u+1}N, \dots, m_vN \rangle = \langle yN, x_2N, \dots, x_uN \rangle \neq G/N.$$

□

By [6, Theorem 5.1], a finite p -group satisfies the MGSE property. We are going to prove that this is the unique case in which a non-cyclic finite nilpotent group satisfies the MGSE property.

Lemma 4.5. *Let G be a non-cyclic finite group which satisfies the MGSE property. If N is a normal subgroup of G and G/N is cyclic, then G/N is a p -group. In particular G/G' is a p -group.*

Proof. Suppose, by contradiction, that $G/N = \langle gN \rangle$ and that $|gN|$ is not a prime-power. Then there exist $g_1, g_2 \in G$ such that $g = g_1g_2$ and neither g_1N nor g_2N generates G/N . Let $d = d(G)$. By Lemma 4.3, there exist $m_1, \dots, m_d, n_1, \dots, n_d \in N$ such that

$$G = \langle gm_1, m_2, \dots, m_d \rangle = \langle g_1n_1, g_2n_2, n_3, \dots, n_d \rangle.$$

Then $A = \{gm_1, m_2, \dots, m_d\}$ and $B = \{g_1n_1, g_2n_2, n_3, \dots, n_d\}$ fail the MGSE property. Indeed there is no $b \in B$ such that $G = \langle b, m_2, \dots, m_d \rangle$. By Lemma 4.4 this contradicts the fact that G satisfies the MGSE property. □

Corollary 4.6. *Let G be a non-cyclic finite nilpotent group. Then G satisfies the MGSE property if and only if G is a p -group.*

Proof. It is easy to see that a finite p -group satisfies the MGSP property (see [6, Theorem 5.1]). Conversely let G be a finite non-cyclic nilpotent group which satisfies the MGSE property. By Lemma 4.5 G/G' is a p -group. This implies that G itself is a p -group. □

Now we need to recall a result proved in [24].

Lemma 4.7. *Let G be a 2-generated solvable non-nilpotent group. If the generating graph of G is a cograph, then $G/\text{Frat}(G) \cong V \rtimes \langle x \rangle$, where x has prime order and V is a faithful irreducible.*

Proof. See [24, Theorem 1.7]. □

Lemma 4.8. *Let G be a 2-generated solvable non-nilpotent group. If G satisfies the MGSE property, then $G/\text{Frat}(G) \cong V \rtimes \langle x \rangle$, where x has prime order and V is a faithful irreducible $\langle x \rangle$ -module.*

Proof. Let G be a 2-generated non-nilpotent solvable group. By Lemma 4.7 either $G/\text{Frat}(G) \cong V \rtimes \langle x \rangle$, where x has prime order and V is a faithful irreducible $\langle x \rangle$ -module, or the generating graph $\Gamma(G)$ of G is not a cograph. In the second case, $\Gamma(G)$ contains four vertices x_1, x_2, x_3, x_4 such that the subgroup of $\Gamma(G)$ induced by them is the the four-vertex path $x_1 - x_2 - x_3 - x_4$. This implies that $A = \{x_1, x_2\}$ and $B = \{x_3, x_4\}$ fail the MGSE property, since there is no $b \in A$ such that $\langle x_1, b \rangle = G$. □

In the following lemma we collect some results on the generating property of the semidirect product $V \rtimes H$, when H is a non-cyclic finite solvable group and V is a faithful irreducible H -module.

Lemma 4.9. *Assume that H is a non-cyclic finite solvable group and let V be a faithful irreducible H -module. Moreover assume that $H = \langle h_1, \dots, h_{d(H)} \rangle$. Then there exists $1 \leq i \leq d$ and $n_1, \dots, n_d \in N$ such that*

$$G = \langle h_1, \dots, h_{i-1}, h_i n_i, h_{i+1}, \dots, h_d \rangle = \langle h_1 n_1, \dots, h_{i-1} n_{i-1}, h_i, h_{i+1} n_{i+1}, \dots, h_d n_d \rangle.$$

Proof. It follows from [22, Lemma 2] and [23, Proposition 2.2] and its proof. □

Lemma 4.10. *Assume that H is a non-cyclic finite solvable group and let V be a faithful irreducible H -module. Then $G = V \rtimes H$ does not satisfy the MGSE property.*

Proof. Let $d = d(H)$. By Lemma 4.9, it is not restrictive to assume that there exist $n_1, \dots, n_d \in N$ such that $G = \langle h_1 n_1, h_2, \dots, h_d \rangle = \langle h_1, h_2 n_2, \dots, h_d n_d \rangle$. We claim that the generating sets $X = \{h_1 n_1, h_2, \dots, h_d\}$ and $Y = \{h_1, h_2 n_2, \dots, h_d n_d\}$ fail the MGSE property. Indeed assume that there exists $y \in Y$ such that $\langle y, h_2, \dots, h_d \rangle = G$. Since $\langle h_1, \dots, h_d \rangle = H < G$, it must be $y = h_i n_i$ for some $2 \leq i \leq d$. But this would imply $\langle h_i n_i, h_2, \dots, h_d \rangle \leq \langle h_2, \dots, h_d \rangle N < HN = G$. □

Lemma 4.11. *Let G be a finite non-nilpotent solvable group. If G satisfies the MGSE property, then $G/\text{Frat}(G) \cong N^\delta \rtimes H$, where δ is a positive integer, H is cyclic of prime order, N is an irreducible H -module and $C_H(V) = 1$.*

Proof. It follows from Lemma 4.2 and the fact that G is nilpotent if and only if $G/\text{Frat}(G)$ is nilpotent, that we may assume $\text{Frat}(G) = 1$.

But then we may write $\text{Fit}(G) = N_1 \times \dots \times N_t$ as a product of abelian minimal normal subgroups of G . Since $\text{Frat}(G) = 1$, $\text{Fit}(G)$ has a complement H in G . Let $C_i = C_H(N_i)$ and $M_i = \prod_{i \neq j} N_j$. Then $M_i C_i = C_G(N_i)$ is a normal subgroup of G and $G/M_i C_i \cong N_i \rtimes H/C_i$. By Lemma 4.10, H/C_i is cyclic. Since $C_G(\text{Fit}(G)) = \text{Fit}(G)$,

$$H \cong \frac{G}{\text{Fit}(G)} \cong \frac{G}{\bigcap_{1 \leq i \leq t} C_G(N_i)} \leq \prod_{1 \leq i \leq t} \frac{G}{C_G(N_i)} \cong \prod_{1 \leq i \leq t} \frac{H}{C_i}.$$

Hence H is abelian and it follows from Lemma 4.5 that H is a p -group.

There exists $1 \leq i \leq t$ such that $[H, N_i] \neq 1$ (otherwise G would be abelian and consequently $G = \text{Fit}(G)$). This implies in particular $(|N_i|, p) = (|N_i|, |H|) = 1$. As we noticed before, H/C_i is cyclic, so we may assume $H = \langle h_1, h_2, \dots, h_u \rangle$ with $h_j \in C_i$ if $j \geq 2$. Let $X = N_i H$ and $1 \neq n \in N$. Notice that, since $(|C_i|, |N_i|) = 1$, $\langle hm \rangle = \langle h, m \rangle$ for every $h \in H$ and $m \in N$. In particular, if $u := d(H) \geq 3$, then $A = \{h_1, \dots, h_{u-2}, h_{u-1}n, h_u\}$ and $B = \{h_1, \dots, h_{u-2}, h_{u-1}, h_u n\}$ are two minimal generating sets of X which fail the MGSE property, since there is no $b \in B$ such that $G = \langle h_1, \dots, h_{u-2}, b, h_u \rangle$. Since X is an epimorphic image of G , it follows from Lemma 4.4 that the assumption $u \geq 3$ leads to a contradiction. Hence $d(H) \leq 2$.

Let J be a subset of $I = \{1, \dots, t\}$ with the property that for any $i \in I$ there exists one and only one $j \in J$ such that N_i is H -isomorphic to N_j . Let $Y = (\prod_{j \in J} N_j)H$. By Lemma 4.4, Y satisfies the MGSE property. The maximal subgroups of Y are:

- (a) $(\prod_{j \in J} M_j) \text{Frat}(H)$;
- (b) $(\prod_{j \neq k} M_j) H^y$ with $k \in J$ and $y \in M_j$.

This implies that

$$\text{Frat}(Y) = \bigcap_{j \in J} C_{\text{Frat}(H)}(N_j) = \bigcap_{i \in I} C_{\text{Frat}(H)}(N_i) = \text{Frat}(G) = 1.$$

Moreover it follows from [14, Satz 2 and Satz 3] that $d(Y) = 2$. But then we may deduce from Lemma 4.8 that $|J| = 1$ and $|H| = p$. □

Lemma 4.12. *Let V be a faithful irreducible $\langle x \rangle$ -module with $|x| = p$ a prime. Then, for every positive integer δ , the semidirect product $G = V^\delta \rtimes \langle x \rangle$ satisfies the MGSE property.*

Proof. We may identify V with the additive group of a finite field F and x with an element of order p of the multiplicative group of F . Notice that $d(G) = \delta + 1$. Moreover it follows from [19, Proposition 2.1 and 2.2] that

$$\langle (v_{1,1}, \dots, v_{1,\delta+1})x_1, \dots, (v_{\delta+1,1}, \dots, v_{\delta+1,\delta+1})x_{\delta+1} \rangle = G,$$

if and only if

$$\det \begin{pmatrix} x_1 - 1 & \cdots & x_{\delta+1} - 1 \\ v_{1,1} & \cdots & v_{\delta+1,1} \\ \vdots & \vdots & \vdots \\ v_{1,\delta+1} & \cdots & v_{\delta+1,\delta+1} \end{pmatrix} \neq 0.$$

In particular, to prove that two generating sets

$$\begin{aligned} X_1 &= \{(v_{1,1,1}, \dots, v_{1,\delta+1,1})x_{1,1}, \dots, (v_{1,\delta+1,1}, \dots, v_{1,\delta+1,1})x_{1,\delta+1}\}, \\ X_2 &= \{(v_{2,1,1}, \dots, v_{2,\delta+1,1})x_{2,1}, \dots, (v_{2,\delta+1,1}, \dots, v_{2,\delta+1,1})x_{2,\delta+1}\}, \end{aligned}$$

satisfy the MGSE property it suffices to notice that, given the two-matrices

$$A_1 = \begin{pmatrix} x_{1,1} - 1 & \cdots & x_{1,\delta+1} - 1 \\ v_{1,1,1} & \cdots & v_{1,\delta+1,1} \\ \vdots & \vdots & \vdots \\ v_{1,1,\delta+1} & \cdots & v_{1,\delta+1,\delta+1} \end{pmatrix}, \quad A_2 = \begin{pmatrix} x_{2,1} - 1 & \cdots & x_{2,\delta+1} - 1 \\ v_{2,1,1} & \cdots & v_{2,\delta+1,1} \\ \vdots & \vdots & \vdots \\ v_{2,1,\delta+1} & \cdots & v_{2,\delta+1,\delta+1} \end{pmatrix},$$

for any $i \in \{1, \dots, \delta + 1\}$ there exists $j \in \{1, \dots, \delta + 1\}$ such that the matrix obtained replacing the i -th column of A_1 with the j -th column of A_2 is still invertible. \square

We may summarize the results obtained in this section in the following statement:

Theorem 4.13. *A non-cyclic finite solvable group G satisfies the MGSE property if and only if $G/\text{Frat}(G) \cong N^\delta \rtimes H$, where δ is a positive integer, H is cyclic of prime order, N is an irreducible H -module and $C_H(V) = 1$.*

Combining the previous theorem with the main result of [1], it turns out that if a finite group G satisfies the MGSE property, then all the minimal generating sets of G have the same size, or equivalently $\Delta(G) = \Gamma(G)$. However the converse is not true. For example, let G be the solvable subgroup of $Sym(5)$ generated by $x = (2, 3, 4, 5)$ and $y = (1, 2, 3, 5, 4)$. Then all the minimal generating sets of G have size 2, however $A = \{x^2, xy\}$ and $B = \{x, y\}$ are two generating sets that fail the MGSE property, since $\langle x^2, x \rangle$ and $\langle x^2, y \rangle$ are proper subgroups of G .

We conjecture that a finite group which satisfies the MGSE property must be solvable. The following are partial results supporting this conjecture.

Theorem 4.14. *A finite non-abelian simple group does not satisfy the MGSE property.*

Proof. It follows from the fact that the generating graph of a finite non-abelian simple group contains a 4-path [24, Corollary 1.10]. \square

Corollary 4.15. *Let G be a non-nilpotent finite group. If G satisfies the MGSE property, then G contains a unique maximal normal subgroup, say N , and G/N is cyclic of order p . In particular G is non-perfect and G/G' is a cyclic p -group.*

Proof. It follows from Lemmas 4.4, 4.5 and the previous theorem. \square

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